Fractional Hadamard transform with continuous variables in the context of quantum optics*

Li-yun Hu†, Xue-xiang Xu, and Shan-jun Ma
College of Physics and Communication Electronics, Jiangxi Normal University, Nanchang 330022, China

October 5, 2010

Abstract

We introduce the quantum fractional Hadamard transform with continuous variables. It is found that the corresponding quantum fractional Hadamard operator can be decomposed into a single-mode fractional operator and two single-mode squeezing operators. This is extended to the entangled case by using the bipartite entangled state representation. The new transformation presents more flexibility to represent signals in the fractional Hadamard domain with extra freedom provided by an angle and two-squeezing parameters.

Keywords: fractional Hadamard transform, fractional Hadamard operator, the additivity of operator

PACC: 0367, 4250

1 Introduction

Fractional Fourier transform (FrFT) is a generalization of the ordinary Fourier transform, which has been used in signal processing and image manipulations [1, 2]. The concept of the FrFT was originally described by Condon [3] and was later introduced for signal processing by Namias [4] as a Fourier transform of fractional order. The 1-dimension FrFT of α-order is defined in Refs.[5, 6] as

$$g\left(x'\right)=\sqrt{\frac{e^{\mathtt{i}\left(\frac{\pi}{2}-\alpha\right)}}{2\pi\sin\alpha}}\int e^{-\frac{\mathtt{i}\left(x'^{2}+x^{2}\right)}{2\tan\alpha}+\frac{\mathtt{i}xx'}{\sin\alpha}}f\left(x\right)\mathtt{d}x. \tag{1}$$

The usual Fourier transform is a special case with order $\alpha=\pi/2$. On the other hand, many orthogonal transform have been successfully used in signal processing, such as discrete cosine transform [7], discrete Hartley transform [8] and Hadamard transform.

Hadamard transform is not only an important tool in classical signal processing, but also is of great importance for quantum computation applications [9]. This transform, used to go from the position basis $|x\rangle$ to the momentum basis, is defined as [10, 11]

$$\mathcal{F}\left|x\right\rangle=\frac{1}{\sqrt{\pi}\sigma}\int_{-\infty}^{\infty}e^{2\mathtt{i}xy/\sigma^{2}}\left|y\right\rangle\mathtt{d}y, \tag{2}$$

where σ is the scale length (also makes the expression in the exponential dimensionless), $|x\rangle$ and $|y\rangle$ are the eigenvector of coordinate operator X. In Ref.[12], the explicit form of $\mathcal{F}$ has been derived by using the technique of integration within an ordered product (IWOP) of operators [13, 14, 15], and it

*Project supported by the National Natural Science Foundation of China (Grant Nos 10775097 and 10874174) and the Research Foundation of the Education Department of Jiangxi Province of China.
†Corresponding author. E-mail: hlyun2008@126.com; hlyun@jxnu.edu.cn.

is found that it can be decomposed into a single-mode squeezing operator and a position-momentum mutual transform operator, i.e., $\mathcal{F} = S_1^{-1}(-1)^{i\pi a^\dagger a/2}$. In addition, the two-mode Hadamard transform with continuous variables is also introduced by using the bipartite entangled state representation, whose Hadamard operator involves a two-mode squeezing operator and a mutual transform operator.

In this paper, we shall introduce the continuous fractional Hadamard transform (CFrHT), which is a generalization of the usual Hadamard transform in Eq.(2). The development of the CFrHT is based upon the same spirit of continuous fractional Fourier transform (CFrFT). Then the CFrHT operator (CFrHTO) is derived by using the IWOP technique, and its properties are analyzed. It is found that the CFrHTO can be decomposed into a single-mode fractional operator $e^{\mathtt{i}\alpha a^\dagger a}$ and two single-mode squeezing operators. On the other hand, since the publication of the paper of Einstein, Podolsky and Rosen (EPR) in 1935 [16], arguing the incompleteness of quantum mechanics, the conception of entanglement has become more and more fascinating and important as it plays a central role in quantum imformation and quantum computation, we also shall introduce the two-mode CFrHO in bipartite entangled state representation, which turns out to involve the two fractional operators and two two-mode squeezing operators.

