Timeless path integral for relativistic quantum mechanics

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Abstract

Starting from the canonical formalism of relativistic (timeless) quantum mechanics, the formulation of timeless path integral is rigorously derived. The transition amplitude is reformulated as the sum, or functional integral, over all possible paths in the constraint surface specified by the (relativistic) Hamiltonian constraint, and each path contributes with a phase identical to the classical action divided by $\hbar$. The timeless path integral manifests the timeless feature as it is completely independent of the parametrization for paths. For the special case that the Hamiltonian constraint is a quadratic polynomial in momenta, the transition amplitude admits the timeless Feynman's path integral over the (relativistic) configuration space.

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I. INTRODUCTION

The idea that quantum mechanics can be well defined even if the notion of time is absent has been proposed [1, 2] and developed in a number of different strategies [3–7]. The motivation for formulating quantum mechanics in *timeless* description comes from the research on quantum gravity, as in the quantum theory of general relativity, the spacetime background is not fixed and generally it is not possible to make sense of quantum variables "at a moment of time". This is closely related to the "problem of time" in quantum gravity [8].

In particular, a comprehensive formulation for the relativistic (timeless) quantum mechanics and its probabilistic interpretation are presented in Chapter 5 of [9]. The formulation is based on the canonical (Hilbert spaces and self-adjoint operators) formalism and we wonder whether it also admits the covariant (sum-over-histories) formalism. In the conventional nonrelativistic (with time) quantum mechanics, the transition amplitudes are the matrix elements of the unitary evolution generated by the Hamiltonian and can be reformulated as the sum over histories, called the path integral (see [10] for a detailed derivation). In the relativistic quantum mechanics, however, the concept of time evolution is not well defined in the fundamental level; therefore, conceptual issues and technical subtleties arise when one tries to derive the timeless path integral from the canonical formalism.

The aim of this paper is to rigorously derive the timeless path integral for relativistic quantum mechanics, starting from the canonical formulation in [9]. It turns out the transition amplitude can be reformulated as the sum, or functional integral, over all possible paths in the constraint surface Σ specified by the (relativistic) Hamiltonian constraint $H(q^a, p_a) = 0$ for the configuration variables q^a and their conjugate momenta p_a, and each path contributes with a phase identical to the classical action divided by $\hbar$. Unlike the conventional path integral in which every path is parameterized by the time variable t, the timeless path integral is completely independent of the parametrization for paths, manifesting the timeless feature. Furthermore, for the special case that the Hamiltonian constraint is a quadratic polynomial in p_a, the timeless path integral over Σ reduces to the timeless Feynman's path integral over the (relativistic) configuration space.

The timeless path integral for relativistic quantum mechanics is appealing both conceptually and technically. Conceptually, timeless path integral offers an alternative interpretation of relativistic quantum fluctuations and is more intuitive than the canonical formalism for many aspects. It can give a new point of view about how the conventional quantum mechanics with time emerges within a certain approximation and thus may help to resolve the problem of time. Technically, timeless path integral provides tractable tools to compute (at least numerically or approximately) the transition amplitudes which otherwise remain formal in the canonical formalism. For example, the semiclassical approximation for the timeless path integral can be developed *à la* the Wentzel-Kramers-Brillouin (WKB) method.

In the research of loop quantum gravity (LQG), the sum-over-histories formulation is an active research area that goes under the name "spin foam models" (SFMs) (see [9] and references therein for LQG and SFMs). In particular, over the past years, SFMs in relation to the kinematics of LQG have been clearly established [11–14]. However, the Hamiltonian dynamics of LQG is far from fully understood, and although well motivated, SFMs have not been systematically derived from any well-established theories of canonical quantum gravity. Meanwhile, loop quantum cosmology (LQC) has recently been cast in a sum-over-histories formulation, providing strong support for the general paradigm underlying SFMS [15, 16]. In this paper, the timeless path integral is systematically derived from the canonical formalism of relativistic quantum mechanics, and we hope it will shed new light on the issues of the interplay between LQG/LQC and SFMs.

This paper is organized as follows. We begin with a review on the classical theory of relativistic mechanics in Sec. II and then a review on the quantum theory of relativistic mechanics in Sec. III. The main topic is presented in Sec. IV, where the timeless path integral is derived and investigated in detail. Finally, conclusions and outlooks are summarized and discussed in Sec. V.

II. CLASSICAL THEORY OF RELATIVISTIC MECHANICS

The conventional formulation of classical mechanics treats the time t on a special footing and therefore is not broad enough for general-relativistic systems, which treat time on the equal footing as other variables. To include general-relativistic systems, we need a a more general formulation with a new conceptual scheme. A timeless formulation for relativistic classical mechanics is proposed for this purpose and described in detail in Chapter 3 of [9], excerpts from which are presented in this section with some new materials added to give a review and define notations.

A. Hamiltonian formalism

Let $\mathcal{C}$ be the *relativistic configuration space* coordinated by q^a for $a = 1, 2, \cdots, d$ with q^a being the *partial observables* and d being the dimension of $\mathcal{C}$. In nonrelativistic mechanics, one of the partial observables can be singled out and treated specially as the time t, i.e. $q^a = (t, q^i)$, but this separation is generally not possible for general-relativistic systems. An observation yields a complete set of q^a, which is called an *event*. In nonrelativistic mechanics, an observation is a reading of the time t together with other readings q^i.

Consider the cotangent space $\Omega = T^*\mathcal{C}$ coordinated by q^a and their momenta p_a. The space Ω carries a natural one-form $\tilde{\theta} = p_a\, dq^a$. Once the kinematics (i.e. the space $\mathcal{C}$ of the partial observables q^a) is known, the dynamics is fully determined by giving a *constraint surface* Σ in the space Ω. The constraint surface Σ is specified by $H = 0$ with a function $H : \Omega \to \mathbb{R}^k$. Denote $\tilde{\gamma}$ an unparameterized curve in Ω (observables and momenta) and γ its projection to $\mathcal{C}$ (observables only). The physical motion is determined by the function H via the following

> **Variational principle.** A curve γ in $\mathcal{C}$ is a *physical motion* connecting the events q_1^a and q_2^a, if $\tilde{\gamma}$ extremizes the action
>
> $$S[\tilde{\gamma}] = \int_{\tilde{\gamma}} p_a\, dq^a \tag{2.1}$$
>
> in the class of the curves $\tilde{\gamma}$ which satisfy
>
> $$H(q^a, p_a) = 0, \tag{2.2}$$
>
> (i.e. $\tilde{\gamma} \in \Sigma$) and whose projection γ to $\mathcal{C}$ connect q_1^a and q_2^a.

If $k = 1$, H is a scalar function and called the *Hamiltonian constraint.* If $k > 1$, there is gauge invariance and H is called the *relativistic Hamiltonian.* The pair $(\mathcal{C}, H)$ describes a relativistic dynamical system. All (relativistic and nonrelativistic) Hamiltonian systems can be formulated in this timeless formalism.

By parameterizing the curve $\tilde{\gamma}$ with a parameter τ, the action (2.1) reads as

$$S[q^a, p_a, N_i] = \int d\tau \left(p_a(\tau)\, \frac{dq^a(\tau)}{d\tau} - N_i(\tau) H^i(q^a, p_a) \right), \tag{2.3}$$

where the constraint (2.2) has been implemented with the Lagrange multipliers $N_i(\tau)$. Varying this action with respect to $N_i(\tau)$, $q^a(\tau)$ and $p_a(\tau)$ yields the constraint equation(s) (2.2) together with the Hamilton equations:

$$\frac{dq^a(\tau)}{d\tau} = N_j(\tau)\frac{\partial H^j(q^a,p_a)}{\partial p_a}, \tag{2.4a}$$

$$\frac{dp_a(\tau)}{d\tau} = -N_j(\tau)\frac{\partial H^j(q^a,p_a)}{\partial q^a}. \tag{2.4b}$$

For $k>1$, a motion is determined by a k-dimensional surfaces in $\mathcal{C}$ and different choices of the k arbitrary functions $N_j(\tau)$ determine different curves and parametrizations on the single surface that defines a motion. For $k=1$, a motion is a 1-dimensional curve in $\mathcal{C}$ and different choices of $N(\tau)$ correspond to different parametrizations for the same curve. Different solutions of $q^a(\tau)$ and $p_a(\tau)$ for different choices of $N_j(\tau)$ are gauge-equivalent representations of the same motion and different choices of $N_j(\tau)$ have no physical significance.

Along the solution curve, the change rate of H with respect to τ is given by

$$\begin{aligned}\frac{dH^i}{d\tau} &= \frac{dq^a}{d\tau}\frac{\partial H^i}{\partial q^a}+\frac{dp_a}{d\tau}\frac{\partial H^i}{\partial p_a} = N_j\frac{dH^j}{dp_a}\frac{\partial H^i}{\partial q^a}+N_j\frac{dH^j}{dq^a}\frac{\partial H^i}{\partial p_a}\\ &\equiv N_j\,\{H^i,H^j\}.\end{aligned} \tag{2.5}$$

To be consistent, the physical motion should remain on the constraint surface Σ. That is, $dH/d\tau$ has to vanish along the curve. Therefore, we must have the condition

$$\{H^i,H^j\}\big|_\Sigma = 0, \qquad \text{abbreviated as} \quad \{H^i,H^j\}\approx 0 \tag{2.6}$$

for all i and j. A function $F(q^a,p_a)$ defined in a neighborhood of Σ is called *weakly zero* if $F|_\Sigma=0$ (abbreviated as $F\approx 0$) and called *strongly zero* if

$$F|_\Sigma = 0 \quad \text{and} \quad \left(\frac{\partial F}{\partial q^a},\frac{\partial F}{\partial p_a}\right)\Bigg|_\Sigma = 0, \qquad \text{abbreviated as} \quad F\simeq 0. \tag{2.7}$$

It can be proven that $F\approx 0$ implies $F\simeq f_iH^i$ for some functions $f_i(q^a,p_a)$. Consequently, we have

$$\{H^i,H^j\}\simeq f^{ij}{}_k(q^a,p_a)\,H^k. \tag{2.8}$$

The condition (2.6) ensures all constraints H^i to be *first class*. (See [17] for more about constrained systems and the concept of first class constraints.)

