Socio-economic utility and chemical potential

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Abstract. - In statistical physics, the conservation of particle number results in the equalization of the chemical potential throughout a system at equilibrium. In contrast, the homogeneity of utility in socio-economic models is usually thought to rely on the competition between individuals, leading to Nash equilibrium. We show that both views can be reconciled by introducing a notion of chemical potential in a wide class of socio-economic models, and by relating it in a direct way to the equilibrium value of the utility. This approach also allows the dependence of utility across the system to be determined when agents take decisions in a probabilistic way. Numerical simulations of a urban economic model also suggest that our result is valid beyond the initially considered class of solvable models.

Socio-economic sciences and statistical physics are both interested in the evolution of systems characterized by a large number of interacting entities. These entities can for instance be economic or social agents in social sciences [1–3], atoms or molecules in statistical physics [4–6]. The question of the emergence of macroscopic patterns from the interactions of a large number of microscopic agents is studied by both fields of science. In statistical physics, a quantitative framework has been developed over the last century, allowing the equilibrium behaviour of large assemblies of atoms or molecules to be handled precisely [6].

In socio-economic models, the preferences of individuals are usually characterized by a utility function, which describes their welfare with respect to their current situation or environment. Each individual or agent wants to maximize his own welfare. Decisions (e.g., moving to a more convenient place) are thus taken in a purely selfish way, while in physics the motion of particles is governed by the variation of the total energy. Recently, a global function linking individual decisions to the variation of a global quantity has been introduced to describe some classes of socio-economic models [7]. This approach then allows such models to be described with statistical physics tools. Importantly, the equilibrium state can then be calculated by maximizing a state function (akin to a free energy) instead of having to solve a complicated Nash equilibrium of strategically interacting agents.

The question we investigate in this letter is whether this physical description of socio-economic models can be extended to other basic concepts of statistical physics, such as the equalization of thermodynamic parameters like temperature or chemical potential. The equalization of these quantities throughout the system precisely results from the conservation of the conjugated extensive quantities, namely the energy or the number of particles. Although there is no notion of energy in socio-economic models, the dynamics indeed conserves the number of agents. A natural question is thus to know whether a chemical potential can be defined in such models, and what would be its relation to standard socio-economic concepts. This question is further motivated by the following remark. In spatial socio-economic models, the individual dynamics leads to a Nash equilibrium, where no agent has an incentive to move. If all agents are of the same type, the Nash equilibrium results in a spatially uniform utility, even if the environment is spatially inhomogeneous like in cities, where the center plays a specific role. This uniformity property is also expected from the chemical potential (if such a quantity can be defined), suggesting a possible relation between these two notions.

Here, we investigate this issue in the framework of a generic class of exactly solvable models involving a population of locally interacting agents. We define in a precise way a chemical potential for this class of models, and pro-

vide a direct link between the chemical potential and the socio-economic utility. Two explicit examples from the field of urban economics are also presented.

Model and dynamics. – This work deals with socio-economic models characterized by a large number of agents interacting on a network. Agents reside on the nodes of the network, labeled by an index $q = 1, \ldots, Q$ (Q is the total number of nodes) and are able to move from one node to another in order to increase their utility. They interact only with other agents on the same node. In addition, agents belong to m different groups, according for instance to their income, or to their cultural preferences. The variables used to describe the system are the numbers n_{qi} of agents of each group $i = 1, \ldots, m$ at each node q. The configuration of the system is described by the set $x = \{n_{qi}\}$. We assume that agents can move from one node to another, but cannot change group, so that for all i, the total number $N_i = \sum_q n_{qi}$ of agents of group i is fixed. The satisfaction of agents of type i on node q is characterized by a utility $U_{qi}(n_{q1}, ..., n_{qm})$ that depends only on the numbers of agents of each group on the same node q.

