A NEW METHOD IN FANO GEOMETRY

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Abstract. We give some bounds on the anticanonical degrees of Fano varieties with Picard number 1 and mild singularities. The proof is based on a study of positivity properties of sheaves of differential operators on ample line bundles.

1. Introduction

1.1. **Fano varieties.** The purpose of this paper is to bound the degrees of large classes of Fano varieties.

Definition 1.1. A *unipolar* $\mathbb{Q}$-*Fano variety* is an n-dimensional complex projective variety X such that
i) X is normal and $\mathbb{Q}$-factorial,
ii) the set of Weil divisors modulo numerical equivalence forms a group

$$\mathbb{Z}\cdot\{\mathbb{D}_{\mathbb{X}}\}\cong\mathbb{Z}$$

with D_X a Weil divisor,
iii) the $\mathbb{Q}$-Cartier $(-K_X)$ is ample. We write

$$-K_X=i_X\{D_X\},$$

for some positive integer i_X.

The positive integer i_X is called the *Weil index* of X. Also we define

$$t_X$$

to be the smallest positive integer such that

$$t_X D_X$$

is Cartier. Then, of course,

$$(K_X)^n=\frac{(t_X K_X)^n}{t_X^n}.$$

By the Appendix to §1 of [Re], the group of Weil divisors is isomorphic to the set of saturated, torsion-free rank-one sheaves on X, which in turn is the same as the set of rank-one reflexive sheaves. These sheaves are called *divisorial sheaves.* For example, on a Cohen-Macaulay, normal variety, the dualizing sheaf is always divisorial.

We let

$$X'$$

Date: January, 2000.

denote the smooth points of X. Then any divisorial sheaf is equal to the push-forward of its restriction to any subset of X whose complement has codimension at least two, in particular from X'.

1.2. **One-canonical singularities.**

Definition 1.2. A normal, Cohen-Macaulay variety X is said to be 1-*canonical* if, for any resolution

$$\epsilon : Y \to X$$

the differential

$$d\epsilon : \epsilon^* (\Omega_X) \to \Omega_Y$$

factors through a map

$$\epsilon^* (\Omega_X^{\vee\vee}) \to \Omega_Y,$$

that is, 1-forms on X' lift to holomorhic forms on Y.

Notice that this is the precise analogue with respect to one-forms of the condition on top forms which defines canonical singularities. Note also that the condition is automatically satisfied whenever the natural map

$$\Omega_X \to \Omega_X^{\vee\vee}$$

is surjective. It can be shown that any locally finite quotient of a smooth in codimension 2 complete intersection is 1-canonical (see [Ra]).

1.3. **The theorem.** We choose a very ample polarization

$$H \equiv \textit{high multiple of } \ \{D_X\}$$

and let

$$C \equiv H^{n-1} \subseteq X' \tag{1}$$

be a generic linear section curve of the embedding of X given by H.

Theorem 1.1. *i) Let X be a unipolar $\mathbb{Q}$-Fano variety with only log-terminal, 1-canonical singularities. Then*

$$(-K)_X^n \leq \left(\max.\left\{\frac{2\,(C\cdot(-K_X))}{\mu_{\min.}(T_X)}, i_X\right\}\right)^n \leq \left(\max.\left\{\frac{2\,(C\cdot(-K_X))}{\mu_{\min.}(T_X)}, t_X\,(n+1)\right\}\right)^n$$

where

$$\mu_{\min.}(T_X)$$

is the minimum slope of the subquotients in a Harder-Narasimhan filtration of the tangent bundle T_X of X with respect to the polarization H (see §5 below), and we always have

$$\frac{C\cdot(-K_X)}{\mu_{\min.}(T_X)} \leq i_X.$$

ii) Let X be a unipolar $\mathbb{Q}$-Fano variety. Suppose that T_X is semi-stable. Then

$$\frac{C\cdot(-K_X)}{\mu_{\min.}(T_X)} = n$$

so that

$$(-K)_X^n \leq (\max.\{2n, t_X\,(n+1)\})^n.$$

In this paper we give a complete proof of this theorem. A slightly more general result is proved with a slightly different viewpoint in [Ra], which also contains some ancillary definitions, examples and technical details, as well as a number of applications. Here our purpose has been to clarify the main ideas of the proof, and we have thus steered a direct course to the main result while trying to make the argument comprehensible to nonexperts. At this point, we suggest that the first-time reader skip immediately to §11 below in order to get an idea of the strategy of the proof. Suffice it to say here that the proof is based on positivity properties of sheaves of differential operators and in particular is completely independent of rational curves and bend-and-break which were used heavily in earlier approaches to boundedness of Fano varieties (see [Ko] and references therein).

Both authors would like to thank Paul Burchard, János Kollár, and Robert Lazarsfeld for helpful communications and especially to thank James McKernan for several key suggestions and at least one important correction.