B. Nonrelativistic mechanics as a special case

The conventional nonrelativistic mechanics can also be formulated in the timeless framework as a special case. For the nonrelativistic systems, the relativistic configuration space has the structure $\mathcal{C} = \mathbb{R}\times\mathcal{C}_0$, where $\mathcal{C}_0$ is the conventional nonrelativistic configuration space; i.e., $q^a = (t, q^i)$ as one of the partial observables is identified as the time t. Correspondingly, the momenta read as $p_a = (p_t, p_i)$ with p_t being the conjugate momentum of t and p_i being the conjugate momenta of q^i. The Hamiltonian constraint is given by

$$H(t, q^i, p_t, p_i) = p_t + H_0(q^i, p_i, t), \tag{2.9}$$

where $H_0(q^i, p_i, t)$ is the conventional nonrelativistic Hamiltonian function. Given the Hamiltonian constraint in the form of (2.9), the Hamilton equations (2.4) lead to

$$\frac{dt}{d\tau} = N(\tau), \qquad \frac{dp_t}{d\tau} = -N(\tau)\frac{\partial H_0}{\partial t}, \tag{2.10a}$$

$$\frac{dq^i}{d\tau} = N(\tau)\frac{\partial H_0}{\partial p_i}, \qquad \frac{dp_i}{d\tau} = -N(\tau)\frac{\partial H_0}{\partial q^i}, \tag{2.10b}$$

which read as

$$\frac{dp_t}{dt} = -\frac{\partial H_0}{\partial t} \tag{2.11}$$

and

$$\frac{dq^i}{dt} = \frac{\partial H_0}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H_0}{\partial q^i}, \tag{2.12}$$

if particularly we use t to parameterize the curve of solutions. Furthermore, the constraint (2.2) dictates $p_t = -H_0$. Thus, the momentum p_t is the negative of energy and it is a constant of motion if $H_0 = H_0(q^i, p_i)$ has no explicit dependence on t. The equations in (2.12) are precisely the conventional Hamilton equations for nonrelativistic mechanics.

The Hamilton equations in (2.12) form a system of first-order ordinary differential equations. Given the initial condition $q^i(t_0) = q^i_0$ and $p_i(t_0) = p_{i0}$ at the time t_0, the existence and uniqueness theorem for ordinary differential equations states that there exists a solution of (2.12) given by $q^i = q^i(t)$ and $p_i = p_i(t)$ for $t \in \mathbb{R}$, and furthermore the solution is unique.[1] As a consequence, q^i and p_i evolve as functions of t, and a physical motion is an *open* curve in $\mathcal{C} = \mathbb{R} \times \mathcal{C}_0$, along which the observable t is *monotonic*.

A dynamical system in which a particular partial observable can be singled out as t such that the Hamiltonian is separated as in the form of (2.9) is called *deparametrizable*. For deparametrizable systems, the change of t is in accord with the ordinary notion of time,

[1] In order to apply the existence and uniqueness theorem, we assume $\partial H_0/\partial q^i$, $\partial H_0/\partial p_i$, $\partial^2 H_0/\partial {q^i}^2$, $\partial^2 H_0/\partial {p_i}^2$ and $\partial^2 H_0/\partial q^i \partial p_j$ all continuous.

which does not turn around but grows monotonically along the physical motion. Generically, however, relativistic systems might be non-deparametrizable — no preferred observable can serve as the time such that other variables are described as functions of time along the physical motion. The classical theory predicts the physical motion as an unparameterized curve, which gives *correlations* between physical variables, not the way physical variables evolve with respect to a preferred time variable. In the next subsection, we will introduce the timeless double pendulum as an example to illustrate the timeless feature.

C. Example: Timeless double pendulum

Let us now introduce a genuinely timeless system as a simple model to illustrate the mechanics without time. This model was first introduced in [3, 4] and used repeatedly as an example in [9].

Consider a mechanical system with two partial observables, a and b, whose dynamics is specified by the relativistic Hamiltonian

$$H(a, b, p_a, p_b) = -\frac{1}{2}\left(p_a^2 + p_b^2 + a^2 + b^2 - 2E\right) \tag{2.13}$$

with a given constant E. The relativistic configuration space is $\mathcal{C} = \mathbb{R}^2$ coordinated by a and b, and the cotangent space $\Omega = T^*\mathcal{C}$ is coordinated by (a, b, p_a, p_b). The constraint surface Σ is specified by $H = 0$; it is a 3-dimensional sphere of radius $\sqrt{2E}$ in Ω.

In the $N(\tau) = 1$ gauge, the Hamilton equations (2.4) give

$$\frac{da}{d\tau} = p_a, \qquad \frac{db}{d\tau} = p_b, \qquad \frac{dp_a}{d\tau} = -a, \qquad \frac{dp_b}{d\tau} = -b, \tag{2.14}$$

and the Hamiltonian constraint (2.2) gives

$$a^2 + b^2 + p_a^2 + p_b^2 = 2E. \tag{2.15}$$

The general solution is given by

$$a(\tau) = A_a \sin(\tau), \qquad b(\tau) = A_b \sin(\tau + \beta), \tag{2.16}$$

where $A_a = \sqrt{2E}\sin\alpha$ and $A_b = \sqrt{2E}\cos\alpha$, and α and β are constants.

Therefore, physical motions are closed curves (ellipses) in $\mathcal{C} = \mathbb{R}^2$. (Choosing different gauges for N yields the same curve with different parametrizations.) This system is non-deparametrizable and does not admit a conventional Hamiltonian formulation, because, as discussed in Sec. II B, physical motions in $\mathcal{C} = \mathbb{R} \times \mathcal{C}_0$ for a nonrelativistic system are monotonic in t and thus cannot be closed curves.

D. Lagrangian formalism

Consider the special case that the relativistic Hamiltonian is given in the form:[2]

$$H(q^a, p_a) = \sum_a \alpha_a p_a^2 + \sum_a \beta_a p_a q^a + \sum_a \gamma_a p_a + V(q^a), \tag{2.17}$$

where α_a, β_a and γ_a are constant coefficients, and $V(q^a)$ is the potential which depends only on q^a. This form is quite generic and many examples of interest belong to this category such as the relativistic particle (free or subject to an external potential), the timeless double pendulum (harmonic or anharmonic) and the nonrelativistic system as described by (2.9) with $H_0 = \sum_i p_i^2/2m_i + V(q^i, t)$. The Hamilton equations (2.4) yields

$$\frac{dq^a}{d\tau} = N\left(2\alpha_a p_a + \beta_a q^a + \gamma_a\right), \tag{2.18a}$$

$$\frac{dp_a}{d\tau} = -N\left(\beta_a p_a + \frac{\partial V}{\partial q^a}\right). \tag{2.18b}$$

Equation (2.18a) gives the relation between the momenta p_a and the "velocities" $\dot{q}^a := dq^a/d\tau$, through which the inverse Legendre transform recasts the action (2.3) in terms of the Lagrangian function:

$$\begin{aligned} S[q^a, \dot{q}^a, N; \tau] &= \int d\tau\, L(q^a, \dot{q}^a, N) \\ &= \int d\tau \left(\sum_a \frac{N}{4\alpha_a}\left[\frac{\dot{q}^a}{N} - \beta_a q^a - \gamma_a\right]^2 - NV(q^a)\right). \end{aligned} \tag{2.19}$$

Variation with respect to N yields

$$\begin{aligned} \frac{\delta S}{\delta N} &\equiv \frac{\partial L}{\partial N} = 0 \quad \Rightarrow \\ 0 &= \sum_a \frac{1}{4\alpha_a}\left[\frac{\dot{q}^a}{N} - \beta_a q^a - \gamma_a\right]^2 - \sum_a \frac{\dot{q}^a}{2\alpha_a N}\left[\frac{\dot{q}^a}{N} - \beta_a q^a - \gamma_a\right] - V \\ &= -\left(\sum_a \alpha_a p_a^2 + \sum_a \beta_a p_a q^a + \sum_a \gamma_a p_a + V\right) = -H, \end{aligned} \tag{2.20}$$

[2] In this subsection, the repeated index a is not summed unless $\sum_a$ is explicitly used.

which is precisely the Hamiltonian constraint (2.2). On the other hand, variation with respect to q^a gives the equation of motion as a second-order differential equation:

$$\begin{aligned} \frac{\delta S}{\delta q^a} &\equiv \frac{\partial L}{\partial q^a} - \frac{d}{d\tau}\frac{\partial L}{\partial \dot{q}^a} = 0 \\ \Rightarrow \quad \frac{d}{Nd\tau}\left(\frac{dq^a}{Nd\tau}\right) &= \beta_a^2 q^a + \beta_a\gamma_a - 2\alpha_a\frac{\partial V}{\partial q^a}, \end{aligned} \tag{2.21}$$

which is equivalent to (2.18).

III. QUANTUM THEORY OF RELATIVISTIC MECHANICS

The timeless formulation for relativistic classical mechanics is reviewed in Sec. II. Based on the Hamiltonian framework of the classical theory, the quantum theory of relativistic mechanics can be formulated in canonical formalism. Unlike the conventional quantum theory, relativistic quantum mechanics does not describe evolution in time, but correlations between observables. The timeless formulation for relativistic quantum mechanics is described in detail in Chapter 5 of [9], excerpts from which are presented in Sec. III A and Sec. III B to give a review. Issues on the physical Hilbert space are detailed in Sec. III C and the physical interpretation of quantum measurements and collapse is discussed in Sec. III D.

A. General scheme

Let $\mathcal{C}$ be the relativistic configuration space for the classical theory as described in the Sec. II A. The corresponding quantum theory can be formulated timelessly in the following scheme:

Kinematical states. Let $\mathcal{S} \subset \mathcal{K} \subset \mathcal{S}'$ be the Gelfand triple defined over $\mathcal{C}$ with the measure $d^dq_a \equiv dq^1dq^2\cdots dq^d$.[3] The kinematical states of a system are represented by vectors $|\psi\rangle \in \mathcal{K}$, and $\mathcal{K}$ is called the *kinematical Hilbert space.*

Partial observables. A partial observable is represented by a self-adjoint operator in $\mathcal{K}$.