Our work is arranged as follows. In section 2, for the single-mode case, the normally ordered fractional Hadamard operator is derived by using the IWOP technique. The properties of fractional Hadamard operator is discussed in section 3, such as the unitarity, the decomposition of the CFrHO and its transform relation. Then the single-mode case is extended to two-mode case in section 4 and some similar discussions to singled-mode case are presented. Section 5 is devoted to exploring the measurements for the output states from the CFrHT. Conclusions are involved in the last section.

2 Normally Ordered Fractional Hadamard Operator

In this section, we first introduce the continuous fractional Hadamard transform (CFrHT), i.e.,

$$\mathcal{H}_\alpha(\mu,\nu)\left|x\right\rangle = \sqrt{\frac{e^{\mathtt{i}\left(\frac{\pi}{2}-\alpha\right)}}{2\pi\mu\nu\sin\alpha}}\int_{-\infty}^{\infty}\exp\left\{-\frac{\mathtt{i}\left(x^2/\mu^2+y^2/\nu^2\right)}{2\tan\alpha}+\frac{\mathtt{i}xy}{\mu\nu\sin\alpha}\right\}\left|y\right\rangle \mathtt{d}y, \tag{3}$$

where μ,ν are the scale length (also make the expression in the exponential dimensionless), α is an angle, and $\mathcal{H}^\alpha(\mu,\nu)$ is called the CFrHT operator (CFrHO). In particular, when $\alpha=\pi/2$ and $\mu=\nu=\sigma/\sqrt{2}$, Eq.(3) just reduces to Eq.(2).

In order to find the explicit expression of the CFrHO, multiplying Eq.(3) by the bra $\int \mathtt{d}x\left\langle x\right|$ from the rights in two-side, where $\left|y\right\rangle$ and $\left|x\right\rangle$ are coordinate eigenvectors, $X\left|x\right\rangle = x\left|x\right\rangle$, and

$$\left|x\right\rangle = \pi^{-1/4}\exp\left\{-\frac{x^2}{2}+\sqrt{2}xa^\dagger-\frac{a^{\dagger 2}}{2}\right\}\left|0\right\rangle, \tag{4}$$

we can recast the CFrHO $\mathcal{H}_\alpha(\mu,\nu)$ into the following integral form,

$$\mathcal{H}_\alpha(\mu,\nu) = \sqrt{\frac{e^{\mathtt{i}\left(\frac{\pi}{2}-\alpha\right)}}{2\pi\mu\nu\sin\alpha}}\int_{-\infty}^{\infty}\exp\left\{-\frac{\mathtt{i}\left(x^2/\mu^2+y^2/\nu^2\right)}{2\tan\alpha}+\frac{\mathtt{i}xy}{\mu\nu\sin\alpha}\right\}\left|y\right\rangle\left\langle x\right| \mathtt{d}x\mathtt{d}y. \tag{5}$$

Then using the vacuum projector's normal ordering form $\left|0\right\rangle\left\langle 0\right| =: e^{-a^\dagger a}:$ (where the symbol : : denotes the normally ordering) and the IWOP technique to directly perform the integration, we finally obtain

$$\begin{aligned}\mathcal{H}_\alpha(\mu,\nu) &= \frac{1}{\pi}\sqrt{\frac{e^{\mathtt{i}\left(\frac{\pi}{2}-\alpha\right)}}{2\mu\nu\sin\alpha}}\int_{-\infty}^{\infty}:\exp\left\{-\frac{A}{2\mu^2}x^2+\sqrt{2}xa+\frac{\mathtt{i}xy}{\mu\nu\sin\alpha}\right.\\ &\quad\left.-\frac{B}{2\nu^2}y^2+\sqrt{2}ya^\dagger-\frac{\left(a^\dagger+a\right)^2}{2}\right\}:\mathtt{d}x\mathtt{d}y\\ &= \sqrt{\frac{2\mu\nu e^{\mathtt{i}\left(\frac{\pi}{2}-\alpha\right)}}{u\sin\alpha}}\exp\left\{\left(\frac{\nu^2A}{u}-\frac{1}{2}\right)a^{\dagger 2}\right\}\\ &\quad\times\exp\left\{a^\dagger a\ln\frac{\mathtt{i}2\mu\nu}{u\sin\alpha}\right\}\exp\left\{\left(\frac{\mu^2B}{u}-\frac{1}{2}\right)a^2\right\},\end{aligned}\tag{6}$$