2. Good resolution of X

We will need to understand pull-back of divisorial sheaves under "nice" resolution of X. Let

$$\epsilon : Y \to X$$

be a resolution obtained by a succession of "modifications," where a modification

$$X_{i+1} \subseteq \mathbb{P}^{n_{i+1}}$$

of

$$X_i \subseteq \mathbb{P}^{n_i}$$

is obtained by blowing up $\mathbb{P}^{n_i}$ along a smooth center inside the singular locus of X_i, then embedding the proper transform X_{i+1} of X_i into a projective space $\mathbb{P}^{n_{i+1}}$ via a sufficiently high multiple of the ample divisor

$$m_i \mathcal{O}_{\mathbb{P}^{n_i}}(1) - F_i,$$

F_i being the exceptional divisor of the blow-up. Repeating this process as necessary, we arrive at a smooth projective manifold Y such that the exceptional locus $E = \bigcup_i E_i$ lies over the singular set of X and is a is a simple-normal-crossing divisor. Y has Neron-Severi group (modulo numerical equivalence, given by

$$NS(Y) = \mathbb{Z} \cdot \tilde{\mathbb{D}}_{\mathbb{X}} \oplus \sum\nolimits_{\beth} \mathbb{Z} \cdot \mathbb{E}_{\beth}$$

where $\tilde{D}_X$ denotes the proper transform of the Weil divisor D_X.

If M is a divisorial sheaf on X, then locally at any singular point of X,

$$M = I \cdot L$$

with I the ideal sheaf of an effective Weil divisor and L locally free of rank one. So we define the "integral-divisorial" pull-back $\epsilon^*_{id.} M$ of M to be the line bundle on Y given by sections of $\epsilon^* L$ whose order along each divisor B is greater than or equal to

$$\min \{ ord_B (f \circ \epsilon) : f \in I \}.$$

along that divisor. If M is Cartier, then pull-back is just pull-back of line bundles. However, if M is not Cartier, the natural map

$$m\epsilon_{id.}^{*}M\rightarrow\epsilon^{*}mM$$

is not necessarily an isomorphism. Later we will need the fact that, for any morphism

$$M\rightarrow F$$

for M divisorial and F locally free, there is an induced morphism

$$\epsilon_{id.}^{*}M\rightarrow\epsilon^{*}F.$$

We have, for example, for any positive integer m',

$$\epsilon_{id.}^{*}m'D_{X}=\tilde{D}_{X}+\sum\nolimits_{i}\tilde{a}_{i}\left(m'\right)E_{i} \tag{2}$$

where the $\tilde{a}_{i}\left(m'\right)$ are integers. On the other hand, for M divisorial, one has the divisorial pull-back

$$\epsilon_{div.}^{*}M:=\frac{1}{m}\epsilon^{*}mM$$

where mM is Cartier. So we have

$$\epsilon_{div.}^{*}D_{X}=\tilde{D}_{X}+\sum\nolimits_{i}\frac{a_{i}}{t_{X}}E_{i} \tag{3}$$

with the a_{i} non-negative integers. Since there is a standard multiplication map

$$\left(\epsilon_{id.}^{*}M\right)^{m}\rightarrow\left(\epsilon_{div.}^{*}M\right)^{m}.$$

we have

$$\tilde{a}_{i}\left(m'\right)\geq\frac{m'a_{i}}{t_{X}}. \tag{4}$$

By construction, we have that the divisor

$$m\tilde{D}_{X}+\sum\nolimits_{i}\left(\frac{a_{i}}{t_{X}}-t_{i}\right)E_{i}.$$

is ample for $0<t_{i}<<1$ and $m>0$. So, by the Kawamata-Viehweg Vanishing Theorem,

$$H^{j}\left(-\epsilon^{*}mt_{X}D_{X}\right)=0$$

for $j\leq dimX$. Also we have

$$\begin{aligned}K_{Y}&=\epsilon_{div.}^{*}K_{X}+\sum\nolimits_{i}b_{i}E_{i}\\&=-i_{X}\tilde{D}_{X}+\sum\nolimits_{i}\left(b_{i}-\frac{i_{X}}{t_{X}}a_{i}\right)E_{i}.\end{aligned}$$

So by duality

(5)

$$H^{j}\left(K_{Y}+m\epsilon^{*}t_{X}D_{X}\right)=H^{j}\left(\left(mt_{X}-i_{X}\right)\tilde{D}_{X}+\sum\nolimits_{i}\left(b_{i}+\left(m-\frac{i_{X}}{t_{X}}\right)a_{i}\right)E_{i}\right)=0$$

for all $j,m>0$.

3. Bounding the Weil index

Referring to (5)

$$\begin{aligned}\chi(a) &= h^0\left(K_Y + \epsilon^* a t_X D_X\right) \\ &= h^0\left(\left(a t_X - i_X\right)\tilde{D}_X + \sum\nolimits_i \left(b_i + \left(a - \frac{i_X}{t_X}\right) a_i\right) E_i\right)\end{aligned}$$

is zero for $1 \leq a t_X \leq (i_X - 1)$, since, in that case, the push-forward of a section would be negative along D_X. To see that $\chi(a)$ is not identically zero, choose large a such that the pullback of $a t_X D_X$ to Y vanishes to order at least i_X on $\tilde{D}_X$ and order at least b_i on each E_i. Thus

$$\left(\frac{i_X - 1}{t_X}\right) \leq \deg \chi(a) \leq n.$$

so that

$$\begin{aligned}\frac{i_X - 1}{t_X} &< n + 1, \\ i_X &\leq t_X (n+1).\end{aligned}$$

(This argument in the smooth case is due to Kobayashi-Ochiai.)