The simultaneous eigenstates $|s\rangle$ of a complete set of commuting partial observables

[3] That is, $\mathcal{S}$ is the space of the smooth functions $f(q^a)$ on $\mathcal{C}$ with fast decrease, $\mathcal{K} = L^2[\mathcal{C}, d^dq^a]$ is a Hilbert space, and $\mathcal{S}'$ is formed by the tempered distributions over $\mathcal{C}$.

are called *quantum events*. In particular, $\hat{q}^a$ and $\hat{p}_a$ are partial observables acting respectively as multiplicative and differential operators on $\psi(q^a)$; i.e., $\hat{q}^a\psi(q^a) = q^a\psi(q^a)$ and $\hat{p}_a\psi(q^a) = -i\hbar\,\partial\psi(q^a)/\partial q^a$. Their eigenstates $|q^a\rangle$ (defined as $\hat{q}^a|q^a\rangle = q^a|q^a\rangle$) and $|p_a\rangle$ (defined as $\hat{p}_a|p_a\rangle = p_a|p_a\rangle$) are both quantum events.

Dynamics. Dynamics is defined by a self-adjoint operator $\hat{H}$ in $\mathcal{K}$, called *relativistic Hamiltonian.* The operator from $\mathcal{S}$ to $\mathcal{S}'$ schematically defined as

$$\hat{P} = \int d\tau\, e^{-i\tau\hat{H}} \tag{3.1}$$

is called the "projector".[4] The matrix elements

$$W(s,s') := \langle s|\hat{P}|s'\rangle \tag{3.2}$$

are called *transition amplitudes*, which encode entire physics of the dynamics.

Physical states. A *physical state* is a solution of the quantum Hamiltonian constraint equation:

$$\hat{H}|\psi\rangle = 0, \tag{3.3}$$

which is the quantum counterpart of (2.2). Given an arbitrary kinematical state $|\psi_\alpha\rangle \in \mathcal{S}$, we can associate an element $(\Psi_{\psi_\alpha}| \in \mathcal{S}'$, defined by its (linear) action on arbitrary states $|\psi_\beta\rangle \in \mathcal{S}$ as

$$(\Psi_{\psi_\alpha}|\psi_\beta\rangle = \int d\tau\,\langle e^{i\tau\hat{H}}\psi_\alpha|\psi_\beta\rangle \equiv \langle\psi_\alpha|\hat{P}|\psi_\beta\rangle, \tag{3.4}$$

such that $(\Psi_{\psi_\alpha}|$ is a physical state, namely, a solution to (3.3). The solution space is endowed with the Hermitian inner product:

$$(\Psi_{\psi_\alpha}|\Psi_{\psi_\beta}) := (\Psi_{\psi_\alpha}|\psi_\beta\rangle, \tag{3.5}$$

which is called the *physical inner product*. The Cauchy completion of the solution space with respect to the physical inner product $(\cdot|\cdot)$ is called the *physical Hilbert space* and denoted as $\mathcal{H}$.

[4] The integration range depends on the system: It is over a compact space if the spectrum of $\hat{H}$ is discrete and over a noncompact space if the spectrum is continuous. The operator $\hat{P}$ is a projector in the precise sense only if zero is a part of the discrete spectrum of $\hat{H}$.

Measurements and collapse. If the measurement corresponding to a partial observable $\hat{A}$ is performed, the outcome takes the value of one of the eigenvalues of $\hat{A}$ if the spectrum of $\hat{A}$ is discrete, or in a small spectral region (with uncertainty) if the spectrum is continuous. Measuring a complete set of partial observables $\hat{A}_i$ *simultaneously* is called a *complete measurement* at an "instance",[5] the outcome of which gives rise to a kinematical state $|\psi_\alpha\rangle$ (which is a simultaneous eigenstate of $\hat{A}_i$ if the spectra of $\hat{A}_i$ are discrete). The physical state is said to be *collapsed* to $|\Psi_{\psi_\alpha})$ by the complete measurement.

Prediction in terms of probability. If at one instance a complete measurement yields $|\psi_\alpha\rangle$, the probability that at another instance another complete measurement yields $|\psi_\beta\rangle$ is given by

$$\mathcal{P}_{\beta\alpha} = \left| \frac{(\Psi_{\psi_\beta}|\Psi_{\psi_\alpha})}{\sqrt{(\Psi_{\psi_\beta}|\Psi_{\psi_\beta})}\,\sqrt{(\Psi_{\psi_\alpha}|\Psi_{\psi_\alpha})}} \right|^2 = \left| \frac{W[\psi_\beta,\psi_\alpha]}{\sqrt{W[\psi_\beta,\psi_\beta]}\,\sqrt{W[\psi_\alpha,\psi_\alpha]}} \right|^2, \tag{3.6}$$

where

$$W[\psi_\beta,\psi_\alpha] := \langle\psi_\beta|\hat{P}|\psi_\alpha\rangle = \int ds \int ds'\ \overline{\psi_\beta(s)}\ W(s,s')\,\psi_\alpha(s'). \tag{3.7}$$

In particular, if the quantum events s make up a discrete spectrum, the probability of the quantum event s given the quantum event s' is

$$\mathcal{P}_{ss'} = \left| \frac{W(s,s')}{\sqrt{W(s,s)}\,\sqrt{W(s',s')}} \right|^2. \tag{3.8}$$

If the spectrum is continuous, the probability of a quantum event in a small spectral region R given a quantum event in a small spectral region R' is

$$\mathcal{P}_{RR'} = \left| \frac{W(R,R')}{\sqrt{W(R,R)}\,\sqrt{W(R',R')}} \right|^2, \tag{3.9}$$

where

$$W(R,R') := \int_R ds \int_{R'} ds'\ W(s,s'). \tag{3.10}$$

[5] In the timeless language, a complete measurement is said to be conducted at some "instance", not at some "instant".

It should be noted that, unlike the classical theory, the relativistic quantum mechanics described above is *not* equivalent to the conventional quantum theory, even if the system is deparametrizable. In conventional quantum mechanics, the time t is treated as a parameter and not quantized as an operator. Thus, the measurement of t is presumed to have zero uncertainty ($\Delta t = 0$). In relativistic quantum mechanics, t is on the same footing as other observables q^i and the measurement of t will yield nonzero Δt. For a simple harmonic oscillator govern by the relativistic Hamiltonian $H = p_t + H_0 = p_t + p_\alpha^2/2m + m\omega^2\alpha^2/2$, it was shown in [5, 6] that, if $\Delta t \ll m\Delta\alpha^2/\hbar$, we can ignore the temporal resolution Δt and idealize the measurement of t as *instantaneous*, and the conventional nonrelativistic quantum theory is recovered as a good approximation of the relativistic quantum mechanics.

In the following, we will first take the timeless double pendulum as an example to see how the above scheme is carried out, and then elaborate on some intricate issues.

B. Example: Timeless double pendulum

Take the timeless double pendulum introduced in Sec. II C as an example. The kinematical Hilbert space is $\mathcal{K} = L^2(\mathbb{R}^2, dadb)$, and the quantum Hamiltonian equation reads as

$$\hat{H}\psi(a,b) = \frac{1}{2}\left(-\hbar^2\frac{\partial^2}{\partial a^2} - \hbar^2\frac{\partial^2}{\partial b^2} + a^2 - b^2 - 2E\right)\psi(a,b) = 0. \tag{3.11}$$

Since $\hat{H} = \hat{H}_a + \hat{H}_b - E$, where $\hat{H}_a$ (resp. $\hat{H}_b$) is the nonrelativistic Hamiltonian for a simple harmonic oscillator in the variable a (resp. b), this equation can be easily solved by using the basis that diagonalizes $\hat{H}_a$ and $\hat{H}_b$. Let

$$\psi_n(a) \equiv \langle a|n\rangle = \frac{1}{\sqrt{n!}}\,H_n(a)\,e^{-a^2/2\hbar} \tag{3.12}$$

be the normalized nth eigenfunction for the harmonic oscillator with eigenvalue $E_n = \hbar(n + 1/2)$, where $H_n(a)$ is the nth Hermite polynomial. Clearly, the function

$$\psi_{n_a,n_b}(a,b) := \psi_{n_a}(a)\,\psi_{n_b}(b) \equiv \langle a,b|n_a,n_b\rangle \tag{3.13}$$

solves (3.11) if

$$\hbar\,(n_a + n_b + 1) = E, \tag{3.14}$$

which implies the quantum theory exists only if $E = \hbar(N+1)$ with $N \in \mathbb{Z}^+ \cup \{0\}$.

Consequently, for a given N, the general solution of (3.11) is given by

$$\Psi(a,b) = \sum_{n=0}^{N} c_n\, \psi_n(a)\, \psi_{N-n}(b), \tag{3.15}$$

and thus the physical Hilbert space $\mathcal{H}$ is an $(N+1)$-dimensional proper subspace of $\mathcal{K}$ spanned by an orthonormal basis $\{|n, N-n\rangle\}_{n=0,\cdots,N}$.

The projector $\hat{P} : \mathcal{S} \to \mathcal{H}$ is a true projector as $\mathcal{H}$ is a proper subspace of $\mathcal{K}$ for the case that the spectrum of $\hat{H}$ is discrete. Obviously, $\hat{P}$ is given by

$$\hat{P} = \sum_{n=0}^{N} |n, N-n\rangle\langle n, N-n\,|, \tag{3.16}$$

which can be obtained (up to an irrelevant overall factor) from (3.1):

$$\begin{aligned} \int_0^{2\pi/\hbar} d\tau\, e^{-i\tau\hat{H}} \;&\propto\; \frac{1}{2\pi}\int_0^{2\pi} d\tau \sum_{n_a,n_b} |n_a, n_b\rangle\, e^{-i\tau(n_a+n_b+1-E)}\langle n_a, n_b\,| \\ &= \sum_{n_a,n_b} \delta_{n_a+n_b+1,E}|n_a, n_b\rangle\langle n_a, n_b\,| = \hat{P}. \end{aligned} \tag{3.17}$$

Here, the integration range is so chosen because $\exp(-i\tau\hat{H})$ is periodic in τ with period $2\pi/\hbar$ if $E = \hbar(N+1)$.