where we have set $A=\mathtt{i}\cot\alpha+\mu^2, B=\mathtt{i}\cot\alpha+\nu^2, u=\csc^2\alpha+AB$,and used the operator identity in the last step of Eq.(6),

$$\exp\left\{fa^\dagger a\right\}=:\exp\left\{\left(e^f-1\right)a^\dagger a\right\}:,\tag{7}$$

Eq.(6) is the normally ordered form of the CFrHO. In particular, when $\alpha=\pi/2$ and $\mu=\nu=\sigma/\sqrt{2}$, leading to $A=B=\sigma^2/2, u=1+\sigma^4/4$, then Eq.(6) becomes

$$\begin{aligned}\mathcal{H}_{\pi/2}\left(\sigma/\sqrt{2},\sigma/\sqrt{2}\right) &= \frac{2\sigma}{\sqrt{\sigma^4+4}}\exp\left\{\frac{\sigma^4-4}{\sigma^4+4}\frac{a^{\dagger 2}}{2}\right\}\\ &\quad\times\exp\left\{a^\dagger a\ln\frac{4\mathtt{i}\sigma^2}{\sigma^4+4}\right\}\exp\left\{\frac{\sigma^4-4}{\sigma^4+4}\frac{a^2}{2}\right\},\end{aligned}\tag{8}$$

which is just the result Eq.(7) in Ref.[12].

3 Properties of Fractional Hadamard Operator

From Eq.(5) one can see that the CFrHO is a unitary one, i.e., $\mathcal{H}_\alpha(\mu,\nu)\mathcal{H}_\alpha^\dagger(\mu,\nu)=\mathcal{H}_\alpha^\dagger(\mu,\nu)\mathcal{H}_\alpha(\mu,\nu)=1$. In fact, uisng Eq.(5) and the orthogonality of coordinate state, $\langle x'|x\rangle=\delta(x-x')$, we have

$$\begin{aligned}\mathcal{H}_\alpha(\mu,\nu)\mathcal{H}_\alpha^\dagger(\mu,\nu) &= \frac{1}{2\pi\mu\nu\sin\alpha}\int_{-\infty}^{\infty}\exp\left\{\frac{\mathtt{i}\left(y'^2-y^2\right)}{2\nu^2\tan\alpha}+\mathtt{i}x\frac{y-y'}{\mu\nu\sin\alpha}\right\}|y\rangle\langle y'|\,\mathtt{d}x\mathtt{d}y\mathtt{d}y'\\ &= \frac{1}{\mu\nu\sin\alpha}\int_{-\infty}^{\infty}\delta\left(\frac{y-y'}{\mu\nu\sin\alpha}\right)\exp\left\{\frac{\mathtt{i}\left(y'^2-y^2\right)}{2\nu^2\tan\alpha}\right\}|y\rangle\langle y'|\,\mathtt{d}y\mathtt{d}y'\\ &= \int_{-\infty}^{\infty}|y\rangle\langle y|\,\mathtt{d}y=\mathcal{H}_\alpha^\dagger(\mu,\nu)\mathcal{H}_\alpha(\mu,\nu)=1.\end{aligned}\tag{9}$$

In order to see clearly its transform relation under the CFrHO, next we examine its decomposition. Performing the change of variables, $x/\mu\to x$, $y/\nu\to y$, we can be recast Eq.(5) into the following form,

$$\mathcal{H}_\alpha(\mu,\nu)=\sqrt{\frac{\mu\nu e^{\mathtt{i}\left(\frac{\pi}{2}-\alpha\right)}}{2\pi\sin\alpha}}\int_{-\infty}^{\infty}\exp\left\{-\frac{\mathtt{i}\left(x^2+y^2\right)}{2\tan\alpha}+\frac{\mathtt{i}xy}{\sin\alpha}\right\}|\nu y\rangle\langle\mu x|\,\mathtt{d}x\mathtt{d}y.\tag{10}$$