Referring to §1 we therefore have

$$C \cdot D_X = \frac{-C \cdot K_X}{i_X} \geq \frac{-C \cdot K_X}{t_X (n+1)}.$$

4. Atiyah class

Given line bundles L and L' on X', the set of smooth points of X, let

$$\mathfrak{D}^n(L, L')$$

denote the sheal of holomorphic differential operators of order $\leq n$ on sections of L with values in sections of L'. (If $L = L'$, we simply write $\mathfrak{D}^n(L)$.) The sequence

$$0 \to \mathcal{O}_{X'} \to \mathfrak{D}^1(\mathcal{O}_{X'}) \to T_{X'} \to 0$$

splits as a sequence of left $\mathcal{O}_{X'}$-modules but not as a sequence of right $\mathcal{O}_{X'}$-modules. In fact, if we tensor on the right by a line bundle L^* to obtain the exact sequence

$$0 \to L^* \to \mathfrak{D}^1(L, \mathcal{O}_{X'}) \to T_{X'} \otimes L^* \to 0$$

of left $\mathcal{O}_{X'}$-modules and then tensoring this last sequence on the left by L, we obtain the exact sequence

$$0 \to \mathcal{O} \to \mathfrak{D}^1_{T_{X'}}(L) \to T_{X'} \to 0.$$

The obstruction to splitting this last sequence (as a sequence of left modules) is given by taking a meromorphic section l_0 on L and splitting the above sequence over the set where $l_0 \neq 0, \infty$ via

$$\frac{\partial}{\partial x} \longmapsto \left(l \mapsto l_0 \frac{\partial (l/l_0)}{\partial x}\right).$$

Writing patching data $\{z_U\}$ for the divisor of l_0 we have

$$z_U^{-1} \frac{\partial (f \cdot z_U)}{\partial x} - z_{U'}^{-1} \frac{\partial (f \cdot z_{U'})}{\partial x} = f\left(\frac{\partial \log z_U}{\partial x} - \frac{\partial \log z_U}{\partial x}\right)$$

so that the obstruction to splitting is given by

$$c_1(L) \in H^1\left(\Omega^1_{X'}\right).$$

5. Harder-Narasimhan filtration

Throughout we will deal with vector bundles and sheaves modulo codimension-two phenomena. Thus "torsion" means torsion along a divisor, "vector bundle" means locally free through codimension one, etc.

Since the Picard number of X is one and singularities are of codimension two, there is an unambiguous notion of stability (H-stability) of bundles on X'. We will use in an essential way the Harder-Narasimhan filtration

$$E_1 < \ldots < E_{l(E)-1} < E_{X'}$$

of a vector bundle E with

$$\frac{E_i}{E_{i-1}}$$

semi-stable locally free sheaves such that the slopes

$$\mu_i = \frac{c_1\left(\frac{E_i}{E_{i-1}}\right)\cdot C}{rk\left(\frac{E_i}{E_{i-1}}\right)}$$

form a strictly decreasing sequence whose extremal elements are denoted as

$$\mu_{\max.}(E)\,,\ \mu_{\min.}(E)$$

respectively. By results of Mehta-Ramanathan ([MehR]), the above filtration restricts to a Harder-Narasimhan filtration on a generic curve $C \subseteq X'$ and conversely, that is, a filtration that restricts to a HN-filtration on generic C, is an HN-filtration.

If

$$E = T_{X'},$$

then

$$\begin{aligned}\mu_{\max.}(T_X) &= \mu(T_1) = \frac{c_1(T_1)\cdot C}{t_1} \geq \frac{-K_X\cdot C}{n}\\ \mu_{\min.}(T_X) &= \frac{c_1\left(T_{X'}/T_{l(T_{X'})-1}\right)\cdot C}{t_{l(T_{X'})}}\end{aligned}$$

where $t_i = rk\,(T_i/T_{i-1})$. Notice that, if $T_{X'}$ is non-negative, then all the T_i are integrable since the map

$$\begin{aligned}T_i \otimes T_i &\rightarrow \frac{T_{X'}}{T_i}\\ \xi\otimes\eta &\mapsto [\xi,\eta]\end{aligned}$$

is $\mathcal{O}_{X'}$-bilinear and

$$2\mu_{\min.}(T_i) = 2\mu\left(\frac{T_i}{T_{i+1}}\right) > \mu_{\max.}\left(\frac{T_{X'}}{T_i}\right).$$

Similarly suppose that, for some line bundle L, $\mathfrak{D}^1(L)$ is non-negative. Let

$$D_1 \leq \ldots \leq D_{l(D)} = \mathfrak{D}^1(L)$$

be an HN-filtration. Then the map

$$\begin{array}{rcl} D_i \otimes D_i & \rightarrow & \dfrac{\mathfrak{D}^1(L)}{D_i} \\ \xi \otimes \eta & \mapsto & [\xi, \eta] \end{array}$$

is also zero.