The transition amplitudes are given by

$$\begin{aligned} W(a,b,a',b') \;&:=\; \langle a,b|\hat{P}|a',b'\rangle = \sum_{n=0}^{N} \langle a,b|n, N-n\rangle\langle n, N-n|a',b'\rangle \\ &= \sum_{n=0}^{N} \frac{e^{-(a^2+b^2+a'^2+b'^2)/2\hbar}}{n!(N-n)!}\, H_n(a)\, H_{N-n}(b)\, H_n(a')\, H_{N-n}(b')\,, \end{aligned} \tag{3.18}$$

which is the probability density of measuring (a,b), given (a',b') measured at another instance. Furthermore, the probability of the quantum event (n_a, n_b) given the quantum event (n'_a, n'_b) is

$$W[\psi_{n_a,n_b}, \psi_{n'_a,n'_b}] := \langle n_a, n_b|\hat{P}|n'_a, n'_b\rangle = \delta_{N,n_a+n_b}\delta_{n_a,n'_a}\delta_{n_b,n'_b}. \tag{3.19}$$

C. More on the physical Hilbert space

The operator $\hat{P} : \mathcal{S} \to \mathcal{S}'$ maps an arbitrary element of $\mathcal{S}$ to its dual space S'. If zero is in the continuous spectrum of $\hat{H}$, $\hat{P}$ maps S to a larger space S' and thus is *not* really a projector. In this case, the physical state $(\Psi_{\psi_\alpha}|$ mapped from $|\psi_\alpha\rangle$ is a tempered distribution.

$\hat{P}$ becomes a true projector only if zero is a part of the discrete spectrum of $\hat{H}$ such as in the timeless double pendulum.

The construction in (3.1) is a special case for the *group averaging* procedure [18, 19], the idea of which is to averaging over all states along the gauge flow (generated by the constraint operator) to yield the physical solution which satisfies the constraint equation. In the special case, let $|E\rangle$ be the eigenstate of $\hat{H}$ with eigenvalue E, then schematically we have

$$\begin{aligned}\hat{H}|\Psi_{\psi_\alpha}) &= \int d\tau\, \hat{H} e^{-i\tau\hat{H}}|\psi_\alpha\rangle = \int d\tau \int dE\, \hat{H} e^{-i\tau\hat{H}}|E\rangle\langle E|\psi_\alpha\rangle \\ &= \int d\tau \int dE\, E\, e^{-i\tau E}|E\rangle\langle E|\psi_\alpha\rangle = \int dE\, \delta(E)\, E|E\rangle\langle E|\psi_\alpha\rangle = 0, \end{aligned} \tag{3.20}$$

thus showing that $\hat{P}$ maps an arbitrary kinematical state $|\psi_\alpha\rangle$ to a physical state which satisfies the constraint equation (3.3). Furthermore, it can be easily shown that $(\Psi_{\psi_\alpha}|\psi_\beta\rangle = (\Psi_{\psi_\alpha}|\psi'_\beta\rangle$ if $(\Psi_{\psi_\beta}| = (\Psi_{\psi'_\beta}|$, and therefore the physical inner product in (3.5) is well defined.

If there are multiple constraints, we have to solve the multiple constraint equations simultaneously:

$$\hat{H}^i|\psi\rangle = 0, \qquad \text{for } i = 1, \cdots, k. \tag{3.21}$$

In the simplest case that $[\hat{H}^i, \hat{H}^j] = 0$ for all i, j, the projector can be easily constructed via

$$\hat{P} = \int d\tau_1 \cdots \int d\tau_k\, e^{-i\tau_i \hat{H}^i} \tag{3.22}$$

as a direct extension of (3.1). In general, however, $\hat{H}^i$ do not commute, as classically the Poisson brackets $\{H^i, H^j\}$ vanish only *weakly* [see (2.6) and (2.8)].

In the case that $\hat{H}^i$ do not commute but form a closed Lie algebra, i.e.,

$$[\hat{H}^i, \hat{H}^j] = f^{ij}{}_k\, \hat{H}^k \tag{3.23}$$

with $f^{ij}{}_k$ being constants, the exponentials of $\hat{H}^i$ form a Lie group G and the physical state can be obtained by group averaging:

$$|\Psi_{\psi_\alpha}) = \int_G d\mu(\hat{U})\, \hat{U}|\psi_\alpha\rangle, \tag{3.24}$$

where $d\mu$ is the Haar measure. It follows

$$\begin{aligned}\hat{U}'|\Psi_{\psi_\alpha}) &= \int_G d\mu(\hat{U})\, \hat{U}'\hat{U}|\psi_\alpha\rangle = \int_G d\mu(\hat{U}'^{-1}\hat{U}'')\, \hat{U}''|\psi_\alpha\rangle \\ &= \int_G d\mu(\hat{U}'')\, \hat{U}''|\psi_\alpha\rangle = |\Psi_{\psi_\alpha}) \end{aligned} \tag{3.25}$$

for any $\hat{U}' \in G$. The fact that $|\Psi_{\psi_\alpha})$ is invariant under any $\hat{U}' \in G$ implies that it is annihilated by the generators of G, namely, $\hat{H}^i|\Psi_{\psi_\alpha}) = 0$. Furthermore, the physical inner product in (3.5) is again well defined. (See [19] for more details and subtleties.) The averaging in (3.22) is indeed a special case of (3.24).

Generically, ${f^{ij}}_k$ are functions of q^a and p_a in (2.8), and, correspondingly, $\hat{H}^i$ do not form a closed Lie algebra in the kinematical space $\mathcal{K}$. In this case, it is much more difficult to obtain the physical solutions and to construct the quantum theory which is free of quantum anomalies (see [20] for the issues of anomalies).

D. Remarks on measurements and collapse

Imagine that a quantum system is measured by Alice and Bob at two different instances, yielding two outcomes corresponding to $|\psi_\alpha\rangle$ and $|\psi_\beta\rangle$, respectively. From the perspective of Alice, the physical state is collapsed to $|\Psi_{\psi_\alpha})$ by her measurement and Bob's measurement affirms her prediction. Bob, on the other hand, regards the physical state to be collapsed to $|\Psi_{\psi_\beta})$ by his measurement and predicts what Alice can measure. The striking puzzle arises: Who, Alice or Bob, causes the physical state to collapse in the first place?

In the timeless framework, it turns out to be an invalid question to ask who collapses the physical state *first*, since we cannot make any sense of time. The seemingly puzzle is analogous to the Einstein-Podolsky-Rosen (EPR) paradox, in which a pair of entangled particles are measured separately by Alice and Bob. In the context of special relativity, if the two measurements are conducted at two spacetime events which are spacelikely separated, the time-ordering of the two events can flip under a Lorentz boost and thus has no physical significance. Alice and Bob can both claim that the entangled state is collapsed by her/his measurement and thus have different knowledge about what the physical state should be, yet the predictions by Alice and Bob are consistent to each other. In our case, the measurement at an instance is analogous to the measurement on a single particle of the EPR pair; the kinematical state is analogous to the (local) state of a single particle; and the physical state is analogous to the (global) entangled state of the EPR pair. A complete knowledge (usually from measurement) about the local state will collapse the global state at once through the entanglement, which is analogous to the dynamics (or say, transition amplitudes) in our case. Consistency also holds in our case as $(\Psi_{\psi_\alpha}|\psi_\beta\rangle = \overline{(\Psi_{\psi_\beta}|\psi_\alpha\rangle}$. (See [21] for more on the

EPR paradox in the relational interpretation of quantum mechanics and also Section 5.6 of [9] for more on the philosophical issues.)

As a side remark, exploiting further the close analogy between the EPR pair and the timeless formalism of relativistic quantum mechanics, one might be able to conceive an analog of the Bell's inequality, which would help to elaborate on the interpretational and conceptual issues of relativistic quantum mechanics at the level of thought experiments.

IV. TIMELESS PATH INTEGRAL

The canonical formalism for relativistic quantum mechanics is described in Sec. III. All information of the quantum dynamics is encoded by the transition amplitudes (3.2). In particular, by choosing $|s\rangle = |q^a\rangle$ and $|s'\rangle = |q'^a\rangle$, all physics can be obtained from the following transition amplitudes

$$W(q^a, q'^a) = \langle q^a|\hat{P}|q'^a\rangle \sim \int d\tau\, \langle q^a|e^{-i\tau\hat{H}}|q'^a\rangle. \tag{4.1}$$

From now on, we will use the notation $\sim$ to denote the equality up to an overall constant factor which has no physical significance, as any overall constant is canceled out in the numerator and denominator in (3.8).

As a special case of group averaging, the integration range of τ is taken to be a compact interval if $\exp(-i\tau\hat{H})$ forms a compact Lie group $U(1)$ (timeless double pendulum is an example) and it is taken to be $(-\infty,\infty)$ if the group of $\exp(-i\tau\hat{H})$ is noncompact. For the case that $\exp(-i\tau\hat{H})$ gives $U(1)$, we can unwrap $U(1)$ to its covering space $\mathbb{R}$ and correspondingly integrate τ over $(-\infty,\infty)$. The unwrapping only gives rise to an overall multiplicative factor (which is divergent if not properly regulated). Therefore, in any case, up to an irrelevant overall factor, transition amplitudes can be computed by

$$W(q^a, q'^a) \sim \int_{-\infty}^{\infty} d\tau\, \langle q^a|e^{-i\tau\hat{H}}|q'^a\rangle, \tag{4.2}$$

where $\langle q^a|e^{i\tau\hat{H}}|q'^a\rangle$ can be thought as the transition amplitude for a kinematical state $|q'^a\rangle$ to "evolve' to the state $|q^a\rangle$ by the "parameter time" τ. Equation (4.2) sums over $\langle q^a|e^{i\tau\hat{H}}|q'^a\rangle$ for all possible values of τ, suggesting that $W(q^a, q'^a)$ is intrinsically timeless as the parameter time τ has no essential physical significance.