The model is defined with a continuous time dynamics following the standard logit rule [8]. Agents can move from a node q to a node q' with a probability per unit time

$$W = \frac{\nu_0}{1 + e^{-\Delta U/T}}, \tag{1}$$

where $\Delta U = U'_{q'i} - U_{qi}$ is the variation of the agent's own utility, with

$$U'_{q'i} = U_{q'i}(n_{q'1}, ..., n_{q'i} + 1, ..., n_{q'm}) \tag{2}$$
$$U_{qi} = U_{qi}(n_{q1}, ..., n_{qi}, ..., n_{qm}). \tag{3}$$

The parameter T plays the role of an effective temperature, introducing noise in the decision process to take into account other factors influencing choices [8], and ν_0 is a characteristic transition frequency.

In order to obtain analytical results, we assume that the utility function is such that the change of individual utility experienced by an agent during a move can be expressed as the variation of a function of the global configuration $x = \{n_{qi}\}$ [7]. More precisely, we assume that there exists a function $L(x)$ such that for each agent in group i, moving from node q to node q',

$$U'_{q'i} - U_{qi} = L(y) - L(x) \tag{4}$$

where $y = (n_{11}, \ldots, n_{qi} - 1, \ldots, n_{q'i} + 1, \ldots, n_{Qm})$ and $x = (n_{11}, \ldots, n_{Qm})$ are the configurations of the system after and before the move respectively. Such a function $L(x)$ thus provides a link between the individual behaviour of agents and the evolution of the whole system. In physical terms, it can be thought of as an effective energy.

The stationary probability distribution $\mathcal{P}_{\rm s}(\{n_{qi}\}) = \mathcal{P}_{\rm s}(x)$ is obtained by solving the master equation governing the dynamics of the system [9]. If Eq. (4) holds, detailed balance is satisfied [9,10], and we obtain the following expression for the distribution $\mathcal{P}_{\rm s}(x)$:

$$\mathcal{P}_{\rm s}(x) = \frac{1}{Z_{\rm s}} \frac{e^{L(x)/T}}{\prod_{q,i} n_{qi}!} \prod_i \delta\left(\sum_q n_{qi} - N_i\right) \tag{5}$$

where $Z_{\rm s}$ is a normalization constant. The product of Kronecker δ functions accounts for the conservation of the total number of agents in each group. The different factors appearing in Eq. (5) can be given a simple interpretation. The exponential factor directly comes from the detailed balance associated to the logit rule Eq. (1), while the product of factorials appearing at the denominator in Eq. (5) results from the coarse-graining of configurations. Namely, given the numbers of agents $\{n_{qi}\}$, there are for each group $N_i! / \prod_q n_{qi}!$ ways to arrange the agents of the group. As the numbers N_i are fixed, $N_i!$ can be reabsorbed into the normalization constant.

We now define a density of agents $\rho_{qi} = n_{qi}/H$ by dividing the number of agents by a characteristic number $H \gg 1$, for instance a maximal number of agents on a node. The utility U_{qi} then becomes a function $u_{qi}(\rho_{q1}, ..., \rho_{qm})$ of the densities on node q. We further assume that the function $L(x)$ can be written in the form

$$L(x) = H\tilde{L}(\{\rho_{qi}\}). \tag{6}$$

In addition, the logarithm of the product of factorials in Eq. (5) can be expanded for large H using Stirling's approximation, leading to

$$\ln \prod_{q,i} n_{qi}! \approx H \sum_{q,i} \big(\rho_{qi} \ln \rho_{qi} + (\ln H - 1)\rho_{qi}\big). \tag{7}$$

The stationary probability distribution eventually takes the form

$$\mathcal{P}(\{\rho_{qi}\}) = \frac{1}{Z} e^{H[\tilde{L}(\{\rho_{qi}\}) + TS(\{\rho_{qi}\})]/T} \times \prod_i \delta\Big(\sum_q \rho_{qi} - Q\overline{\rho}_i\Big) \tag{8}$$

where $\overline{\rho}_i = N_i/(QH)$ is the average density of agents from group i. The 'entropic' term $S(\{\rho_{qi}\})$ is defined as

$$S(\{\rho_{qi}\}) = -\sum_{q,i} \rho_{qi} \ln \rho_{qi}. \tag{9}$$

Note that linear terms in ρ_{qi} have been reabsorbed into the normalization constant Z.