We begin by examining the slopes of the HN-filtration of

$$\mathfrak{D}^1(L, \mathcal{O}_{X'})$$

the sheaf of first-order differential operators from a line bundle L on X' to the structure sheaf $\mathcal{O}_{X'}$. Again recalling that all sheaves are taken "modulo codimension two," suppose that T' is a torsion-free subbundle of $T_{X'}$. Then restricting the symbol map

$$\mathfrak{D}^1(\mathcal{O}_{X'}) \rightarrow T_{X'}$$

to the preimage of T', we obtain the sequence

$$0 \rightarrow \mathcal{O}_{X'} \rightarrow \mathfrak{D}^1_{T'}(\mathcal{O}_{X'}) \rightarrow T' \rightarrow 0$$

is exact, so that the sequence

$$0 \rightarrow L^* \rightarrow \mathfrak{D}^1_{T'}(L, \mathcal{O}_{X'}) \rightarrow T' \otimes L^* \rightarrow 0$$

obtained by tensoring *on the right* with L^* is also exact.

6. Relative positivity of first order operators

Lemma 6.1. *Suppose $T_{X'}$ is positive and L is a line bundle with $L \cdot C \neq 0$. Then*

$$\mathfrak{D}^1(L)$$

is positive, in fact

$$\mu_{\min.}\left(\mathfrak{D}^1(L)\right) \geq \min\left\{C \cdot D_X, \frac{1}{2}\mu_{\min.}(T_{X'})\right\} =: b.$$

Proof. Consider the semi-stable quotient

$$\mathfrak{D}^1(L) \rightarrow \frac{D_l}{D_{l-1}} =: F.$$

in an HN-filtration for $\mathfrak{D}_1(L)$. Thus

$$\mu_{\min.}\left(\mathfrak{D}^1(L)\right) = \mu(F).$$

The composition

$$\mathcal{O}_{X'} \rightarrow \mathfrak{D}^1(L) \rightarrow F \tag{6}$$

is either zero or injective (through codimension one). If it is zero then we have a quotient map

$$T_{X'} \rightarrow F$$

so that $\mathfrak{D}_1(L)$ is positive since $T_{X'}$ is. In fact, in that case,

$$\mu_{\min.}\left(\mathfrak{D}^1(L)\right) \geq \mu_{\min.}(T_{X'}).$$

If the composition (6) is injective, let

$$M$$

denote the saturation of the image of $\mathcal{O}$ in F. If

$$\mathcal{O} \neq M, \tag{7}$$

then

$$c_1(M) \geq D_X \cdot C$$

so that, by the semi-stability of F,

$$\mu(F) \geq \frac{1}{2}\mu_{\min.}(T_{X'}).$$

If

$$\mathcal{O} = M,$$

then

$$\frac{F}{M}$$

is a torsion-free quotient of F and

$$\begin{aligned} c_1(F) &= c_1\left(\frac{F}{M}\right), \\ rkF &= 1 + rk\left(\frac{F}{M}\right) \end{aligned}$$

so that

$$\mu(F) = \frac{c_1\left(\frac{F}{M}\right)}{1 + rk\left(\frac{F}{M}\right)}$$

where

$$\mu\left(\frac{F}{M}\right) = \frac{c_1\left(\frac{F}{M}\right)}{rk\left(\frac{F}{M}\right)} \geq \mu_{\min.}(T_{X'}).$$

Now, $M \neq F$ by §4, so

$$\mu_{\min.}\left(\mathfrak{D}^1(L)\right) \geq \frac{rk\left(\frac{F}{M}\right)}{1 + rk\left(\frac{F}{M}\right)} \cdot \mu_{\min.}(T_{X'}) \geq \frac{1}{2}\mu_{\min.}(T_{X'}).$$

□

7. Extending to the case of vector bundles

Lemma 7.1. *If E is a positive vector bundle on X', then*

$$\mu_{\min.}(\mathfrak{D}^1(E, \mathcal{O})) \geq \mu_{\min.}(E^*) + b$$

where

$$b = \min\left\{D_X \cdot C, \frac{1}{2}\mu_{\min.}(T_{X'})\right\}.$$

Proof. Via a HN-filtration for E and the isomorphism

$$\mathfrak{D}^1(E', \mathcal{O}) \to \frac{\mathfrak{D}^1(E, \mathcal{O})}{\mathfrak{D}^1(E/E', \mathcal{O})}$$

reduce to the case E semi-stable. Let

$$M = \det E.$$

We know from §6 that

$$\mu_{\min.}\left(\mathfrak{D}^{1}\left(M,\mathcal{O}\right)\right)\geq -M\cdot C+b.$$