Rigorously, the integration should be regularized via

$$W(q^a, q'^a) \sim \lim_{M\to\infty} \frac{\int_{-M}^{M} d\tau \langle q^a | e^{-i\tau\hat{H}} | q'^a \rangle}{\int_{-M}^{M} d\tau}, \tag{4.3}$$

as a cut-off M is introduced to regulate the integral and the irrelevant overall factor to be finite. As we will see, the variable τ corresponds to the parametrization of curves in the path integral and integrating over all τ indicates that the parametrization of curves has no physical significance. The above regularization scheme is physically well justified, as it cuts off the curves in the path integral which are "too wild" (noncompact curves), given the endpoints q'^a and q^a fixed.

In the following, starting from (4.2), we will first derive the timeless path integral for the case of a single Hamiltonian constraint and then investigate it in more detail. In the end, we will study the path integral with multiple relativistic constraints.

A. General structure

For a given τ, let us introduce a *parametrization sequence*: $\tau_0 = 0,\ \tau_1,\ \tau_2, \cdots, \tau_{N-1},\ \tau_N = \tau$ with $\tau_i \in \mathbb{R}$, and define $\Delta\tau_n := \tau_n - \tau_{n-1}$. The conditions on the endpoints ($\tau_0 = 0$ and $\tau_n = \tau$) correspond to $\sum_{n=1}^{N} \Delta\tau_n = \tau$. The *mesh* of the parameter sequence is defined to be $\max_{n=1,\cdots,N}\{|\Delta\tau_n|\}$. The parameter sequence is said to be fine enough if its mesh is smaller than a given small number ϵ.[6]

As τ is fixed now, identifying $q^a = q_N^a$ and $q'^a = q_0^a$, and using $\sum_{n=1}^{N} \Delta\tau_n = \tau$, we can

[6] Let X be a topological space and $s \in [0,1]$. A continuous map $\gamma:\ s \mapsto \gamma(s) \in X$ is called a *path* with an initial point $s(0) = x_0$ and an end point $s(1) = x_1$. The image of γ is called a *curve*, which can be reparameterized with respect to a new variable τ as $\gamma:\ \tau \mapsto \gamma(\tau)$ by introducing an arbitrary continuous function $\tau:\ s \mapsto \tau(s) \in \mathbb{R}$. The parametrization sequence $\tau_0 = 0,\ \tau_1,\ \tau_2, \cdots, \tau_{N-1},\ \tau_N = \tau$ can be viewed as a discrete approximation for the reparametrization function $\tau(s)$ with $\tau(s=0) = 0$ and $\tau(s=1) = \tau$ if we identify $\tau_n = \tau(n/N)$. For the case that $\tau(s)$ is injective, the parametrization sequence is ordered (i.e. $0 = \tau_0 < \tau_1 < \cdots < \tau_{N-1} < \tau_N = \tau$ and $\Delta\tau_n > 0$ if $\tau > 0$) and called a *partition* of the interval $[0, \tau]$, which is used to define the Riemann integral as the continuous limit: $\int_0^\tau f(\tau')d\tau' = \lim_{\text{mesh}\to 0} \sum_{n=0}^{N-1} f(\tau_n)\Delta\tau_n$. In the timeless formulation of relativistic mechanics, a dynamical solution is an *unparameterized* curve in Ω and its parametrization has no physical significance. In order to exploit the timeless feature, we should keep the parametrizations generic and not restrict ourselves to injective ones. Correspondingly, the partition is generalized to a parametrization sequence and the Riemann integral is generalized to the Riemann-Stieltjes integral as $\int_0^1 f(s)d\tau(s) = \lim_{\text{mesh}\to 0} \sum_{n=0}^{N-1} f(n/N)\Delta\tau_n$, which is well defined even if $\tau(s)$ is not injective.

rewrite $\langle q^a|e^{-i\tau\hat{H}}|q'^a\rangle$ as

$$\begin{aligned}
&\langle q^a|e^{-i\tau\hat{H}}|q'^a\rangle \equiv \langle q^a_N|e^{-i\Delta\tau_N\hat{H}}\, e^{-i\Delta\tau_{N-1}\hat{H}}\cdots e^{-i\Delta\tau_1\hat{H}}|q^a_0\rangle \\
=& \left(\prod_{n=1}^{N-1}\int d^dq^a_n\right)\langle q^a_N|e^{-i\Delta\tau_N\hat{H}}|q^a_{N-1}\rangle\langle q^a_{N-1}|e^{-i\Delta\tau_{N-1}\hat{H}}|q^a_{N-2}\rangle\cdots\langle q^a_1|e^{-i\Delta\tau_1\hat{H}}|q^a_0\rangle, \quad (4.4)
\end{aligned}$$

where we have inserted $N-1$ times the completeness relation

$$\int d^dq^a\,|q^a\rangle\langle q^a\,| := \int dq^1\cdots dq^d\,|q^1,\cdots,q^d\rangle\langle q^1,\cdots,q^d\,|\,. \tag{4.5}$$

For a given arbitrary small number ϵ, by increasing N, we can always make the parameter sequence fine enough such that $\mathrm{mesh}\{\tau_i\} < |\tau|/N < \epsilon$.[7] Consequently, we can approximate each $\langle q^a_n|e^{-i\Delta\tau_n\hat{H}}|q^a_{n-1}\rangle$ to the first order in ϵ as

$$\langle q^a_{n+1}|e^{-i\Delta\tau_{n+1}\hat{H}}|q^a_n\rangle = \langle q^a_{n+1}|1-i\Delta\tau_{n+1}\hat{H}(\hat{q}^a,\hat{p}_a)|q^a_n\rangle + \mathcal{O}(\epsilon^2). \tag{4.6}$$

For the generic case that the Hamiltonian operator $\hat{H}$ is a polynomial of $\hat{q}^a$ and $\hat{p}_a$ and is *Weyl ordered*, with the use of the completeness relation for the momenta

$$\int \frac{d^dp_a}{(2\pi\hbar)^d}\,|p_a\rangle\langle p_a\,| := \int \frac{dp_1\cdots dp_d}{(2\pi\hbar)^d}\,|p_1,\cdots,p_d\rangle\langle p_1,\cdots,p_d\,|, \tag{4.7}$$

it can be shown that

$$\langle q^a|\hat{H}(\hat{q}^q,\hat{p}_a)|q'^a\rangle = \int \frac{d^dp_a}{(2\pi\hbar)^d}\,\exp\left[\frac{i}{\hbar}\,p_a(q^a-q'^a)\right]H\left(\frac{q^a+q'^a}{2},p_a\right). \tag{4.8}$$

(See Exercise 11.2 in [10] for the proof.) Applying (4.8) to (4.6), we have

$$\begin{aligned}
\langle q^a_{n+1}|e^{-i\Delta\tau_{n+1}\hat{H}}|q^a_n\rangle &= \int \frac{d^dp_{na}}{(2\pi\hbar)^d}\,e^{ip_{na}(q^a_{n+1}-q^a_n)/\hbar}\left[1-i\Delta\tau_{n+1}H((q^a_{n+1}+q^a_n)/2,p_{na}\right] + \mathcal{O}(\epsilon^2) \\
&= \int \frac{d^dp_{na}}{(2\pi\hbar)^d}\,e^{ip_{na}\Delta q^a_n/\hbar}\,e^{-i\Delta\tau_{n+1}H(\bar{q}^a_n,p_{na})} + \mathcal{O}(\epsilon^2),
\end{aligned} \tag{4.9}$$

where we define $\bar{q}^a_n := (q^a_{n+1}+q^a_n)/2$ and $\Delta q^a_n := q^a_{n+1}-q^a_n$.

Making the parametrization sequence finer and finer (by decreasing ϵ or equivalently by increasing N) and at the end going to the limit $\epsilon\to 0$ or $N\to\infty$, we can cast (4.4) as

$$\begin{aligned}
\langle q^a|e^{-i\tau\hat{H}}|q'^a\rangle &= \lim_{N\to\infty}\left(\prod_{n=1}^{N-1}\int d^dq^a_n\right)\left(\prod_{n=0}^{N-1}\int\frac{d^dp_{na}}{(2\pi\hbar)^d}\right)\exp\left(\frac{i}{\hbar}\sum_{n=0}^{N-1}p_{na}\Delta q^a_n\right) \\
&\quad\times\exp\left(-i\sum_{n=0}^{N-1}\Delta\tau_{n+1}H(\bar{q}^a_n,p_{na})\right).
\end{aligned} \tag{4.10}$$

[7] More rigorously, for a given ϵ, the large number N should be chosen to satisfy $\mathrm{mesh}\{\tau_i\} < |\tau|/N < M/N < \epsilon$, where M is the cut-off regulator defined in (4.3), so that the $\mathcal{O}(\epsilon^2)$ term in (4.9) can be dropped for any value of $|\tau|$. In the end, we have to integrate (4.10) over all possible values of τ to obtain $W(q^a,q'^a)$, and the regularization is essential to keep the $\mathcal{O}(\epsilon^2)$ terms under control for arbitrary values of τ.

