Time–lagged covariance estimator for i.i.d. Gaussian assets

Andrzej Jarosz*

The Henryk Niewodniczański Institute of Nuclear Physics, Polish Academy of Sciences, Radzikowskiego 152, 31–342 Kraków, Poland

I apply the method of planar diagrammatic expansion to solve the problem of finding the mean spectral density of the non–Hermitian time–lagged covariance estimator for a system of i.i.d. Gaussian random variables. I confirm the result in a much simpler way using a recent conjecture about non–Hermitian random matrix models with rotationally–symmetric spectra. I conjecture and test numerically a form of finite–size corrections to the mean spectral density featuring the complementary error function.

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I. INTRODUCTION

An important problem in various fields of science investigating systems of time–dependent random variables is to unravel correlations between these variables at distinct time moments. Speaking quantitatively, suppose one looks at a system of N random variables (labeled by $i = 1, 2, \ldots, N$), and for each one of them one possesses a historical time series of length T of its values at consecutive time moments $a = 1, 2, \ldots, T$; let these values be denoted by X_{ia}, which constitutes an $N \times T$ matrix $\mathbf{X}$. A typical example would be a system of financial assets [1], though manifold other applications are also possible, such as brain science [2] or meteorology [3].

A convenient estimator, based on the historical data contained in $\mathbf{X}$, of the covariance which exists between asset i and asset j over time span t is defined as

$$c_{ij}^{\text{lag } t} \equiv \frac{1}{T-t} \sum_{a=1}^{T-t} X_{ia} X_{j,a+t}. \qquad (1)$$

This estimates the "true" covariance, stemming from the j.p.d.f. $P(\mathbf{X})$ of the variables,

$$C_{ij}^{\text{lag } t} \equiv \langle X_{ia} X_{j,a+t} \rangle , \qquad (2)$$

where translational invariance in time is assumed, hence (2) does not depend on a, only on the lag t. The estimator (1) reproduces the true value (2), however, it is additionally marred by measurement noise due to finiteness of the time series; only for $T \to \infty$ would the measured and true quantities coincide. However, a more practically relevant regime is when both N and T are large and of comparable magnitude, such as several hundred financial assets sampled daily over a few years,

$$N \to \infty, \quad T \to \infty, \quad r \equiv \frac{N}{T} = \text{finite}. \qquad (3)$$

This "thermodynamic limit" will be assumed throughout the paper, with r quantifying the noise–to–signal ratio.

A crucial question is to devise means to unveil the true covariance from behind the statistical blur present in the estimator. This challenge can be helped by solving the inverse problem: Assuming a certain form of the j.p.d.f. (possibly described by some parameters), $P(\mathbf{X}; \{\text{parameters}\})$, and deriving statistical properties of the estimator (1). Eventually, by fitting these properties to experimental data, one may assess the parameters of the considered form of the underlying j.p.d.f.

II. PROBLEM

A. Probability distribution

In this letter, I show how to solve the inverse problem in the simplest case of the assets being i.i.d. Gaussian random numbers. The same technique can then be applied to more complicated situations, which will be the subject of subsequent articles. This present case serves thus as an introduction to the method of solution, as well as may be considered as the "zeroth–order hypothesis": even if a system exhibits correlations between its constituents, it is profitable to compare it to a system of independent assets, any deviation providing information about the correlations.

I also make a simplifying assumption that the assets are complex (all the real and imaginary parts being i.i.d. Gaussian of mean zero and variance denoted by $\sigma^2/2$); I will explain (see the end of appendix A) that at the leading order in the thermodynamic limit (3), the results are the same as for real assets. Hence I take

$$P(\mathbf{X}) \propto \prod_{i=1}^{N} \prod_{a=1}^{T} \mathrm{e}^{-\frac{1}{\sigma^2}|X_{ia}|^2} = \mathrm{e}^{-\frac{1}{\sigma^2}\mathrm{Tr}\mathbf{X}^{\dagger}\mathbf{X}}, \qquad (4)$$

which is the well–known Girko–Ginibre random matrix model [4].