To determine $\tilde{L}$, we combine Eqs. (4) and (6), and expand $\tilde{L}$ to leading order in $1/H$, yielding

$$\frac{\partial \tilde{L}}{\partial \rho_{q'i}} - \frac{\partial \tilde{L}}{\partial \rho_{qi}} = u_{q'i} - u_{qi}. \tag{10}$$

By identification, we get for all q

$$\frac{\partial \tilde{L}}{\partial \rho_{qi}} = u_{qi}(\rho_{q1}, ..., \rho_{qm}). \tag{11}$$

As the r.h.s. of Eq. (11) only depends on densities of agents on node q, $\tilde{L}$ necessarily takes the form

$$\tilde{L}(\{\rho_{qi}\}) = \sum_q l_q(\rho_{q1}, ..., \rho_{qm}), \tag{12}$$

and one has

$$\frac{\partial l_q}{\partial \rho_{qi}} = u_{qi}. \tag{13}$$

If there is a single group ($m = 1$), $l_q(\rho_q)$ is simply obtained by integrating $u_q(\rho_q)$. In contrast, if $m > 1$, l_q (and thus $\tilde{L}$) only exists if the following condition, resulting from the equality of cross-derivatives of l_q, is satisfied:

$$\frac{\partial u_{qi}}{\partial \rho_{qj}} = \frac{\partial u_{qj}}{\partial \rho_{qi}}, \qquad i \neq j. \tag{14}$$

If this condition holds, the stationary distribution reads

$$\mathcal{P}(\{\rho_{qi}\}) = \frac{1}{Z} \prod_q e^{H f_q(\rho_{q1},...,\rho_{qm})/T} \prod_i \delta\Big(\sum_q \rho_{qi} - Q\overline{\rho}_i\Big) \tag{15}$$

where f_q is given by

$$f_q(\rho_{q1}, ..., \rho_{qm}) = l_q(\rho_{q1}, ..., \rho_{qm}) + T s(\rho_{q1}, ..., \rho_{qm}), \tag{16}$$

with

$$s(\rho_{q1}, ..., \rho_{qm}) = -\sum_i \rho_{qi} \ln \rho_{qi}. \tag{17}$$

In analogy to physical systems, $f_q(\rho_{q1}, ..., \rho_{qm})$ can be interpreted as a local free energy (up to a change of sign), and the term $s(\rho_{q1}, ..., \rho_{qm})$, which is multiplied by the 'temperature' T, may be seen as an entropic contribution associated to the node q.

Utility and chemical potential. – We now turn to the main result of this letter. The configurations $\{\rho^*_{qi}\}$ which maximize $F = \sum_q f_q$ under the constraints of fixed global density $\sum_q \rho_{qi} = Q\overline{\rho}_i$ are the most probable (or equilibrium) configurations. Finding the equilibrium densities of agents is then a constrained maximization problem. Let us introduce a Lagrangian

$$\mathcal{L}(\{\rho_{qi}\}, \{\lambda_i\}) = \sum_q f_q(\rho_{q1}, ..., \rho_{qm}) - \sum_i \lambda_i \Big(\sum_q \rho_{qi} - Q\overline{\rho}_i\Big), \tag{18}$$

where the parameters λ_i are Lagrange multipliers associated to the conservation of the number of agents in each group. In physical terms, λ_i corresponds to the chemical potential[1] of the agents of group i. The equilibrium densities $\{\rho^*_{qi}\}$ are then determined from the conditions $\partial\mathcal{L}/\partial\rho_{qi} = 0$ for all (q, i), yielding