In the limit $N \to \infty$, the points q_n and p_n can be viewed as the sampled points of a continuous curve in $\Omega = T^*\mathcal{C}$ given by $\tilde{\gamma}(\tau') = (q^a(\tau'), p_a(\tau'))$, which is parameterized by τ' and with the endpoints projected to $\mathcal{C}$ fixed by $q^a(\tau' = 0) = q'^a$ and $q^a(\tau' = \tau) = q^a$. That is, q_n and p_n are the sampled points of $\tilde{\gamma}$ as $q_n^a = q^a(\tau_n)$ and $p_{na} = p_a(\tau_n)$. In the treatment of functional integral, it is customary to introduce the special notations for path integrals:

$$\prod_{n=1}^{N-1} \int d^d q_n^a \quad \to \quad \int \mathcal{D}q^a, \tag{4.11a}$$

$$\prod_{n=0}^{N-1} \int \frac{d^d p_{na}}{(2\pi\hbar)^d} \quad \to \quad \int \mathcal{D}p_a. \tag{4.11b}$$

Meanwhile, in the continuous limit ($N \to \infty$), the finite sums appearing in the exponents in (4.10) also converge to the integrals:

$$\frac{i}{\hbar}\sum_{n=0}^{N-1} p_{na}\Delta q_n^a \quad \to \quad \frac{i}{\hbar}\int_{\tilde{\gamma}} p_a dq^a \equiv \frac{i}{\hbar}\int_{\tilde{\gamma}} \left(p_a \frac{dq^a}{d\tau'}\right) d\tau' \tag{4.12}$$

and

$$-\, i \sum_{n=0}^{N-1} \Delta\tau_{n+1} H(p_{na},\, \bar{q}_n^a) \quad \to \quad -i \int_{\tilde{\gamma}} H(q^a(\tau'), p_a(\tau'))\, d\tau'. \tag{4.13}$$

Note that the continuous limit above is defined via the Riemann-Stieltjes integral as an extension of the Riemann integral (see Footnote 6). With the new notations, (4.10) can be written in a concise form:

$$\langle q^a | e^{-i\tau\hat{H}} | q'^a \rangle = \int \mathcal{D}q^a \int \mathcal{D}p_a \;\exp\left(\frac{i}{\hbar}\int_{\tilde{\gamma}} p_a dq^a\right) \exp\left(-i\int_{\tilde{\gamma}} H(q^a(\tau'), p_a(\tau'))\, d\tau'\right). \tag{4.14}$$

It is remarkable to note that (up to the factor $i/\hbar$) the continuous limit in (4.12) is simply the line integral of the one-form $\tilde{\theta} = p_a dq^a$ over the curve $\tilde{\gamma}$, identical to (2.1), and is independent of the parametrization τ. On the other hand, the integral in (4.13) depends on the parametrization of τ. Thus, to compute $W(q^a, q'^a))$ in (4.2), the integration over τ only hits the second exponential in (4.14) and the first exponential simply factors out. The integration of the second exponential over τ yields

$$\begin{aligned} &\int_{-\infty}^{\infty} d\tau \exp\left(-i\int_{\tilde{\gamma}} H(q^a(\tau'), p_a(\tau'))\, d\tau'\right) \\ = &\int_{-\infty}^{\infty} d\tau \exp\left(-i\tau\int_{\tilde{\gamma}} H(q^a(\bar{\tau}), p_a(\bar{\tau}))\, d\bar{\tau}\right) = \delta\left(\int_{\tilde{\gamma}} H(q^a(\bar{\tau}), p_a(\bar{\tau}))\, d\bar{\tau}\right), \end{aligned} \tag{4.15}$$

where we have rescaled the parametrization τ' to $\bar{\tau} = \tau'/\tau$ so that the endpoints now read as $q'^a = q^a(\bar{\tau} = 0)$ and $q^a = q^a(\bar{\tau} = 1)$.[8] The appearance of the Dirac delta function indicates that only the paths which satisfy $\int_{\tilde{\gamma}} H(\bar{\tau})d\bar{\tau} = 0$ will contribute to the path integral for $W(q^a, q'^a)$. The condition $\int_{\tilde{\gamma}} H(\bar{\tau})d\bar{\tau} = 0$ is, however, still not geometrical, since $\bar{\tau}$ can be further reparameterized to $\bar{\tau}' = \bar{\tau}'(\bar{\tau})$ to yield $\int_{\tilde{\gamma}} H(\bar{\tau}')d\bar{\tau}' \neq 0$ even with the initial and final values fixed, i.e., $\bar{\tau}'(\bar{\tau} = 0) = 0$ and $\bar{\tau}'(\bar{\tau} = 1) = 1$. On the other hand, $W(q^a, q'^a)$ cast in (4.2) has no dependence on the parametrization whatsoever, which implies that, in the continuous limit, the contribution of a path $\tilde{\gamma}$ satisfying the condition $\int_{\tilde{\gamma}} H(\bar{\tau})d\bar{\tau} = 0$ for a specific (rescaled) parametrization $\bar{\tau}$ is somehow exactly canceled by that of another path satisfying the same condition. In the end, only the paths restricted to the constraint surface (i.e., $\tilde{\gamma} \in \Sigma$, or equivalently $H(\tau') = 0$ for all τ' along the path) contribute to the path integral for $W(q^a, q'^a)$. The constraint $\tilde{\gamma} \in \Sigma$ is now geometrical.

How the aforementioned cancelation takes place is obscure. To elucidate this point, we exploit the fact that $W(q^a, q'^a)$ is independent of the parametrization and play the trick by averaging over all possible parametrizations. That is, up to an overall factor of no physical significance, we can recast $W(q^a, q'^a)$ by summing over different parametrizations as follows:

$$\begin{aligned} W(q^a, q'^a) &\sim \int d\tau \int [\mathcal{D}\Delta\tau]_{\sum\Delta\tau_n=\tau} \langle q^a | e^{i\tau\hat{H}} | q'^a \rangle \qquad (4.16) \\ &\sim \int d\tau \int [\mathcal{D}\Delta\tau]_{\sum\Delta\tau_n=\tau} \int \mathcal{D}q^a \int \mathcal{D}p_a \, \exp\left(\frac{i}{\hbar}\int_{\tilde{\gamma}} p_a dq^a\right) \exp\left(-i\sum_{n=0}^{N-1} \Delta\tau_{n+1} H(\bar{q}_n^a, p_{na})\right), \end{aligned}$$

where the notation $\int [\mathcal{D}\Delta\tau]_{\sum\Delta\tau_n=\tau}$ is a shorthand for

$$\underbrace{\int_{-\tau/N}^{\tau/N} d\Delta\tau_1 \int_{-\tau/N}^{\tau/N} d\Delta\tau_2 \cdots \int_{-\tau/N}^{\tau/N} d\Delta\tau_N}_{\sum_{n=1}^N \Delta\tau_n = \tau} \quad \rightarrow \quad \int [\mathcal{D}\Delta\tau]_{\sum\Delta\tau_n=\tau}\,, \qquad (4.17)$$

which sums over all fine enough (namely, $\mathrm{mesh}\{\tau_i\} < |\tau|/N$) parametrization sequences for a given τ. It is easy to show that

$$\int_{-\infty}^{\infty} d\tau \underbrace{\int_{-\tau/N}^{\tau/N} d\Delta\tau_1 \int_{-\tau/N}^{\tau/N} d\Delta\tau_2 \cdots \int_{-\tau/N}^{\tau/N} d\Delta\tau_N}_{\sum_{n=1}^N \Delta\tau_n = \tau} = \prod_{n=0}^{N-1} \int_{-\infty}^{\infty} d\Delta\tau_{n+1}, \qquad (4.18)$$

[8] In the expression of (4.15), we have removed the cut-off regulator (i.e. the limit $M \to \infty$ has been taken). More rigorously, we have removed the regulator *before* the limit $N \to \infty$ is taken. The Dirac delta function in (4.15) would have been a *nascent delta function* if the regulator had not been removed.

when the cut-off regulator M is removed (also see Footnote 8). Consequently, for a given arbitrary parametrization τ', renaming the varying $\Delta\tau_n$ as $\Delta\tau_n = \hbar^{-1} N_n \Delta\tau'_n$, we can rewrite (4.16) as

$$\begin{aligned} & W(q^a, q'^a) \\ \sim & \int \mathcal{D}q^a \int \mathcal{D}p_a \int \mathcal{D}N \; \exp\left(\frac{i}{\hbar}\int_{\tilde{\gamma}} p_a dq^a\right) \exp\left(-\frac{i}{\hbar}\sum_{n=0}^{N-1} \Delta\tau'_{n+1} N_{n+1} H(\bar{q}^a_n, p_{na})\right), \end{aligned} \tag{4.19}$$

where we introduce the notation

$$\prod_{n=0}^{N-1} \int_{-\infty}^{\infty} dN_{n+1} \quad \rightarrow \quad \int \mathcal{D}N. \tag{4.20}$$

Again, in the continuous limit, the finite sum converges to the Riemann-Stieltjes integral:

$$-\frac{i}{\hbar}\sum_{n=0}^{N-1} \Delta\tau'_{n+1} N_{n+1} H(\bar{q}^a_n, p_{na}) \quad \rightarrow \quad -\frac{i}{\hbar}\int_{\tilde{\gamma}} N(\tau') H(q^a(\tau'), p_a(\tau')) d\tau', \tag{4.21}$$

and (4.19) can be neatly written as the *path integral*:

$$\begin{aligned} W(q^a, q'^a) &\sim \int \mathcal{D}q^a \int \mathcal{D}p_a \int \mathcal{D}N \; \exp\left[\frac{i}{\hbar}\left(\int_{\tilde{\gamma}} p_a dq^a - \int_{\tilde{\gamma}} N(\tau') H d\tau'\right)\right] \\ &\equiv \int \mathcal{D}q^a \int \mathcal{D}p_a \int \mathcal{D}N \; \exp\left[\frac{i}{\hbar}\int_{\tilde{\gamma}} \left(p_a \frac{dq^a}{d\tau'} - N(\tau') H\right) d\tau'\right]. \end{aligned} \tag{4.22}$$

Integration over N can be carried out to obtain the delta functional:

$$\int \mathcal{D}N \exp\left(\frac{i}{\hbar}\int N(\tau') H d\tau'\right) \sim \delta[H] \equiv \lim_{N\to\infty} \prod_{n=0}^{N-1} \delta(H(\bar{q}^a_n, p_{an})), \tag{4.23}$$

and thus the path integral (4.22) can be written in an alternative form as

$$W(q^a, q'^a) \sim \int \mathcal{D}q^a \int \mathcal{D}p_a \; \delta[H] \exp\left[\frac{i}{\hbar}\int_{\tilde{\gamma}} p_a dq^a\right], \tag{4.24}$$

where insertion of the delta functional $\delta[H]$ confines the path to be in the constraint surface (i.e. $\tilde{\gamma} \in \Sigma$). Note that the phase in the exponent in (4.24) is identical to the classical action defined in (2.1) (divided by $\hbar$) and that in (4.22) is identical to the classical action in (2.3) with $k = 1$. Therefore, each path in Σ contributes with a phase, which is the classical action divided by $\hbar$.