*Electronic address: jedrekjarosz@gmail.com

B. Estimator

I will consider the estimator (1) for time lag $t = 1$, since I will explain (see the end of appendix B) that any $t \ll T$ is equivalent to $t = 1$. Since T is large (3), I will change the normalization $1/(T-1)$ into just $1/T$, and moreover, extend the sum in (1) to run from 1 to T, where the index $(T+1)$ is understood cyclically as 1. We should also take into account that the assets are complex numbers. In total, the estimator I consider reads, in matrix notation,

$$\mathbf{c} \equiv \frac{1}{T}\mathbf{X}\mathbf{D}\mathbf{X}^{\dagger}, \quad \text{where} \quad D_{ab} \equiv \delta_{a+1,b}, \tag{5}$$

and the index $(T+1)$ in the Kronecker delta means 1.

C. Mean spectral density

I will aim at computing a basic statistical property of this $N \times N$ random matrix $\mathbf{c}$, the "mean spectral density,"

$$\rho_{\mathbf{c}}(\lambda, \overline{\lambda}) \equiv \frac{1}{N}\sum_{i=1}^{N}\left\langle \delta^{(2)}\left(\lambda - \lambda_i, \overline{\lambda} - \overline{\lambda_i}\right)\right\rangle. \tag{6}$$

The complex Dirac delta is used because $\mathbf{c}$ is a non–Hermitian matrix, with complex eigenvalues λ_i, coalescing in the thermodynamic limit (3) into some two–dimensional domain $\mathcal{D}$.

One may conveniently encode the mean spectral density in an object called the "non–holomorphic Green function" $G_{\mathbf{c}}(z, \overline{z})$ (see (A2)), which relates to it as

$$\rho_{\mathbf{c}}(z, \overline{z}) = \frac{1}{\pi}\frac{\partial}{\partial \overline{z}}G_{\mathbf{c}}(z, \overline{z}), \quad \text{for} \quad z \in \mathcal{D}. \tag{7}$$

Equivalently, one defines the "non–holomorphic M–transform,"

$$M_{\mathbf{c}}(z, \overline{z}) \equiv z G_{\mathbf{c}}(z, \overline{z}) - 1. \tag{8}$$

These are extensions of the well–known concepts from Hermitian random matrix theory of the Green function $G_{\mathbf{H}}(z) \equiv \frac{1}{N}\mathrm{Tr}\langle (z\mathbf{1}_N - \mathbf{H})^{-1}\rangle$ and M–transform $M_{\mathbf{H}}(z) \equiv z G_{\mathbf{H}}(z) - 1$.

III. RESULTS

A. Mean spectral density of c

The main result of this paper is that the eigenvalues of the time–lagged covariance estimator $\mathbf{c}$ (5) for the i.i.d. Gaussian assets (4) are scattered on average inside the centered circle of radius $R_{\mathrm{b}} = \sigma^2\sqrt{r(1+r)}$, and their density stems via (7), (8) from the M–transform $M \equiv M_{\mathbf{c}}(z, \overline{z})$ which obeys a third–order polynomial (Cardano) equation,

$$\begin{aligned} &4r^3 M^3 + 4r^2(1+2r)M^2 + \\ &+ r\left((1+r)(1+5r) - \frac{|z|^2}{\sigma^4}\right)M + \\ &+ (1+r)\left(r(1+r) - \frac{|z|^2}{\sigma^4}\right) = 0. \end{aligned} \tag{9}$$

Notice in particular that (9) depends on z only through $|z|$, and hence the density is rotationally–symmetric around zero. Its test against Monte–Carlo simulations is presented in figure 1, with excellent agreement everywhere except the borderline $|z| = R_{\mathrm{b}}$.