Next, deform to the normal cone. Namely blow up $(C\times\{0\})$ in $\left(X\times\mathbb{A}^{\Bbbk}\right)$ and pull E back to the product and let E' be the restriction to the normal bundle $N_{C\backslash X}$ lying inside exceptional divisor. Since the deformation to the normal cone is trivial on a first-order neighborhood of the proper transform of $\left(C\times\mathbb{A}^{\Bbbk}\right)$, we have

$$\mathfrak{D}^{1}\left(E,\mathcal{O}_{C}\right)=\mathfrak{D}^{1}\left(E',\mathcal{O}_{C}\right)=\mathfrak{D}^{1}\left(\nu^{*}E_{C},\mathcal{O}_{C}\right)$$

where $\nu:N_{C\backslash X}\rightarrow C$ is the projection given by the normal bundle. Let $e=rkE$. Taking an unramified e-fold cover

$$\pi:\tilde{C}\rightarrow C,$$

we have

$$\nu^{*}E_{C}\times_{C}\tilde{C}=L\otimes F$$

with L the pullback of a line bundle $L_{\tilde{C}}$.on $\tilde{C}$ and F the pull-back of a semi-stable vector bundle $F_{\tilde{C}}$.on $\tilde{C}$. Also

$$\begin{aligned} e\cdot(c_{1}L) &\equiv \pi^{*}\det E\\ c_{1}F &\equiv 0. \end{aligned}$$

So by the well-known theorem of Narasimhan-Seshadri, $F_{\tilde{C}}$ is given by a locally constant sheaf on $\tilde{C}$. Also the product rule induces an isomorphism

$$\begin{aligned} \mathfrak{D}^{1}\left(L,\mathcal{O}_{\tilde{C}}\right) &\rightarrow \mathfrak{D}^{1}\left(L^{e},L^{e-1}\right)=L^{e-1}\otimes\mathfrak{D}^{1}\left(L^{e},\mathcal{O}_{\tilde{C}}\right)\\ D &\mapsto \left(l_{1}\cdot l_{2}\cdot\ldots\cdot l_{e}\mapsto Dl_{1}\cdot l_{2}\cdot\ldots\cdot l_{e}+l_{1}\cdot Dl_{2}\cdot\ldots\cdot l_{e}+\ldots\right) \end{aligned}$$

so that, since HN-filtrations are preserved under covers, we have by the rank one case that

$$\begin{aligned} \mu_{\min.}\left(\mathfrak{D}^{1}\left(L,\mathcal{O}_{\tilde{C}}\right)\right) &= \mu_{\min.}\left(L^{e-1}\otimes\mathfrak{D}^{1}\left(L^{e},\mathcal{O}_{\tilde{C}}\right)\right)\\ &= (e-1)\,L\cdot\tilde{C}+\mu_{\min.}\left(\pi^{*}\mathfrak{D}^{1}\left(\det\left(\nu^{*}E_{C}\right),\mathcal{O}_{N}\right)\right)\\ &= (e-1)\,L\cdot\tilde{C}+e\cdot\mu_{\min.}\left(\mathfrak{D}^{1}\left(\det\left(\nu^{*}E_{C}\right),\mathcal{O}_{N}\right)\right)\\ &= (e-1)\,L\cdot\tilde{C}+e\cdot\mu_{\min.}\left(\mathfrak{D}^{1}\left(\det E,\mathcal{O}\right)\right)\\ &\geq (e-1)\,L\cdot\tilde{C}+e\cdot\left(-\det E_{C}+b\right)\\ &= -L\cdot\tilde{C}+eb \end{aligned}\tag{8}$$

where $N=N_{C\backslash X}$. So, since E is semi-stable

$$\mu_{\min.}\left(\mathfrak{D}^{1}\left(L,\mathcal{O}_{\tilde{C}}\right)\right)\geq-\frac{\pi^{*}\det E}{e}+eb=e\left(-\mu\left(E\right)+b\right).$$

On the other hand, since $F_{\tilde{C}}$ is locally constant and therefore F is too, we have that

$$\mathfrak{D}^{1}\left(F\otimes L,\mathcal{O}_{\pi^{-1}N}\right)=F\otimes\mathfrak{D}^{1}\left(L,\mathcal{O}_{\pi^{-1}N}\right)$$

so that

$$\mu_{\min.}\left(\mathfrak{D}^{1}\left(E,\mathcal{O}_{C}\right)\right) = \mu_{\min.}\left(\mathfrak{D}^{1}\left(\nu^{*}E_{C},\mathcal{O}_{C}\right)\right)$$
$$= e^{-1}\cdot\mu_{\min.}\left(\pi^{*}\mathfrak{D}^{1}\left(\det\left(\nu^{*}E_{C}\right),\mathcal{O}_{\tilde{C}}\right)\right)$$
$$= e^{-1}\cdot\mu_{\min.}\left(\mathfrak{D}^{1}\left(F\otimes L,\mathcal{O}_{\tilde{C}}\right)\right)$$
$$= e^{-1}\cdot\mu_{\min.}\left(F^{*}\otimes\mathfrak{D}^{1}\left(L,\mathcal{O}_{\tilde{C}}\right)\right)$$
$$= e^{-1}\cdot\mu_{\min.}\left(\mathfrak{D}^{1}\left(L,\mathcal{O}_{\tilde{C}}\right)\right). \quad (9)$$