The path integral formalism is intuitively appealing. It gives us an intuitive picture about the transition amplitudes: $W(q^a, q'^a)$ is described as the sum, with the weight $\exp(iS/\hbar)$

(where S is the classical action of $\tilde{\gamma}$), over all arbitrary paths $\tilde{\gamma}$ which are restricted to Σ and whose projection γ to $\mathcal{C}$ connect q'^a and q^a. None of q^a is restricted to be monotonic along the paths, and in this sense the formulation is called *timeless* path integral. The parametrization for the paths has no physical significance as can be seen in the expression of (4.24), which is completely geometrical and independent of parametrizations. On the other hand, the continuum notation of (4.22) is really a schematic for the discretized version:

$$\begin{aligned} W(q^a, q'^a) &\sim \lim_{N\to\infty} \prod_{n=1}^{N-1} \int d^d q_n^a \prod_{n=0}^{N-1} \int \frac{d^d p_{na}}{(2\pi\hbar)^d} \prod_{n=0}^{N-1} \int dN_{n+1} \\ &\qquad\qquad \times \exp\left(-\frac{i}{\hbar} \sum_{n=0}^{N-1} \Delta\tau'_{n+1} N_{n+1} H(\bar{q}_n^a, p_{na}) \right) \qquad (4.25a) \\ &\sim \lim_{N\to\infty} \prod_{n=1}^{N-1} \int d^d q_n^a \prod_{n=0}^{N-1} \int \frac{d^d p_{na}}{(2\pi\hbar)^d} \prod_{n=0}^{N-1} \int dN_{n+1} \\ &\qquad\qquad \times \exp\left(-\frac{i}{\hbar} \sum_{n=0}^{N-1} N_{n+1} H(\bar{q}_n^a, p_{na}) \right), \qquad (4.25b) \end{aligned}$$

where $\Delta\tau'_n$ in (4.25a) is absorbed to N_n in (4.25b) and this only results in an irrelevant overall factor. The expression of (4.25b) is explicitly independent of parametrizations.[9]

The contributing paths in the path integral can be very "wild" — not necessarily smooth or even continuous. This calls into question whether the path integral can achieve convergence. We do not attempt to present a rigorous derivation here but refer to [22] for the legitimacy issues and subtleties of the path integral.

Each path in Σ contributes with a different phase, and the contributions from the paths far away from the stationary solution essentially cancel one another through destructive interference. As a result, most contributions come from the paths close to the stationary solution. The stationary solution can be obtained by taking the functional variation on (4.22) with respect to N, q^a and p_a, which yield the classical Hamiltonian constraint (2.2) and the Hamilton equations (2.4). Therefore, the stationary solution is the classical solution and we thus have the approximation

$$W(q^a, q'^a) \approx \sum_i e^{\frac{i}{\hbar} S[\tilde{\gamma}_i]}, \qquad (4.26)$$

[9] Perhaps, more appropriately, the "timeless path integral" should be renamed "timeless 'curve' integral", as in the rigorous terminology, a *curve* is defined as the unparameterized image of a *path*, which is specified by a parameter. However, we keep the name of "path integral" to conform the conventional nomenclature.

where $\tilde{\gamma}_i$ are the classical solutions which connect q'^a and q^a and S is the action.[10] Based on the path integral formalism, a semiclassical theory can be developed in the vicinity of the classical solutions *à la* the WKB method.

B. Deparametrizable systems as a special case

If the Hamiltonian happens to be deparametrizable, the classical Hamiltonian is in the form of (2.9), and the path integral (4.24) reads as

$$\begin{aligned} W(q^a, q'^a) &\sim \int \mathcal{D}t \int \mathcal{D}q^i \int \mathcal{D}p_t \int \mathcal{D}p_i\ \delta[p_t + H_0] \exp\left[\frac{i}{\hbar}\int_{\tilde{\gamma}} \left(p_t dt + p_i dq^i\right)\right] \\ &= \int \mathcal{D}t \int \mathcal{D}q^i \int \mathcal{D}p_i\ \exp\left[\frac{i}{\hbar}\int_{\tilde{\gamma}} \left(p_i \frac{dq^i}{dt} - H_0\right) dt\right] \qquad (4.27a) \\ &\equiv \lim_{N\to\infty} \left(\prod_{n=1}^{N-1} \int dt_n\right) \left(\prod_{n=1}^{N-1} \int d^{d-1} q_n^i\right) \left(\prod_{n=0}^{N-1} \int \frac{d^{d-1} p_{ni}}{(2\pi\hbar)^{d-1}}\right) \\ &\quad \times \exp\left[\frac{i}{\hbar}\sum_{n=0}^{N-1} \left(p_{ni}\frac{\Delta q_n^i}{\Delta t_n} - H_0(\bar{q}_n^i, p_{in})\right) \Delta t_n\right]. \qquad (4.27b) \end{aligned}$$

On the other hand, in the conventional nonrelativistic quantum mechanics, the transition amplitude is given by the conventional path integral (see Chapter 11 of [10]):

$$\begin{aligned} & G(q^i, t; q'^i, t') := \langle q^i | e^{-i\hat{H}_0(t-t')} | q'^i \rangle \\ &= \int \mathcal{D}q^i \int \mathcal{D}p_i\ \exp\left[\frac{i}{\hbar}\int_{\tilde{\gamma}_0} \left(p_i \frac{dq^i}{dt} - H_0\right) dt\right] \qquad (4.28a) \\ &\equiv \lim_{N\to\infty} \left(\prod_{n=1}^{N-1} \int d^{d-1} q_n^i\right) \left(\prod_{n=0}^{N-1} \int \frac{d^{d-1} p_{ni}}{(2\pi\hbar)^{d-1}}\right) \\ &\quad \times \exp\left[\frac{i}{\hbar}\sum_{n=0}^{N-1} \left(p_{ni}\frac{\Delta q_n^i}{\Delta t_n} - H_0(\bar{q}_n^i, p_{in})\right) \Delta t_n\right], \qquad (4.28b) \end{aligned}$$

where the path $\tilde{\gamma}_0$ is in the cotangent space $T^*\mathcal{C}_0$ and its projection γ_0 on $\mathcal{C}_0$ has the endpoints fixed at $q^i(t') = q'^i$ and $q^i(t) = q^i$.

Equations (4.27) looks almost the same as (4.28) except for the functional integral $\int \mathcal{D}t$. In (4.27), all paths given by $q^a(\tau) = (t(\tau), q^i(\tau))$ (parameterized by an arbitrary parameter τ) are summed and the quantum fluctuation in t is also included; by contrast, in (4.28), all

[10] Generally, there could be multiple classical solutions connecting q'^a and q^a (as in the case of the timeless double pendulum), if the system is not deparametrizable.

paths are given by $q^i(t)$ as t is treated as a parameter and with no fluctuation at all. In other words, (4.28) sums over only the paths which are monotonic in t, whereas (4.27) sums over all possible paths whether t is monotonic or not. The difference is profound and shows that the relativistic quantum mechanics is *not* equivalent to the conventional quantum mechanics even if the system is deparametrizable, as already commented in the end of Sec. III A.

However, for most systems, we have the good approximation (4.26) and only the paths in the vicinity of the classical solution are important. Meanwhile, as discussed in Sec. II B, the classical solution for a deparametrizable system is always monotonic in t. Therefore, in (4.27), it is a good approximation to sum over only the paths which are not too deviated from the classical solution and thus monotonic in t. In this approximation, (4.27) reduces to the conventional path integral (4.28) as $\int \mathcal{D}t$ factors out as an irrelevant overall factor. Therefore, the conventional path integral, although not equivalent to, is a good approximation for the timeless path integral. Further research is needed to investigate when the approximation remains good and when it fails; this is closely related to the question when the fluctuation (uncertainty) of t can be ignored and thus the measurement of it can be idealized as instantaneous (see [5, 6] for the case of a simple harmonic oscillator).

C. Timeless Feynman's path integral

Consider the special case that the classical Hamiltonian is given in the form of (2.17) and the Hamiltonian operator is Weyl ordered. As the Hamiltonian is a quadratic polynomial in p_a, the path integral over $\mathcal{D}p_a$ in (4.22) can be integrated out. That is, in the expression:[11]

$$\begin{aligned} &W(q^a, q'^a) \\ &\sim \int \mathcal{D}q^a \int \mathcal{D}N \prod_{n=0}^{N-1} \int \frac{d^d p_n}{(2\pi\hbar)^d}\ \exp\left[\frac{i}{\hbar}\sum_{n=0}^{N-1}\left(\sum_a p_{na}\Delta q_n^a - \Delta\tau'_{n+1}N_{n+1}H(\bar{q}_n^a, p_{na})\right)\right], \end{aligned} \tag{4.29}$$

the integration over each p_{na} can be explicitly carried out:

$$\begin{aligned} &\int_{-\infty}^{\infty} dp_{na} \exp\left(\frac{i}{\hbar}\left[p_{na}\Delta q_n^a - \Delta\tau'_{n+1}N_{n+1}\left(\alpha_a p_{na}^2 + \beta_a p_{na}\bar{q}_n^a + \gamma_a p_{na}\right)\right]\right) \\ &\propto \frac{1}{\sqrt{N_{n+1}}} \exp\left(\frac{i}{\hbar}\frac{\Delta\tau'_{n+1}N_{n+1}}{4\alpha_a}\left[\frac{\Delta q_n^a}{\Delta\tau'_{n+1}N_{n+1}} - \beta_a\bar{q}_n^a - \gamma_a\right]^2\right) \end{aligned} \tag{4.30}$$

[11] In this subsection, the repeated index a is not summed unless $\sum_a$ is explicitly used.

by the Gaussian integral $\int_{-\infty}^{\infty} dx\, e^{-\alpha x^2+\beta x} = (\pi/\alpha)^{1/2} e^{\beta^2/4\alpha}$. Noting that $dN_{n+1}/\sqrt{N_{n+1}} = 2\, d\sqrt{N_{n+1}}$ and introducing the shorthand notation:

$$\prod_{n=0}^{N-1} \int_{-\infty}^{\infty} d\sqrt{N_{n+1}} \quad \to \quad \int \mathcal{D}\sqrt{N}, \tag{4.31}$$

we then have

$$\begin{aligned} W(q^a, q'^a) \sim & \int \mathcal{D}q^a \int \mathcal{D}\sqrt{N} \\ & \times \exp\left[\frac{i}{\hbar}\sum_{n=0}^{N-1}\left(\sum_a \frac{N_{n+1}}{4\alpha_a}\left[\frac{\Delta q_n^a}{\Delta\tau'_{n+1}N_{n+1}} - \beta_a \bar{q}_n^a - \gamma_a\right]^2 - N_{n+1}V(\bar{q}_n^a)\right)\Delta\tau'_{n+1}\right], \end{aligned} \tag{4.32}$$

which written in the continuous form reads as

$$W(q^a, q'^a) \sim \int \mathcal{D}q^a \int \mathcal{D}\sqrt{N} \exp\frac{i}{\hbar}\int_\gamma d\tau' \left(\sum_a \frac{N}{4\alpha_a}\left[\frac{\dot{q}^a}{N} - \beta_a q^a - \gamma_a\right]^2 - NV(q^a)\right), \tag{4.33}$$

where the "velocity" $\dot{q}^a := dq^a/d\tau'$ is the continuous limit of $\Delta q_n^a/\Delta\tau'_{n+1}$.