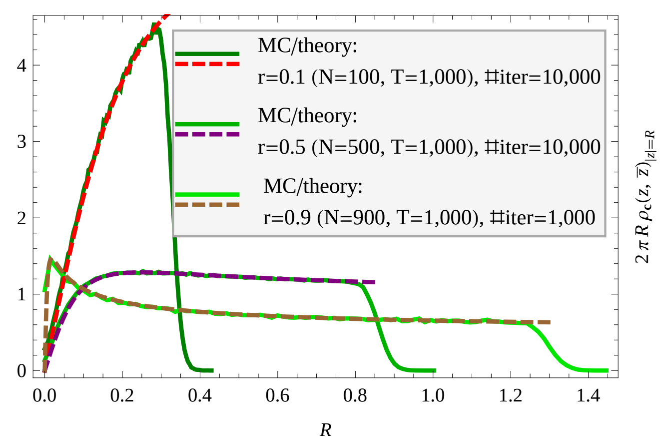

FIG. 1: The radial part of the mean spectral density, $2\pi R \rho_{\mathbf{c}}(z, \overline{z})|_{|z|=R}$, for the estimator (1) with $d = 1$, $\sigma = 1$, obtained from (9) (dashed lines) — verified against numerical Monte–Carlo simulations (solid lines). The matrix dimensions and the number of Monte–Carlo iterations in each case are indicated in the inset.

A solution to this problem has been first presented in [1], however in a form of an inverse Abel–transform plus solving a fourth–order polynomial equation; besides being much more complicated than (9), that prescription seems to be incorrect [5].

B. Finite–size effects

Furthermore, taking after [6] (where the following has been rigorously proven for the Girko–Ginibre model), I conjecture that for finite N, T, the only modification to (the radial part of) the mean spectral density is a multiplicative factor of

$$f_{N,q,R_{\mathrm{b}}}(R) \equiv \frac{1}{2}\mathrm{erfc}\left(q\left(R - R_{\mathrm{b}}\right)\sqrt{N}\right), \tag{10}$$

where q is a parameter to be adjusted by fitting to the Monte–Carlo data, while $\mathrm{erfc}(x) \equiv \frac{2}{\sqrt{\pi}}\int_x^{\infty} \mathrm{d}t \exp(-t^2)$ is the complementary error function. This proposal works very well, see figure 2.

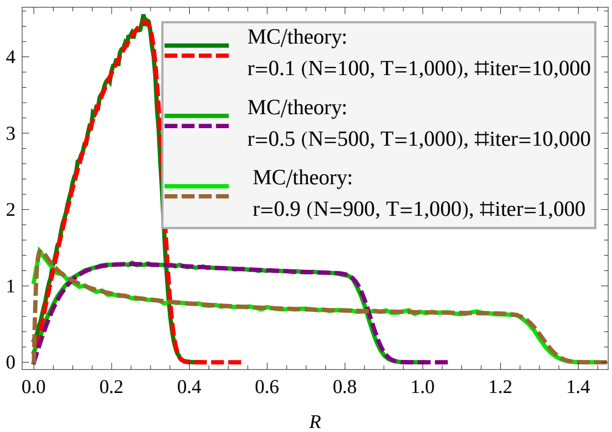

FIG. 2: The radial part of the mean spectral density stemming from (9), multiplied by the finite–size factor (10), $2\pi R\rho_{\mathbf{c}}(z,\overline{z})|_{|z|=R} f_{N,q,R_{\mathrm{b}}}(R)$, for the same estimator as in figure 1 (dashed lines) — verified against numerical Monte–Carlo simulations (solid lines). The least–square fits yield the values of the parameter $q \approx 3.65,\ 1.10,\ 0.66$, from top to bottom.

C. Conjecture about rotationally–symmetric spectra

I will present two derivations of (9). The first one (appendix A) exploits Feynman diagrams and Dyson–Schwinger (DS) equations, and is quite cumbersome. The second one (appendix B) is based on our recent conjecture [7]: If any non–Hermitian random matrix $\mathbf{c}$ has the mean spectrum rotationally–symmetric around zero, which may be restated as $M_{\mathbf{c}}(z,\overline{z}) = \mathfrak{M}_{\mathbf{c}}(|z|^2)$ — then one can define the functional inverse $\mathfrak{M}_{\mathbf{c}}(\mathfrak{N}_{\mathbf{c}}(z)) = z$ (the "rotationally–symmetric non–holomorphic N–transform"), and its relationship to the N–transform of the Hermitian matrix $\mathbf{c}^{\dagger}\mathbf{c}$, defined as $M_{\mathbf{c}^{\dagger}\mathbf{c}}(N_{\mathbf{c}^{\dagger}\mathbf{c}}(z)) = z$, is given by

$$N_{\mathbf{c}^{\dagger}\mathbf{c}}(z) = \frac{z+1}{z}\mathfrak{N}_{\mathbf{c}}(z). \tag{11}$$