By noticing that the single-mode squeezing operator S_1 [17] has its natural expression in coordinate representation [13], i.e.,

$$S_1(\mu)=\frac{1}{\sqrt{\mu}}\int_{-\infty}^{\infty}\mathtt{d}x\left|\frac{x}{\mu}\right\rangle\langle x|,\tag{11}$$

which leads to $|\nu y\rangle = \frac{1}{\sqrt{\nu}} S_1^{-1}(\nu)|y\rangle$, $\langle \mu x| = \frac{1}{\sqrt{\mu}}\langle x| S_1(\mu)$, so Eq.(10) can be decomposed into

$$\mathcal{H}_\alpha(\mu,\nu) = S_1^{-1}(\nu)\mathcal{F}_\alpha S_1(\mu) = S_1^{-1}(\nu)e^{\mathrm{i}\alpha a^\dagger a}S_1(\mu), \tag{12}$$

where $\mathcal{F}_\alpha$ is given by

$$\mathcal{F}_\alpha \equiv \sqrt{\frac{e^{\mathrm{i}\left(\frac{\pi}{2}-\alpha\right)}}{2\pi\sin\alpha}}\int_{-\infty}^{\infty} e^{-\frac{\mathrm{i}(x^2+y^2)}{2\tan\alpha}+\frac{\mathrm{i}xy}{\sin\alpha}}|y\rangle\langle x|\,\mathtt{d}x\mathtt{d}y = e^{\mathrm{i}\alpha a^\dagger a}, \tag{13}$$

this integral result can be obtained by using a similar way to deriving Eq.(6). Thus we see that the CFrHO can be decomposed as a fractional operator and two-single-mode squeezing operators.

Using the decomposition of the CFrHO in Eq.(12), and noticing that $S_1(\mu)XS_1^{-1}(\mu) = \mu X$, $S_1(\mu)PS_1^{-1}(\mu) = P/\mu$, and $e^{\mathrm{i}\alpha a^\dagger a}ae^{-\mathrm{i}\alpha a^\dagger a} = ae^{-\mathrm{i}\alpha}$, which leads to

$$e^{\mathrm{i}\alpha a^\dagger a}Xe^{-\mathrm{i}\alpha a^\dagger a} = X\cos\alpha + P\sin\alpha,\ e^{\mathrm{i}\alpha a^\dagger a}Pe^{-\mathrm{i}\alpha a^\dagger a} = P\cos\alpha - X\sin\alpha, \tag{14}$$

thus we have

$$\begin{aligned}\mathcal{H}_\alpha(\mu,\nu)X\mathcal{H}_\alpha^\dagger(\mu,\nu) &= \mu S_1^{-1}(\nu)(X\cos\alpha + P\sin\alpha)S_1(\nu) \\ &= \frac{\mu}{\nu}X\cos\alpha + \mu\nu P\sin\alpha, \qquad (15)\\ \mathcal{H}_\alpha(\mu,\nu)P\mathcal{H}_\alpha^\dagger(\mu,\nu) &= \frac{\nu}{\mu}P\cos\alpha - \frac{X}{\mu\nu}\sin\alpha, \qquad (16)\end{aligned}$$

from which we see that the CFrHO plays the role of combining the coordinate operator X and momentum operator P in a certain way (15)-(16), i.e., including the squeezing and the rotation. In paticular, when $\alpha = \frac{\pi}{2}$, Eqs.(15)-(16) become

$$\mathcal{H}_{\frac{\pi}{2}}(\mu,\nu)X\mathcal{H}_{\frac{\pi}{2}}^\dagger(\mu,\nu) = \mu\nu P,\ \mathcal{H}_{\frac{\pi}{2}}(\mu,\nu)P\mathcal{H}_{\frac{\pi}{2}}^\dagger(\mu,\nu) = -\frac{X}{\mu\nu}, \tag{17}$$

i.e., the mutual exchanging of coordinate-momentum operators.