Therefore, in the special case that the Hamiltonian is a quadratic polynomial in p_a, the transition amplitude admits a path integral formalism over the configuration space, whereby the functional integration over N is modified as $\int \mathcal{D}\sqrt{N}$. This is called the *configuration space path integral* or *Feynman's path integral*. The configuration space path integral (4.33) sums over all arbitrary paths $\gamma \in \mathcal{C}$ whose endpoints are fixed at q'^a and q^a, and each path contributes with a phase, which is identical to the Lagrangian function as given in (2.19) (divided by $\hbar$). The functional variations on (4.33) with respect to $\sqrt{N}$ and q^a yield the classical Hamiltonian constraint and equation of motion as in (2.20) and (2.21).[12] This shows again that the stationary solution is the classical solution and thus (4.26) is a good approximation.

D. Path integral with multiple constraints

If there are multiple constraints and the constraint operators $\hat{H}^i$ commute, the projector is given by (3.22) and (4.2) can be directly generalized as[13]

$$W(q^a, q'^a) \sim \int_{-\infty}^{\infty} d\tau^1 \cdots \int_{-\infty}^{\infty} d\tau^k \, \langle q^a | e^{-i\sum_{i=1}^k \tau^i \hat{H}^i} | q'^a\rangle. \tag{4.34}$$

[12] Note that $\delta W/\delta\sqrt{N} = 2\sqrt{N}\,\delta W/\delta N$.

[13] In this subsection, the repeated index i is not summed unless $\sum_i$ is explicitly used.

If each $\hat{H}^i$ is a polynomial of $\hat{q}^a$ and $\hat{p}_a$ and Weyl ordered, the linear sum $\hat{H}' = \sum_i \tau^i \hat{H}^i$ is also a polynomial and Weyl ordered. Thus, by replacing τ with 1 and $\hat{H}$ with $\hat{H}'$ in (4.14), it can be shown

$$\begin{aligned} \langle q^a | e^{-i\sum_i \tau^i \hat{H}^i} | q'^a \rangle \;=\; & \int \mathcal{D}q^a \int \mathcal{D}p_a \;\exp\left(\frac{i}{\hbar}\int_{\tilde{\gamma}} p_a dq^a\right) \\ & \times \exp\left(-i\sum_i \int_{\tilde{\gamma}} \tau^i H^i(q^a(\bar{\tau}), p_a(\bar{\tau}))\, d\bar{\tau}\right), \end{aligned} \tag{4.35}$$

where $\bar{\tau}$ is a parameter for the curve $\tilde{\gamma}$ with $q'^a = q^a(\bar{\tau}=0)$ and $q^a = q^a(\bar{\tau}=1)$. Redefining $\tau^i \Delta\bar{\tau}_n$ as $\Delta\tau^i_n$, we then have

$$\begin{aligned} \langle q^a | e^{-i\sum_i \tau^i \hat{H}^i} | q'^a \rangle \;=\; & \int \mathcal{D}q^a \int \mathcal{D}p_a \;\exp\left(\frac{i}{\hbar}\int_{\tilde{\gamma}} p_a dq^a\right) \\ & \times \exp\left(-i\sum_i \int_{\tilde{\gamma}} H^i(q^a(\tau^i), p_a(\tau^i))\, d\tau^i\right), \end{aligned} \tag{4.36}$$

As in the case with a single constraint, the first exponential in (4.36) is independent of parametrizations for the curve $\tilde{\gamma}$, and for the second exponential we can play the same trick by summing over different parametrizations to get rid of the seemingly dependence on parametrizations. Following the same steps in Sec. IV A, for each i, we have

$$\int d\tau^i \int \left[\mathcal{D}\Delta\tau^i\right]_{\sum \Delta\tau^i_n = \tau^i} \langle q^a | e^{i\tau^i \hat{H}^i} | q'^a \rangle \tag{4.37}$$

$$= \int \mathcal{D}q^a \int \mathcal{D}p_a \int \mathcal{D}N_i \;\exp\left[\frac{i}{\hbar}\int_{\tilde{\gamma}} \left(p_a \frac{dq^a}{d\tau'} - N_i(\tau')H^i\right) d\tau'\right] \tag{4.38}$$

for a given arbitrary parametrization τ'. After summed over $[\mathcal{D}\Delta\tau^i]_{\sum \Delta\tau^i_n = \tau^i}$ for each i, (4.34) yields

$$W(q^a, q'^q) \sim \int \mathcal{D}q^a \int \mathcal{D}p_a \prod_{i=1}^{k} \int \mathcal{D}N_i \;\exp\left[\frac{i}{\hbar}\int_{\tilde{\gamma}} \left(p_a \frac{dq^a}{d\tau'} - \sum_{i=1}^{k} N_i H^i\right) d\tau'\right] \tag{4.39a}$$

$$\sim \int \mathcal{D}q^a \int \mathcal{D}p_a \prod_{i=1}^{k} \delta[H^i] \;\exp\left[\frac{i}{\hbar}\int_{\tilde{\gamma}} p_a dq^a\right], \tag{4.39b}$$

which is the direct generalization of (4.22) and (4.24). In the path integral, each path in Σ contributes with a phase, which is the classical action given in (2.3) divided by $\hbar$. Functional variation on (4.39a) with respect to N_i, q^a and p_a again yields the classical equations (2.2) and (2.4).

V. SUMMARY AND DISCUSSION

Starting from the canonical formulation in [9], the timeless path integral for relativistic quantum mechanics is rigorously derived. Given in (4.24), the transition amplitude is formulated as the path integral over all possible paths in the constraint surface Σ (through the confinement by the delta functional $\delta[H]$), and each path contributes with a phase identical to the classical action $\int_{\tilde{\gamma}} p_a dq^a$ divided by $\hbar$. The alternative expression is given in (4.22), which is the functional integral over all possible paths in the cotangent space $\Omega = T^*\mathcal{C}$ as well as over the Lagrange multiplier N. The timeless path integral manifests the timeless feature of relativistic quantum mechanics, as the parametrization for paths has no physical significance. For the special case that the Hamiltonian constraint $H(q^a, p_a)$ is a quadratic polynomial in p_a, the transition amplitude admits the timeless Feynman's path integral over the paths in the configuration space $\mathcal{C}$, as given in (4.33).

The formulation of timeless path integral is intuitively appealing and advantageous in many respects as it generalizes the action principle of relativistic classical mechanics by replacing the classical notion of a single trajectory with a sum over all possible paths. In particular, it is easy to argue that the classical solution contributes most to the transition amplitude and thus (4.26) is a good approximation for generic cases since the stationary solution is identical to the classical one. In the vicinity of the classical trajectory, the semiclassical approximation *à la* the WKB method can be developed. Furthermore, timeless path integral offers a new perspective to see how the conventional quantum mechanics emerges from relativistic quantum mechanics within a certain approximation (as discussed in Sec. IV B) and may provide new insight into the problem of time.

The formulation of timeless path integral can be directly extended for the dynamical systems with multiple constraints as given in (4.39), if the constraint operators $\hat{H}^i$ commute. For the case that $\hat{H}^i$ do not commute but form a closed Lie algebra, the projector is no loner given by (3.22) but we have to invoke (3.24) to obtain the physical state, which leads to

$$W(q^a, q'^a) \sim \int d\mu(\vec{\theta})\, \langle q^a| e^{-i\vec{\theta}\cdot\hat{\vec{H}}} |q'^a\rangle, \tag{5.1}$$

where θ^i are coordinates of the Lie group G generated by $\hat{H}^i$. Starting from (5.1) and following the similar techniques used in this paper, one should be able to obtain the timeless path integral, but the measure of the functional integral $\prod_{i=1}^k \int \mathcal{D}N_i$ appearing in (4.39a) would have to be nontrivially modified as θ^i play the same role of τ^i in (4.34) but now the nontrivial Haar measure $d\mu$ is involved and the nontrivial topology of G has to be taken into account. For the case that $\hat{H}^i$ do not form a closed Lie algebra, it is not clear how to construct the quantum theory which is free of quantum anomalies even in the canonical formalism. The timeless path integral may instead provide a better conceptual framework to start with for the quantum theory.

Throughout this paper, we have focused on simple mechanical systems, but not field theories. In Section 3.3 of [9], the canonical treatment of classical field theories which maintains clear meaning in a general-relativistic context is presented as a direct generalization of the timeless formulation for relativistic classical mechanics (see also [23] and references therein), and the corresponding quantum field theory is formulated in Section 5.3 of [9]. The timeless path integral for relativistic quantum mechanics derived in this paper should be extended for the quantum field theory described in [9]. We leave it for the future research.

Furthermore, as the timeless path integral is systematically derived from the well-controlled canonical formulation of relativistic quantum mechanics, we expect it to provide new insight into the issues of the connection between LQG/LQC and SFMs. Extending timeless path integral to field theories will be particularly helpful.

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