$$u_{qi}(\rho^*_{q1}, ..., \rho^*_{qm}) + T\frac{\partial s}{\partial \rho_{qi}}(\rho^*_{q1}, ..., \rho^*_{qm}) = \lambda_i, \tag{19}$$

which is the main result of this letter. Equation (19) thus provides an answer to the question raised at the beginning of this letter: there is indeed a direct relationship between the socio-economic utility and the chemical potential defined, in analogy to equilibrium physical systems, from the conservation of the number of particles. At zero temperature, both quantities can be identified. This result might come as a surprise: utility is often thought to be the socio-economic concept most similar to the physical concept of energy (or more precisely, the opposite of the energy), because agents seek to maximize their utility in social systems and physical particles minimize the energy in the zero temperature limit. Hence one might have intuitively expected the homogeneity of utility to be linked to a notion of temperature (the thermodynamic variable conjugated to energy), rather than to a chemical potential. Note also that the entropic correction to the utility does not involve the local entropy s, but its derivative $\partial s/\partial\rho_{qi}$.

Interestingly, Eq. (19) not only provides a link between two apparently unrelated concepts, but also yields a non-trivial prediction on the variations of utility across the system at non-zero temperature. As the chemical potential remains uniform at any temperature, one sees from Eq. (19) that the utility can be written as $u_{qi} = \lambda_i - T\partial s/\partial\rho_{qi}$, showing that utility becomes non-uniform if $T > 0$, and that the corrections to uniformity are given by the derivative of the local entropy.

In a statistical physics language, Eq. (15) corresponds to the canonical ensemble, where the number of interacting entities (agents or particles) is fixed. It is sometimes convenient to consider the so-called grand-canonical ensemble, where particles are exchanged with an external reservoir. In the context of agent-based models, the reservoir corresponds to the external world. This means that we implicitly consider a very large network ('the world') and focus only on a small subpart of it ('the system'), still containing a large number of agents. Since the 'world' has a fixed number of agents, it can be described by the stationary distribution Eq. (5). Following standard statistical physics methods [6], the probability distribution of the considered subpart is given by

$$\mathcal{P}_{\rm ow}(\{\rho_{qi}\}) = \frac{1}{Z_{\rm ow}} \prod_q e^{H[f_q(\rho_{q1},...,\rho_{qm}) - \sum_i \lambda_i \rho_{qi}]/T}, \tag{20}$$

where λ_i is the chemical potential of group i imposed by the external world. Finding the most probable densities ρ^*_{qi} is now straightforward since the densities on different nodes are independent. Maximizing the argument of the exponential in Eq. (20), one recovers Eq. (19).

In the following, we give two examples of models belonging to the above generic class, in the context of urban economics.

[1] An equivalent formulation is to define the chemical potential λ_i as the logarithmic derivative of the partition function Z with respect to N_i, a definition that can be extended to some classes of nonequilibrium models [11]. Note also that the standard definition of chemical potential for equilibrium systems differs by a conventional factor $-1/T$ with respect to the one we use here [6].

A simple urban economics model. – The model presented here is a simple model of land use and transport interaction in urban economics [12]. In this model, a city is described as a grid composed of Q blocks. In each block, one or several agents (representing households) can live by paying a rent to the landowner. A central business district (CBD) is placed on the grid and all agents commute there for their work (monocentric city model). A transport cost c per unit distance is associated to this commuting. The distance between a block q and the CBD is denoted by r_q. The size of the city is fixed: a constant radius r_f defines the urban fringe, out of which no agent lives[2]. All agents have the same income Y, which is spent on transport, on housing and on a composite good z representing all other consumer goods. This gives a budget constraint for each agent

$$Y = z + cr_q + \sigma p_q \tag{21}$$

where σ is the surface of housing, and p_q is the rent per unit surface in block q. We first consider a simple model where all agents have the same surface of housing. Each block of the grid is composed of H cells of surface σ_0. A configuration of the city is then given by the number of agents n_q in each block q. We make the further simplifying assumption that the price p_q of housing in a block q only depends on the density $\rho_q = n_q/H$ of agents in this block, namely $p_q = p(\rho_q)$. Let us emphasize that this hypothesis is an important simplification with respect to standard urban economics models, in which the price emerges directly from the competition for land between agents, and the density from their utility maximization with respect to the surface of housing [12]. In cases where an explicit expression is required, we will use a logarithmic form

$$p(\rho_q) = p_0 \ln(1 + \rho_q), \tag{22}$$

where p_0 is a positive constant.