On the other hand, there is a most important feature of the FrFT is that the FrFT obeys the additivity rule, i.e., two successive FrFT of order α and β makes up the FrFT of order $\alpha+\beta$. Then a question naturally arises: Is the two successive CFrHOs still a CFrHO? To answer this question, we examine the direct product $\mathcal{H}_\alpha(\mu,\nu)\otimes\mathcal{H}_\beta(\mu',\nu')$. Using Eq.(12) it is easily seen that when $\mu = \nu'$ there is an additivity of operator as follows

$$\begin{aligned}\mathcal{H}_\alpha(\mu,\nu)\otimes\mathcal{H}_\beta(\mu',\mu) &= S_1^{-1}(\nu)e^{\mathrm{i}\alpha a^\dagger a}S_1(\mu)S_1^{-1}(\mu)e^{\mathrm{i}\beta a^\dagger a}S_1(\mu') \\ &= \mathcal{H}_{\alpha+\beta}(\mu',\nu), \qquad (18)\end{aligned}$$

which can be seen as the additivity property of the CFrHOs. Here it should be pointed out that the condition of additivitive operator for the CFrHOs is that the parameter μ of the prior cascade operator should be equal to the parameter ν' of the next one, i.e., $\mu = \nu'$. This can be clearly seen from the viewpoint of classical optics transform.

4 Two-mode CFrHT

Next, we shall extend the single-mode CFrHT to two-mode case by using the entangled state representation [18],

$$|\eta\rangle = \exp\left\{-\frac{1}{2}|\eta|^2 + \eta a_1^\dagger - \eta^* a_2^\dagger + a_1^\dagger a_2^\dagger\right\}|00\rangle, \tag{19}$$

where $|\eta=\eta_1+\mathtt{i}\eta_2\rangle$ is the common eigenvector of two-particle's relative coordinate X_1-X_2 and total momentum P_1+P_2,

$$(X_1-X_2)|\eta\rangle=\sqrt{2}\eta_1|\eta\rangle, (P_1+P_2)|\eta\rangle=\sqrt{2}\eta_2|\eta\rangle, \tag{20}$$

and $|\eta\rangle$ possesses the completeness and the orthogonality,

$$\int_{-\infty}^{\infty}\frac{\mathtt{d}^2\eta}{\pi}|\eta\rangle\langle\eta|=1,\ \ \langle\eta|\,\eta'\rangle=\pi\delta(\eta-\eta')\,\delta(\eta^*-\eta'^*). \tag{21}$$

In a similar way to introducing Eq.(3), we examine the following transform,

$$\mathcal{H}_\alpha^C(\mu,\nu)|\eta\rangle=\frac{e^{\mathtt{i}\left(\frac{\pi}{2}-\alpha\right)}}{2\mu\nu\sin\alpha}\int\frac{\mathtt{d}^2\eta'}{\pi}e^{-\frac{\mathtt{i}\left(|\eta'|^2/\nu^2+|\eta|^2/\mu^2\right)}{2\tan\alpha}+\frac{\mathtt{i}(\eta'^*\eta+\eta^*\eta')}{2\mu\nu\sin\alpha}}|\eta'\rangle. \tag{22}$$

Using Eq.(21), one can see that $\mathcal{H}_\alpha^C(\mu,\nu)$ is a unitary operator, i.e., $\mathcal{H}_\alpha^C\left[\mathcal{H}_\alpha^C\right]^\dagger=\left[\mathcal{H}_\alpha^C\right]^\dagger\mathcal{H}_\alpha^C=1$. Here we should emphasize that, the exponential item in the right hand side of Eq.(22) can be decomposed into a direct product of two exponential items in the right hand side of Eq.(3), but $|\eta\rangle$ is an entangled state (not the direct product of two single-mode coordinate states, which can be seen clearly from its Schmidt decomposition [19]), thus this is a nontrivial extension from single-mode case to two-mode case.