Taken together (8) and (9) complete the proof in the case E semi-stable. Therefore we are done. □

8. Extending to estimates to higher-order operators

In this section we work over X', the set of smooth points of X. The extension of the above estimates to higher order operators will be made using the elementary fact that (restricting to X') there is a natural surjection

$$\mathfrak{D}^{1}\left(P^{i}\left(E\right),\mathcal{O}\right)\rightarrow\mathfrak{D}^{m+1}\left(E,\mathcal{O}\right). \quad (10)$$

where $P^{i}\left(E\right)$ is the sheaf of i-th order jets of the vector bundle E, that is,

$$p_{*}q^{*}E$$

where p and q are the two projections of the i-th order neighborhood of the diagonal of $X'\times X'$. To see this, assigning to any section its i-th jet to get

$$\mathfrak{D}^{m}\left(E,\mathcal{O}\right)=\mathfrak{Hom}\left(P^{i}\left(E\right),\mathcal{O}\right)$$

where $\mathfrak{Hom}$ is with reference to the left $\mathcal{O}$-linear structure. So we have a left $\mathcal{O}$-linear surjection

$$\mathfrak{D}^{1}\left(P^{i}\left(E\right),\mathcal{O}\right)=\mathfrak{D}^{1}\left(\mathcal{O}\right)\otimes Hom\left(P^{i}\left(E\right),\mathcal{O}\right)=\mathfrak{D}^{1}\left(\mathcal{O}\right)\otimes\mathfrak{D}^{m}\left(E,\mathcal{O}\right)\rightarrow\mathfrak{D}^{m+1}\left(E,\mathcal{O}\right).$$

Lemma 8.1. *Suppose that $T_{X'}$ is positive so that*

$$b=\min.\left\{C\cdot D_{X},\frac{1}{2}\mu_{\min.}\left(T_{X}\right)\right\}>0.$$

If E is a positive vector bundle on X',

$$\mu_{\min.}\left(\mathfrak{D}^{m+1}\left(E,\mathcal{O}\right)\right)\geq\mu_{\min.}\left(\mathfrak{D}^{1}\left(P^{i}\left(E\right),\mathcal{O}\right)\right)\geq\min.\left\{0,\mu_{\min.}\left(E^{*}\right)+\left(m+1\right)b\right\}.$$

Proof. The first inequality is obtained from (10). For the second, we proceed by induction on m. The case $m=0$ comes from Lemma 7.1. Suppose now that, for the quotient

$$Q$$

of minimal slope in a HN-filtration of

$$\mathfrak{D}^{m}\left(E,\mathcal{O}\right),$$

we know that

$$\mu\left(Q\right)\geq\min.\left\{0,\left(\mu_{\min.}\left(E^{*}\right)+mb\right)\right\}.$$

Now if

$$\mu\left(Q\right)<0,$$

we have that the semi-stable bundle Q^* is positive, so by Lemma 8.1,

$$\mu_{\min.}\left(\mathfrak{D}^1\left(Q^*,\mathcal{O}\right)\right)\geq\mu\left(Q\right)+b\geq\min.\left\{b,\left(\mu_{\min.}\left(E^*\right)+\left(m+1\right)b\right)\right\}.$$

On the other hand, if

$$\mu\left(Q\right)\geq 0,$$

then the sequence

$$0\rightarrow Q\rightarrow\mathfrak{D}^1\left(Q^*,\mathcal{O}\right)\rightarrow Q\otimes T_{X'}\rightarrow 0,$$

the positivity of the tangent bundle, and the nice behavior of semi-stability under tensor product gives that

$$\mu_{\min.}\left(\mathfrak{D}^1\left(Q^*,\mathcal{O}\right)\right)\geq 0.$$

Now, dualizing the exact sequence

$$0\rightarrow S\rightarrow\mathfrak{D}^m\left(E,\mathcal{O}\right)\rightarrow Q\rightarrow 0,$$

gives

$$0\rightarrow Q^*\rightarrow P^m\left(E\right)\rightarrow S^*\rightarrow 0$$

from which we obtain the isomorphism

$$\mathfrak{D}^1\left(Q^*,\mathcal{O}\right)\rightarrow\frac{\mathfrak{D}^1\left(P^m\left(E\right),\mathcal{O}\right)}{\mathfrak{D}^1\left(S^*,\mathcal{O}\right)}.$$

Thus

$$\mu_{\min.}\left(\mathfrak{D}^1\left(P^m\left(E\right),\mathcal{O}\right)\right)\geq\mu_{\min.}\left(\mathfrak{D}^1\left(Q^*,\mathcal{O}\right)\right)$$

which completes the proof. □

9. Asymptotic semi-positivity

Specializing Lemma 8.1 to the case of line bundles L, we have

$$\mu_{\min.}\left(\mathfrak{D}^1\left(P^i\left(L\right),\mathcal{O}\right)\right)\geq\min.\left\{0,-L\cdot C+\left(m+1\right)b\right\}$$

from which we immediately conclude:

Lemma 9.1. *If T_X is positive and L is any line bundle on X and*

$$\left(m+1\right)\geq\frac{L\cdot C}{b},$$

then $\mathfrak{D}^1\left(P^m\left(L\right),\mathcal{O}\right)$ and so also $\mathfrak{D}^{m+1}\left(L,\mathcal{O}\right)$ are semi-positive.