The matrix model $\mathbf{c}^{\dagger}\mathbf{c}$ is Hermitian, and therefore may be simpler to solve than the non–Hermitian $\mathbf{c}$, whereupon the conjecture (11) leads to the mean spectral density of $\mathbf{c}$. I will show a quick derivation, using the multiplication law of free probability calculus, of the M–transform of $\mathbf{c}^{\dagger}\mathbf{c}$ for the estimator (5), from which (11) will reproduce the main result (9).

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Appendix A: Diagrammatic derivation

I will now sketch a derivation of the main equation (9) using Feynman–diagrammatic expansion and DS equations; this method is described in detail in appendix A of the second position in [7]. First, I consider instead of $\mathbf{c}$ the random matrix

$$\tilde{\mathbf{c}} \equiv \begin{pmatrix} \mathbf{0}_N & \frac{1}{T}\mathbf{X}\mathbf{D} \\ \mathbf{X}^{\dagger} & \mathbf{0}_T \end{pmatrix}, \tag{A1}$$

which upon squaring yields a matrix with the same eigenvalues as $\mathbf{c}$ (counted twice), plus a number of zero modes, but is linear in $\mathbf{X}$. The non–holomorphic M–transforms relate thus as $M_{\tilde{\mathbf{c}}}(z,\overline{z}) = \frac{2r}{1+r} M_{\mathbf{c}}(z^2,\overline{z}^2)$. The four blocks of this matrix and alike matrices will be denoted by NN, NT, TN, TT, according to their dimensions.

The goal is to find the "Green function matrix,"

$$\mathbf{G}^{\mathrm{D}}_{\tilde{\mathbf{c}}}(z,\overline{z}) \equiv \lim_{\epsilon\to 0}\lim_{N\to\infty}\left\langle \left(\mathbf{Z}^{\mathrm{D}}_{\epsilon} - \tilde{\mathbf{c}}^{\mathrm{D}}\right)^{-1}\right\rangle, \tag{A2}$$

where the "Duplicated" matrices,

$$\mathbf{Z}^{\mathrm{D}}_{\epsilon} \equiv \begin{pmatrix} z\mathbf{1}_{N+T} & \mathrm{i}\epsilon\mathbf{1}_{N+T} \\ \mathrm{i}\epsilon\mathbf{1}_{N+T} & \overline{z}\mathbf{1}_{N+T} \end{pmatrix}, \quad \tilde{\mathbf{c}}^{\mathrm{D}} \equiv \begin{pmatrix} \tilde{\mathbf{c}} & \mathbf{0}_{N+T} \\ \mathbf{0}_{N+T} & \tilde{\mathbf{c}}^{\dagger} \end{pmatrix}. \tag{A3}$$

The four blocks here will be distinguished by placing a bar over the indexes, $\bullet\bullet$, $\bullet\overline{\bullet}$, $\overline{\bullet}\bullet$, $\overline{\bullet}\overline{\bullet}$, from left to right. The normalized trace of the upper left block of (A2) is the desired non–holomorphic Green function.

The matrix $\tilde{\mathbf{c}}$ has random entries being Gaussian, so the full information about them is encoded in the propagators. The Girko–Ginibre (4) propagators, $\langle X_{ia}\overline{X_{jb}}\rangle = \sigma^2\delta_{ij}\delta_{ab}$, translate into

$$\langle[\tilde{\mathbf{c}}^{\mathrm{D}}]_{ia}[\tilde{\mathbf{c}}^{\mathrm{D}}]_{bj}\rangle = \frac{\sigma^2}{T}\delta_{ij}D_{ba}, \tag{A4a}$$