Performing a similar procedure to single-mode case, i.e., noticing that the two-mode squeezing operator has its natural expression in the entangled state representation,

$$S_2(\mu)=\exp\left[\left(a_1^\dagger a_2^\dagger-a_1a_2\right)\ln\mu\right]=\frac{1}{\mu}\int\frac{d^2\eta}{\pi}\left|\frac{\eta}{\mu}\right\rangle\langle\eta|, \tag{23}$$

which leads to $\frac{1}{\nu}\left|\frac{\eta'}{\nu}\right\rangle=S_2(\nu)|\eta'\rangle, \frac{1}{\mu}\left|\frac{\eta}{\mu}\right\rangle=S_2(\mu)|\eta\rangle$, then using the completeness of $|\eta\rangle$ and $|00\rangle\langle00|=:e^{-a^\dagger a-b^\dagger b}:$ and the orthogonality in Eq.(21) we can further decompose the operator $\mathcal{H}_\alpha^C(\mu,\nu)$ into the following form,

$$\mathcal{H}_\alpha^C(\mu,\nu)=S_2^\dagger(\nu)\mathcal{F}_\alpha^C S_2(\mu)=S_2^\dagger(\nu)\exp\left\{\mathtt{i}\alpha\left(a_1^\dagger a_1+a_2^\dagger a_2\right)\right\}S_2(\mu), \tag{24}$$

where the operatror $\mathcal{F}_\alpha^C$ is given by

$$\begin{aligned}\mathcal{F}_\alpha^C &= \frac{e^{\mathtt{i}\left(\frac{\pi}{2}-\alpha\right)}}{2\sin\alpha}\int\frac{\mathtt{d}^2\eta'\mathtt{d}^2\eta}{\mu^2\nu^2\pi}e^{-\frac{\mathtt{i}\left(|\eta'|^2/\nu^2+|\eta|^2/\mu^2\right)}{2\tan\alpha}+\frac{\mathtt{i}(\eta'^*\eta+\eta^*\eta')}{2\mu\nu\sin\alpha}}\left|\frac{\eta'}{\nu}\right\rangle\left\langle\frac{\eta}{\mu}\right| \\ &= \exp\left\{\mathtt{i}\alpha\left(a_1^\dagger a_1+a_2^\dagger a_2\right)\right\}.\end{aligned} \tag{25}$$

Thus we see that the two-mode CFrHO can be decomposed into the form in Eq.(24), i.e., two fractional operators and two two-mode squeezing operators.

This is a convient expression for further deriving the transforms and the condition of additivitive operator. In fact, using Eqs.(24), (14) and Eqs.(20), (21) leading to

$$S_2(\mu)(X_1-X_2)S_2^\dagger(\mu)=\mu(X_1-X_2),\ S_2(\mu)(P_1+P_2)S_2^\dagger(\mu)=\mu(P_1+P_2), \tag{26}$$

$$S_2(\mu)(X_1+X_2)S_2^\dagger(\mu)=\frac{1}{\mu}(X_1+X_2),\ S_2(\mu)(P_1-P_2)S_2^\dagger(\mu)=\frac{1}{\mu}(P_1-P_2), \tag{27}$$

we have

$$\begin{aligned}\mathcal{H}_\alpha^C(X_1-X_2)\left[\mathcal{H}_\alpha^C\right]^\dagger &= \mu S_2^\dagger(\nu)\mathcal{F}_\alpha^C(X_1-X_2)\left[\mathcal{F}_\alpha^C\right]^\dagger S_2(\nu) \\ &= \mu S_2^\dagger(\nu)((X_1-X_2)\cos\alpha+(P_1-P_2)\sin\alpha)S_2(\nu) \\ &= \frac{\mu}{\nu}(X_1-X_2)\cos\alpha+\mu\nu(P_1-P_2)\sin\alpha,\end{aligned} \tag{28}$$

$$\mathcal{H}_\alpha^C(X_1+X_2)\left[\mathcal{H}_\alpha^C\right]^\dagger=\frac{\nu}{\mu}(X_1+X_2)\cos\alpha+\frac{1}{\mu\nu}(P_1+P_2)\sin\alpha, \tag{29}$$

and

$$\mathcal{H}_\alpha^C (P_1 - P_2) \left[\mathcal{H}_\alpha^C\right]^\dagger = \frac{\nu}{\mu}(P_1 - P_2)\cos\alpha - \frac{1}{\mu\nu}(X_1 - X_2)\sin\alpha, \tag{30}$$

$$\mathcal{H}_\alpha^C (P_1 + P_2) \left[\mathcal{H}_\alpha^C\right]^\dagger = \frac{\mu}{\nu}(P_1 + P_2)\cos\alpha - \mu\nu(X_1 + X_2)\sin\alpha. \tag{31}$$