So if

$$\frac{C\cdot\left(-K_X\right)}{b}k\leq\left(m+1\right)$$

we have that

$$\mathfrak{D}^1\left(P^m\left(-kK_X\right),\mathcal{O}\right),\mathfrak{D}^{m+1}\left(\left(-kK_X\right),\mathcal{O}\right)\tag{11}$$

are both semipositive. Recall that

$$b=\min.\left\{C\cdot D_X,\frac{1}{2}\mu_{\min.}\left(T_X\right)\right\}$$

so that

$$\frac{C\cdot\left(-kK_X\right)}{b}=\max.\left\{\frac{2\left(C\cdot\left(-kK_X\right)\right)}{\mu_{\min.}\left(T_X\right)},ki_X\right\}.$$

So if

$$\alpha \geq \max.\left\{\frac{2\left(C\cdot\left(-K_X\right)\right)}{\mu_{\min.}\left(T_X\right)}, i_X\right\},$$

then whenever

$$\alpha k$$

is an integer we have that

$$\mathfrak{D}^{\alpha k}\left(\left(-kK_X\right), \mathcal{O}\right)$$

is semipositive.

Lemma 9.2. *If*

$$\alpha \geq \max.\left\{\frac{2\left(C\cdot\left(-K_X\right)\right)}{\mu_{\min.}\left(T_X\right)}, i_X\right\},$$

then whenever

$$\alpha k$$

is a sufficiently divisible integer,

$$\mathfrak{D}^{\alpha k}\left(-kK_X, \mathcal{O}_C\right)$$

is semi-positive (for sufficiently general C).

10. Positivity of the tangent bundle

¿From this point on we restrict the (normal) singularities we allow on X. The necessity of considering only log terminal X derives from the following:

Lemma 10.1. *If X is a log-terminal, 1-canonical unipolar $\mathbb{Q}$-Fano variety, then*

$$T_X$$

is positive, that is, it has no quotient Q which is locally free in codimension one and has non-positive first Chern class

$$c_1\left(Q\right) \in \left(\mathbb{Z}-\mathbb{N}\right) D_X.$$

Proof. Let

$$\epsilon : Y \rightarrow X$$

be as in 2. Then using (2)-(3)

$$\epsilon_{id.}^{*} m' D_X = \tilde{D}_X + \sum\nolimits_i \tilde{a}_i\left(m'\right) E_i$$

and

$$\epsilon_{div.}^{*} D_X = \tilde{D}_X + \sum\nolimits_i \frac{a_i}{t_X} E_i$$

There is an ample $\mathbb{Q}$-divisor

$$\begin{aligned} A \quad : \quad &= \epsilon_{div.}^{*} D_X - \sum\nolimits_i t_i E_i \\ = \quad & \tilde{D}_X + \sum\nolimits_i \left(\frac{a_i}{t_X} - t_i\right) E_i \end{aligned}$$

with $0 < t_i << 1$. Suppose Q is a torsion-free quotient of

$$T_X$$

with

$$c_1(Q) = -m'D_X,\ m' \geq 0.$$

Then

$$rk\ Q = n' < n$$

since $\bigwedge^n T_X$ is positive. Define

$$M := \left(\bigwedge^{n'} Q^\vee\right),$$

and consider the natural map

$$M \to \bigwedge^{n'} (\Omega_X^{\vee\vee})$$

Then $M^{\vee\vee}$is a divisorial sheaf on X, so, using (4)

$$\begin{aligned} c_1(\epsilon_{id.}^* M^{\vee\vee}) &= m'\tilde{D}_X + \sum\nolimits_i \tilde{a}_i(m')E_i. \\ m'A &= m'\tilde{D}_X + \sum\nolimits_i \left(\frac{m'a_i}{t_X} - t_i\right)E_i \\ c_1(\epsilon_{id.}^* M^{\vee\vee}) - m'A &= \sum\nolimits_i \left(\tilde{a}_i(m') - \frac{m'a_i}{t_X} + t_i\right)E_i \\ &= B + \sum\nolimits_i s_iE_i \end{aligned}$$

with B integral, effective and $0 \leq s_i < 1$. Thus one can write

$$c_1(\epsilon_{id.}^* M^{\vee\vee}) = m'A + B + \sum\nolimits_i s_iE_i \tag{12}$$

while A is the ample divisor given above, and $0 \leq s_i < 1$.

On the other hand, the map

$$Q^\vee \to \Omega_X^{\vee\vee}$$

and so, by the 1-canonical condition, induces a map

$$\epsilon^* Q^\vee \to \epsilon^*(\Omega_X^{\vee\vee}) \to \Omega_Y$$

and so one gets a non-trivial map

$$\epsilon^*(M^{\vee\vee}) \to \Omega_Y^{n'}$$

and therefore a natural map

$$N := \epsilon_{id.}^*(M^{\vee\vee}) \to \Omega_Y^{n'}.$$

Thus one must have $H^0\left(\Omega_Y^{n'}(-N)\right) \neq 0$.