The utility function has to be specified explicitly. It should be an increasing function $U(z)$ of the quantity of composite good z each agent consumes, that we choose to be simply $U(z) = z$. This means that, in the limit $T \to 0$, each agent wants to maximize the share of his income which is left after transport and housing expenses. Using Eqs. (21) and (22), the utility U becomes a function $u_q(\rho_q)$ of the local density,

$$u_q(\rho_q) = Y - cr_q - \sigma_0 p(\rho_q). \tag{23}$$

Urban economics distinguishes closed city models, where the total number N of agents is fixed, and open city models, where N fluctuates due to exchanges with the external world [12]. We start by considering the closed city model. In the continuous limit where H and $N \to \infty$ with the average density $\overline{\rho} = N/(HQ)$ fixed, the stationary probability distribution takes the form Eq. (15), with $f_q(\rho_q)$ given by

$$f_q(\rho_q) = \int_0^{\rho_q} u_q(\rho) d\rho + Ts(\rho_q) \tag{24}$$

and $s(\rho_q) = -\rho_q \ln \rho_q$.

The most probable density ρ_q^* is then obtained as a function of λ from Eq. (19), namely

$$u_q(\rho_q^*) + T\frac{ds}{d\rho_q}(\rho_q^*) = \lambda. \tag{25}$$

In the limit $T \to 0$, often considered in socio-economic models, one finds $\rho_q^* = \rho^*(r_q, \lambda)$, with

$$\rho^*(r_q, \lambda) = p^{-1}\left(\frac{Y - cr_q - \lambda}{\sigma_0}\right), \tag{26}$$

where p^{-1} is the reciprocal function of p. With the specific form Eq. (22), we obtain that the density ρ_q^* decreases exponentially with the distance to the center, as reported in the economics literature [13], which a posteriori justifies the choice made for the function $p(\rho_q)$.

The parameter λ is then determined from the density constraint $\sum_q \rho_q^* = Q\overline{\rho}$. Following standard literature [12], we focus here on the simplest situation of a one-dimensional city. Using the continuous approximation

$$\frac{1}{Q}\sum_q \rho_q^* \approx \frac{1}{r_f}\int_0^{r_f} \rho^*(r, \lambda)\, dr, \tag{27}$$

we compute the average density $\overline{\rho}(\lambda)$, and then determine numerically the reciprocal function $\lambda(\overline{\rho})$.

We now briefly turn to the open city model (similar to the above 'open world' case) where agents can also move to or from a large number of other cities. The stationary distribution is given by Eq. (20), which in the present open city model simplifies to

$$\mathcal{P}_{\rm oc}(\{\rho_{qi}\}) = \frac{1}{Z_{\rm oc}}\prod_q e^{H[f_q(\rho_q) - \lambda\rho_q]/T}. \tag{28}$$

Finding the most probable density is then an unconstrained maximization problem. The relation $df_q/d\rho_q = \lambda$ yields the same equation as (25), resulting in the same density profile (26) in the limit $T \to 0$. For $T > 0$, the density can be obtained from a numerical resolution of Eq. (25). The results are presented on Fig. 1, for different temperatures. One can see that increasing the temperature T progressively blurs the zero temperature exponential pattern given by Eq. (26), eventually leading to a homogeneous density. The same effect has been observed in urban economics models [14]. As a consequence, the city is more spread, leading to a utility gain for agents near the city center, and to a loss for agents in the periphery.