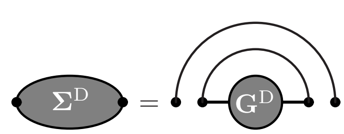
$$\langle[\tilde{\mathbf{c}}^{\mathrm{D}}]_{ia}[\tilde{\mathbf{c}}^{\mathrm{D}}]_{\overline{bj}}\rangle = \frac{\sigma^2}{T^2}\delta_{i\overline{j}}[\mathbf{D}^{\dagger}\mathbf{D}]_{\overline{b}a}, \tag{A4b}$$

$$\langle[\tilde{\mathbf{c}}^{\mathrm{D}}]_{\overline{ia}}[\tilde{\mathbf{c}}^{\mathrm{D}}]_{bj}\rangle = \sigma^2\delta_{\overline{i}j}\delta_{\overline{a}b}, \tag{A4c}$$

$$\langle[\tilde{\mathbf{c}}^{\mathrm{D}}]_{\overline{ia}}[\tilde{\mathbf{c}}^{\mathrm{D}}]_{\overline{bj}}\rangle = \frac{\sigma^2}{T}\delta_{\overline{ij}}[\mathbf{D}^{\dagger}]_{\overline{ba}}. \tag{A4d}$$

The next step is to expand (A2) in a power series around $z = \infty$, and represent its terms as Feynman fat diagrams. In the thermodynamic limit (3), only planar graphs survive. In the Gaussian case, they are rainbow graphs. One introduces the self–energy matrix $\mathbf{G}^{\mathrm{D}} = \left(\mathbf{Z}^{\mathrm{D}} - \mathbf{\Sigma}^{\mathrm{D}}\right)^{-1}$, where $\mathbf{Z}^{\mathrm{D}} \equiv \mathbf{Z}^{\mathrm{D}}_{\epsilon=0}$, which is known as the first DS equation.

The second DS equation pictorially reads



which, together with the propagators (A4a)–(A4d), leads to the following non–zero terms of the self–energy matrix,

$$\Sigma_{ij} = \frac{\sigma^2}{T}\mathrm{Tr}\left(\mathbf{D}\mathbf{G}^{TT}\right)\delta_{ij}, \tag{A5a}$$

$$\Sigma_{ba} = \frac{\sigma^2}{T}\mathrm{Tr}\left(\mathbf{G}^{NN}\right)D_{ba}, \tag{A5b}$$

$$\Sigma_{i\overline{j}} = \frac{\sigma^2}{T^2}\mathrm{Tr}\left(\mathbf{D}^\dagger\mathbf{D}\mathbf{G}^{T\overline{T}}\right)\delta_{i\overline{j}}, \tag{A5c}$$

$$\Sigma_{\overline{b}a} = \frac{\sigma^2}{T^2}\mathrm{Tr}\left(\mathbf{G}^{\overline{N}N}\right)[\mathbf{D}^\dagger\mathbf{D}]_{\overline{b}a}, \tag{A5d}$$

$$\Sigma_{\overline{i}j} = \sigma^2\mathrm{Tr}\left(\mathbf{G}^{\overline{T}T}\right)\delta_{\overline{i}j}, \tag{A5e}$$

$$\Sigma_{b\overline{a}} = \sigma^2\mathrm{Tr}\left(\mathbf{G}^{N\overline{N}}\right)\delta_{b\overline{a}}, \tag{A5f}$$

$$\Sigma_{\overline{i}\overline{j}} = \frac{\sigma^2}{T}\mathrm{Tr}\left(\mathbf{D}^\dagger\mathbf{G}^{\overline{T}\overline{T}}\right)\delta_{\overline{i}\overline{j}}, \tag{A5g}$$

$$\Sigma_{\overline{b}\overline{a}} = \frac{\sigma^2}{T}\mathrm{Tr}\left(\mathbf{G}^{\overline{N}\overline{N}}\right)[\mathbf{D}^\dagger]_{\overline{b}\overline{a}}. \tag{A5h}$$