To contradict the existence of Q, we show that

$$H^0\left(\Omega_Y^{n'}(-N+B)\right) = 0. \tag{13}$$

Case One: $m' = 0$.

If $M^{\vee\vee} = \mathcal{O}_X$, then $N = \mathcal{O}_Y$, and one must show $H^{n'}(\mathcal{O}_Y) = 0$. By log-terminal Kodaira Vanishing (Theorem 1-2-5 of [KMM]), $H^j(\mathcal{O}_X) = 0$ for all $j > 0$, and by the rationality of log-terminal singularities (Theorem 1-3-6 of [KMM]), the higher direct image sheaves $R^j\epsilon_*(\mathcal{O}_Y) = 0$ for all $j > 0$. So by the Leray spectral sequence, $H^{n'}(\mathcal{O}_Y) = 0$. If $M^{\vee\vee}$ and hence N is only numerically trivial, then the fact that $H^1(\mathcal{O}_Y) = 0$ (by the argument just above) implies that $N = \mathcal{O}_Y$.

Case Two: $m' > 0$.

In this case, one shows (13) by using the branched-covering technique employed in the proof of the Kawamata-Viehweg Vanishing Theorem. Namely use Theorem 1-1-1 of [KMM] to construct a smooth finite Galois cover

$$\tau : Z \to Y$$

for which the ample $\mathbb{Q}$-divisor

$$\tau^* A$$

is actually integral (see (12)). Then

$$\Omega_Y^{n'}(-N) = \Omega_Y^{n'}\left(-A - \sum_i s_i E_i\right)$$

is a subsheaf of

$$\tau_*\left(\tau^*\Omega_Y^{n'}(-A)\right).$$

Since we have an injection

$$\tau^*\Omega_Y^{n'} \to \Omega_Z^{n'},$$

and since $\tau^* A$ is ample,

$$H^0\left(\Omega_Z^{n'}(-\tau^* A)\right) = 0$$

by the Nakano Vanishing Theorem, and so

$$H^0\left(\tau_*\left(\tau^*\Omega_Y^{n'}(-A)\right)\right) = 0,$$

which completes the proof. □

11. The strategy/completion of the proof of the Theorem

11.1. **The assumption.** Let

$$\mu_X = \max.\left\{C \cdot D_X, \frac{2C \cdot (-K_X)}{\mu_{\min.}(T_X)}\right\}$$

and suppose

$$(-K_X)^n > \mu_X^n.$$

Choose rational constants α, β with

$$(-K_X)^n > \beta^n > \alpha^n > \mu_X^n.$$

11.2. **Asymptotic lower bound on sections.** For sufficiently divisible $k \in \mathbb{N}$, the Hilbert polynomial

$$\chi(-kK_X) = \frac{(-K_X)^n}{n!}k^n + (\textit{lower powers of } k)$$

gives an asymptotic lower bound on

$$h^0(-kK_X).$$

It is $\leq \binom{m+n-1}{n}$ conditions that a section s of $(-kK_X)$ have a zero of order m at a determined point $x_0 \in X$.

11.3. **Semipositivity.** If $k\alpha$ is an integer and k is sufficiently divisible, the sheaf of differential operators

$$\mathfrak{D}^{\alpha k}\left(-kK_X, \mathcal{O}_X\right)$$

was shown in §9 to be a semipositive bundle, that is, for any sufficiently general complete curve-section

$$C \subseteq X'$$

the vector bundle

$$\mathfrak{D}^{\alpha k}\left(-kK_X, \mathcal{O}_C\right) = Hom\left(\mathcal{O}_X, \mathcal{O}_C\right) \otimes_{\mathcal{O}_X} D^{\alpha k}\left(-kK_X, \mathcal{O}_X\right)$$

has no quotients of negative degree.

11.4. **The contradiction.** For $k >> 0$, suppose that, for the n-th degree polynomial

$$\binom{m+n-1}{n} + 1 = \frac{1}{n!}m^n + \dots ,$$

we let

$$m_k = \alpha k + 1$$

so that

$$\beta k > m_k > m_k - 1 = \alpha k.$$

Then

$$h^0\left(-kK_X\right) > \frac{\beta^n}{n!}k^n \geq \binom{m_k+n-1}{n} + 1.$$

So by 11.2 we have a non-trivial

$$s \in h^0\left(-kK_X\right)$$

with a zero of order at least $\alpha k + 1$ at a given point $x \in C$, our general curve. But the mapping

$$\begin{array}{rcl} \mathfrak{D}^{\alpha k}\left(-kK_X, \mathcal{O}_C\right) & \longrightarrow & \mathcal{O}_C \\ D & \longmapsto & D\left(s\right) \end{array}$$

cannot be trivial because no function in a neighborhood of x in X is annihilated by all differential operator of degree $\leq \alpha k$. On the other hand, this last map factors through

$$\mathcal{O}_C\left(-x\right),$$

a negative line bundle, contradicting 11.3.

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