[2] In standard urban economics models, land is used for agriculture outside the city, and the landowners then earn an agricultural rent [12]. These landowners rent to the highest bidder, so that all prices must be greater than the agricultural rent. However, to simplify the presentation, we have dropped the agricultural rent parameter by introducing a fixed city size.

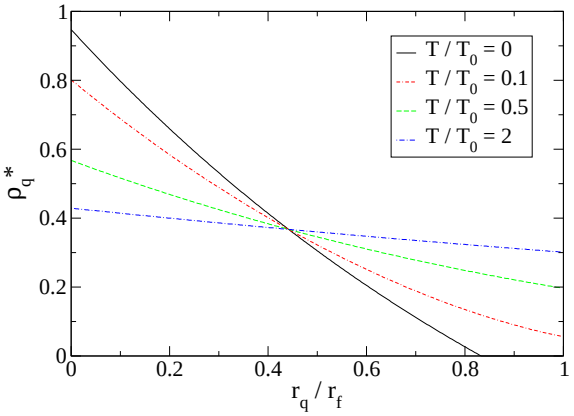

Fig. 1: Evolution with temperature T of the density of agents ρ_q^* as a function of the distance r_q to the center, in the open city model with one type of agents. Parameters: $T_0 \equiv p_0\sigma_0 = 75$, $c = 2$, $Y = 200$, $r_f = 30$, $\lambda = 150$.

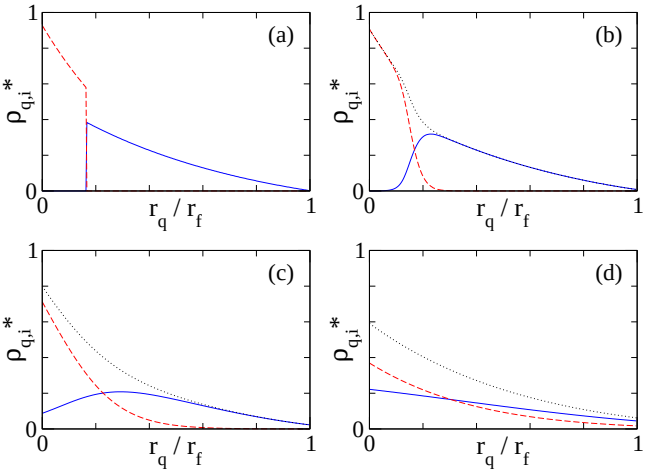

Fig. 2: Density profile ρ_{qi}^* as a function of r_q for both groups of agents (rich, full line; poor, dashed line) for different temperatures: $T/T_0' = 0.0018$ (a), 0.018 (b), 0.089 (c) and 0.36 (d), with $T_0' \equiv p_0\sigma_2$. The dotted lines indicate the total density $\rho_{q1}^* + \rho_{q2}^*$. Parameters: $p_0 = 1.4$, $c = 0.4$, $Y_1 = 452$, $Y_2 = 301$, $\sigma_1 = 6$, $\sigma_2 = 4$, $r_f = 30$, $\overline{\rho}_1 = \overline{\rho}_2 = 0.13$.

Urban model with two types of agents. – In this second model, two income groups are distinguished. Rich agents (group 1) have an income Y_1 and a surface of housing σ_1, while poor agents (group 2) have an income $Y_2 < Y_1$ and a surface of housing $\sigma_2 < \sigma_1$. Each block contains at most H agents, irrespective of their group. A configuration of the city is described by the densities $\rho_{q1} = n_{q1}/H$ and $\rho_{q2} = n_{q2}/H$ in each block q. The price an agent pays for housing depends on his surface of housing and on the local density of poor and rich agents:

$$\begin{aligned} P_{q1}(\rho_{q1},\rho_{q2}) &= \sigma_1\,\tilde{p}(a_1\rho_{q1} + b_1\rho_{q2}) \\ P_{q2}(\rho_{q1},\rho_{q2}) &= \sigma_2\,\tilde{p}(a_2\rho_{q1} + b_2\rho_{q2}) \end{aligned} \tag{29}$$

where a_1, b_1, a_2 and b_2 are given constants, and $\tilde{p}$ a function to be determined. The utility function of an agent of group $i = 1, 2$ in block q has the form

$$u_{qi}(\rho_{q1},\rho_{q2}) = Y_i - cr_q - P_{qi}(\rho_{q1},\rho_{q2}). \tag{30}$$