From Eqs.(28)-(31) it is easy to see that when $\alpha = \pi/2$, the role of $\mathcal{H}_{\pi/2}^C$ is just exchanging $(X_1 - X_2)$ and $(P_1 - P_2)$, $(X_1 + X_2)$ and $(P_1 + P_2)$; while for $\alpha = \pi$, $\mathcal{H}_\pi^C$ can be seen as an identity operator.

In addition, from the decomposition (24) one can see that the direct product $\mathcal{H}_\alpha^C(\mu,\nu)\otimes\mathcal{H}_\beta^C(\mu',\nu')$ satisfies the additivity rule when $\mu = \nu'$, i.e.,

$$\mathcal{H}_\alpha^C(\mu,\nu)\otimes\mathcal{H}_\beta^C(\mu',\nu') = \mathcal{H}_{\alpha+\beta}^C(\mu',\nu). \tag{32}$$

5 Measurements for the output states from the CFrHT

The measurement for quantum state plays an important role in quantum computation and quantum imfromation. When a quantum state $|f\rangle$ is transformed by the CFrHO, then what is the measurement result with continuous orthogonal basis? For single-mode case, the output state from the CFrHT is $|g\rangle_{out} = \mathcal{H}_\alpha(\mu,\nu)|f\rangle$. The measurement basis is choosen as a coordiante eigenvector, then the measurement result is given by

$$\begin{aligned}\langle x|g\rangle_{out} &= \langle x|\mathcal{H}_\alpha(\mu,\nu)|f\rangle \\ &= \int_{-\infty}^{\infty}\mathtt{d}x'\langle x|\mathcal{H}_\alpha(\mu,\nu)|x'\rangle\langle x'|f\rangle \\ &= \sqrt{\frac{e^{\mathtt{i}\left(\frac{\pi}{2}-\alpha\right)}}{2\pi\mu\nu\sin\alpha}}\int_{-\infty}^{\infty} f(x')\, e^{-\frac{\mathtt{i}\left(x'^2/\mu^2+x^2/\nu^2\right)}{2\tan\alpha}+\frac{\mathtt{i}x'x}{\mu\nu\sin\alpha}}\mathtt{d}x',\end{aligned} \tag{33}$$

which just corresponds to a generalized fractional Fourier transform of wave function $f(x') = \langle x'|f\rangle$.

For two-mode case, the measurement result by two-mode entangled state Bell basis is

$$\begin{aligned}\langle \eta'|g\rangle_{out} &= \langle\eta'|\mathcal{H}_\alpha^C(\mu,\nu)|f\rangle \\ &= \int_{-\infty}^{\infty}\frac{\mathtt{d}^2\eta}{\pi}\langle\eta'|\mathcal{H}_\alpha^C(\mu,\nu)|\eta\rangle\langle\eta|f\rangle \\ &= \frac{e^{\mathtt{i}\left(\frac{\pi}{2}-\alpha\right)}}{2\mu\nu\sin\alpha}\int_{-\infty}^{\infty}\frac{\mathtt{d}^2\eta}{\pi}e^{-\frac{\mathtt{i}\left(|\eta'|^2/\nu^2+|\eta|^2/\mu^2\right)}{2\tan\alpha}+\frac{\mathtt{i}\left(\eta'^*\eta+\eta^*\eta'\right)}{2\mu\nu\sin\alpha}}f(\eta),\end{aligned} \tag{34}$$

which is just a generalized complex fractional Fourier transform, and the wave function $f(\eta)$ is the projection of quantum state $|f\rangle$ on $\langle\eta|$. From Eqs.(33) and (34) we can clearly see that the generalized FrFT of the wavefunction for any quantum state $|f\rangle$ in coordinate/entangled state corresponds to the wavefunction of Hadamard-transformed $(\mathcal{H}_\alpha(\mu,\nu)|f\rangle)$ in coordinate/entangled state. In other words, the generalized FrFT of wavefunction is just the matrix element of CFrHO in $\langle x|$ $(\langle\eta'|)$ and $|f\rangle$.

6 Conclusion

Based on quantum Hadamard transform, we have introduced the quantum fractional Hadamard transform with continuous variables. It is found that the corresponding quantum fractional Hadamard operator can be decomposed into a single-mode fractional operator $e^{\mathtt{i}\alpha a^\dagger a}$ and two single-mode squeezing operators. The two-mode fractional Hadamard transform is also introduced by using the bipartite entangled state representation. It is shown that the corresponding transform operator involves two single-mode fractional operators and two two-mode squeezing operators. For any quantum state vector $|f\rangle$, the measurement results for the transformed quantum state (for instance $|g\rangle_{out} = \mathcal{H}_\alpha(\mu,\nu)|f\rangle$) by continuous coordinate state $|x\rangle$ (or bipartite entangled state $|\eta\rangle$) just corresponds to a generalized (complex) fractional Fourier transform. In addition, the new transformation gives us more flexibility to represent signals in the fractional Hadamard domain with extra freedom provided by an angle, and two-squeezing parameters. For more discussions about the optical transforms in the context of quantum optics and the discrete fractional Hadmard transform, we refer to Refs.[20, 21, 22].

References

[1] L. B. Almeida 1994 *IEEE* Trans. Signal Process. **42** 3084

[2] Tao R, Deng B, Zhang W Q, et. al. 2008 *IEEE Trans. Signal Process.* **56** 158

[3] Condon E U 1937 Proc. Nat. Acad. Sci. **23** 158

[4] Namias V 1980 *J. Inst. Math. Appl.* **25** 241

[5] McBride A C and Kerr F H 1987 *IMA J. Appl. Math.* **39** 159

[6] Kerr F H 1988 *J. Math. Anal. Appl.* **136** 404

[7] Pei S C, Yeh M H 2001 *IEEE Trans. Signal Process.* **49** 1198

[8] Pei S C, Tseng C C, Yeh M H and Shyu J J 1998 *IEEE Trans. Circuit Ssytem II* **45** 665

[9] Nielsen M A and Chuang I L 2000 The Quantum Computation and Quantum Information, Cambridge University Press, Cambridge

[10] Braunstein S L 1998 *Phys. Rev. Lett.* **80** 4084

[11] Parker S, Bose S and Plenio M B 2000 *Phys. Rev. A.* **61** 032305

[12] Fan H Y and Guo Q 2008 *Commun. Theor. Phys.* **49** 859

[13] Fan H Y, H. R. Zaidi and J. R. Klauder 1987 *Phys. Rev.* D **35** 1831

[14] Fan H Y 2003 *J. Opt. B*: *Quantum. Semiclass. Opt.* **5** R147

[15] Hu L Y and Fan H Y 2009 *Commun. Theor. Phys.* (Beijing, China) **52** 1071; Hu L Y and Fan H Y 2009 Chin. Phys. B **18** 4657

[16] Einstein A, Podolsky B and Rosen N 1935 *Phys. Rev.* **47** 777

[17] Mandel L and Wolf E 1995 *Optical Coherence and Quantum Optics*, Cambridge Press, (London)

[18] Fan H Y and Klauder J R 1994 *Phys. Rev.* A **49** 704; Fan H Y and Fan Y 1996 *Phys. Rev.* A **54** 958

[19] Preskill J 1998 Quantum Information and Computation, California Institute of Technology; Hu L Y and Lu H L. 2007 Chin. Phys. **16** 2200

[20] Hu L Y and Fan H Y 2008 *J. Mod. Opt.* **55** 1835; Fan H Y and Hu L Y 2009 *Chin. Phys. B* **18** 0611

[21] Tao R, Lang J, Wang Y 2009 Opt. Commun. **282** 1531

[22] Hu L Y and Fan H Y 2009 Int. J. Theor. Phys. DOI 10.1007/s10773-009-0008-z