The model is analytically solvable if Eq. (14) is satisfied. For this condition to hold, one can choose $a_1 = a_2$ and $b_1 = b_2$. Then if $\sigma_1 b_1 = \sigma_2 a_2$, the function $\tilde{p}$ can take any form, for instance the logarithmic form Eq. (22) used in the previous model, in which case we get (choosing $b_1 = \sigma_2$ and $a_2 = \sigma_1$)

$$P_{qi}(\rho_{q1},\rho_{q2}) = \sigma_i\,p_0\ln(1 + \sigma_1\rho_{q1} + \sigma_2\rho_{q2}). \tag{31}$$

The stationary distribution is given by Eq. (15), with

$$f_q(\rho_{q1},\rho_{q2}) = l_q(\rho_{q1},\rho_{q2}) + Ts(\rho_{q1},\rho_{q2}). \tag{32}$$

The expression of $s(\rho_{q1},\rho_{q2})$ is given by Eq. (17), with $m = 2$. Expressing $l_q(\rho_{q1},\rho_{q2})$ explicitly, we get

$$l_q(\rho_{q1},\rho_{q2}) = \int_0^{\rho_{q1}} u_{q1}(\rho,0)\,d\rho + \int_0^{\rho_{q2}} u_{q2}(\rho_{q1},\rho)\,d\rho. \tag{33}$$

The validity of Eq. (13), as well as the symmetry of Eq. (33) with respect to ρ_{q1} and ρ_{q2}, can be checked using Eq. (14). The equilibrium densities $(\rho_{q1}^*, \rho_{q2}^*)$ are determined from Eq. (19), yielding a system of two non-linear equations, to be solved numerically. The results of this numerical resolution are shown on Fig. 2. One recovers at low temperature the standard separation, typical of north-american cities, between poor agents in the city center, and rich agents in the periphery [12]. The effect of a temperature increase is mainly to blur the zero temperature pattern, hence avoiding total segregation.

Therefore, Eq. (19) provides a direct prediction for the utility profile at arbitrary temperature T. It would be interesting to know whether this result remains valid beyond its a priori domain of validity, namely for models satisfying Eq. (14) so that a function $\tilde{L}$ can be defined. Considering again the above urban model with two types of agents, we keep the logarithmic form Eq. (22) for $\tilde{p}$, and choose as an example $a_1 = a_2 = 1$ and $b_1 = b_2 = 0$. These values imply $\sigma_1 b_1 \neq \sigma_2 a_2$ so that Eq. (14) is not satisfied, ruling out the possibility to find a potential function $\tilde{L}$ and to get a simple analytical solution of the model.

Performing numerical simulations of this agent-based model with two income groups, in the case of a one-dimensional closed city, we first validate it thanks to a comparison with the above solvable case. Turning to the non-solvable case, we test the validity of Eq. (19), that is, whether the chemical potentials $\lambda_i = u_{qi} + T\partial s/\partial\rho_{qi}$ $(i = 1, 2)$ are uniform over the city for $T > 0$ (when $T \to 0$, the utility should be uniform anyhow). We indeed observe that for a non-zero temperature, the chemical potentials are homogeneous even in this non-solvable model, while the utility is not (see Fig. 3).

The validity of Eq. (19) in this case can be understood

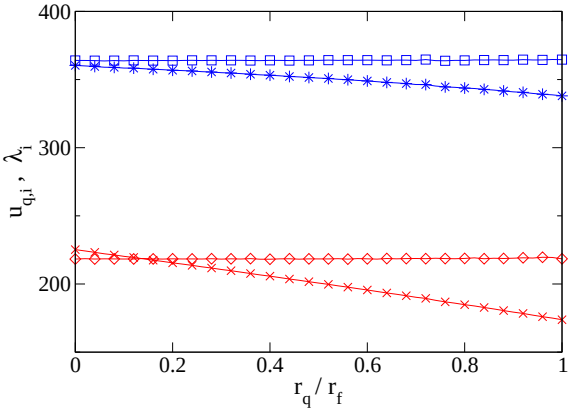

Fig. 3: Numerical simulations of the model with two types of agents, when no potential function $\tilde{L}$ exists (one-dimensional closed city). The utility u_{qi} is plotted as a function of r_q for each group ($i = 1$, $*$; $i = 2$, $\times$). Adding to the utility the term $T\partial s/\partial\rho_{qi}$ yields the chemical potential λ_i ($i = 1$, $\square$; $i = 2$, $\Diamond$), found to be constant throughout the system although the analytical solution is not known. Parameters: $T = 20$, $p_0 = 70$, $c = 2$, $Y_1 = 482$, $Y_2 = 301$, $\sigma_1 = 6.4$, $\sigma_2 = 4$, $r_f = 50$, $N_1 = N_2 = 2000$, $H = 200$, $Q = 51$.

as follows. In this paper, we focused on cases when the probability distribution has the factorized form Eq. (15), which is a consequence of the existence of a potential function $\tilde{L}$. When no function $\tilde{L}$ exists, the stationary distribution is no longer factorized, and we do not know its functional form. However, if the stationary distribution has only short range correlations, a chemical potential can still be introduced, in the same way as a chemical potential can be defined in a physical system with short-range interactions [6].

Discussion. – In this letter, we have provided a clear relationship between the apparently unrelated notions of socio-economic utility and chemical potential. More specifically, we have shown that the uniformity of utility across the social system can be traced back to the conservation of the number of agents. This result not only provides a conceptually interesting link, but also yields non-trivial and testable predictions on the variations of utility among choices (e.g., nodes, blocks) when $T > 0$. We also found numerical evidence that our result extends beyond the class of models in which it was initially derived. It would thus be interesting to explore further its validity through numerical simulations of more realistic models.

The idea of a non-uniform utility at equilibrium (Fig. 3) may be counter-intuitive for economists. Indeed, Nash equilibrium for homogeneous agents implies that all have the same utility, which seems not to be the case here when $T > 0$, since agents in the border of the city have a lower utility than those at the center. However, when noise is introduced in the decision process, a static equilibrium picture is no longer valid. Noise allows agents to explore the city, so that the *time average* value of utility is the same for all agents, leading to a macrostate described by Eq. (19) through the ergodic hypothesis linking time and ensemble averages. The average utility of agents is then a decreasing function of T. Note that this picture of a "time-averaged agent" is close in spirit to the notion of "representative agent" advocated in discrete choice theory [8]. It would be interesting to investigate further the relation between these two approaches.

Another interpretation of our result is to consider the chemical potential λ_i as an effective utility. We first note that the distribution $\mathcal{P}(\{\rho_{qi}\})$ at $T > 0$ can be obtained from the zero-temperature distribution by replacing l_q by $f_q = l_q + Ts$ [see Eqs. (15) and (16)], in the same way as the macroscopic energy is replaced by the free energy in a physical system at finite temperature. Then, changing l_q into f_q in Eq. (13), we get an effective utility $u_{qi}^{\text{eff}} = \partial f_q/\partial\rho_{qi}$. Hence the Nash equilibrium of an assembly of fictitious agents having this utility would precisely correspond to Eq. (19), namely $u_{qi}^{\text{eff}} = \lambda_i$.

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