Plugging this into the first DS equation yields

$$\mathbf{G}^{\mathrm{D}}_{\tilde{\mathbf{c}}} = \begin{pmatrix} z\mathbf{1}_N - \frac{\sigma^2}{T}\mathrm{Tr}\left(\mathbf{D}\mathbf{G}^{TT}\right)\mathbf{1}_N & \mathbf{0} & -\frac{\sigma^2}{T^2}\mathrm{Tr}\left(\mathbf{D}^\dagger\mathbf{D}\mathbf{G}^{T\overline{T}}\right)\mathbf{1}_N & \mathbf{0} \\ \mathbf{0} & z\mathbf{1}_T - \frac{\sigma^2}{T}\mathrm{Tr}\left(\mathbf{G}^{NN}\right)\mathbf{D} & \mathbf{0} & -\sigma^2\mathrm{Tr}\left(\mathbf{G}^{N\overline{N}}\right)\mathbf{1}_T \\ -\sigma^2\mathrm{Tr}\left(\mathbf{G}^{\overline{T}T}\right)\mathbf{1}_N & \mathbf{0} & \overline{z}\mathbf{1}_N - \frac{\sigma^2}{T}\mathrm{Tr}\left(\mathbf{D}^\dagger\mathbf{G}^{\overline{T}\overline{T}}\right)\mathbf{1}_N & \mathbf{0} \\ \mathbf{0} & -\frac{\sigma^2}{T^2}\mathrm{Tr}\left(\mathbf{G}^{\overline{N}N}\right)\mathbf{D}^\dagger\mathbf{D} & \mathbf{0} & \overline{z}\mathbf{1}_T - \frac{\sigma^2}{T}\mathrm{Tr}\left(\mathbf{G}^{\overline{N}\overline{N}}\right)\mathbf{D}^\dagger \end{pmatrix}^{-1}. \tag{A6}$$

The matrix being inverted on the r.h.s. consists of four block, each being a diagonal four–block matrix. It may be proven that a reshuffling of rows and columns is possible so that (A6) is equivalent to two matrix equations for 8 unknown matrices, $\mathbf{G}^{NN}$, ..., $\mathbf{G}^{\overline{T}\overline{T}}$ (the remaining 8 components of $\mathbf{G}^{\mathrm{D}}_{\tilde{\mathbf{c}}}$ are zero),

$$\begin{pmatrix} \mathbf{G}^{NN} & \mathbf{G}^{N\overline{N}} \\ \mathbf{G}^{\overline{N}N} & \mathbf{G}^{\overline{N}\overline{N}} \end{pmatrix} = \begin{pmatrix} z\mathbf{1}_N - \frac{\sigma^2}{T}\mathrm{Tr}\left(\mathbf{D}\mathbf{G}^{TT}\right)\mathbf{1}_N & -\frac{\sigma^2}{T^2}\mathrm{Tr}\left(\mathbf{D}^\dagger\mathbf{D}\mathbf{G}^{T\overline{T}}\right)\mathbf{1}_N \\ -\sigma^2\mathrm{Tr}\left(\mathbf{G}^{\overline{T}T}\right)\mathbf{1}_N & \overline{z}\mathbf{1}_N - \frac{\sigma^2}{T}\mathrm{Tr}\left(\mathbf{D}^\dagger\mathbf{G}^{\overline{T}\overline{T}}\right)\mathbf{1}_N \end{pmatrix}^{-1}, \tag{A7a}$$

$$\begin{pmatrix} \mathbf{G}^{TT} & \mathbf{G}^{T\overline{T}} \\ \mathbf{G}^{\overline{T}T} & \mathbf{G}^{\overline{T}\overline{T}} \end{pmatrix} = \begin{pmatrix} z\mathbf{1}_T - \frac{\sigma^2}{T}\mathrm{Tr}\left(\mathbf{G}^{NN}\right)\mathbf{D} & -\sigma^2\mathrm{Tr}\left(\mathbf{G}^{N\overline{N}}\right)\mathbf{1}_T \\ -\frac{\sigma^2}{T^2}\mathrm{Tr}\left(\mathbf{G}^{\overline{N}N}\right)\mathbf{D}^\dagger\mathbf{D} & \overline{z}\mathbf{1}_T - \frac{\sigma^2}{T}\mathrm{Tr}\left(\mathbf{G}^{\overline{N}\overline{N}}\right)\mathbf{D}^\dagger \end{pmatrix}^{-1}. \tag{A7b}$$

I will now solve this set of equations (A7a), (A7b), with the aim of finding $M_{\tilde{\mathbf{c}}}(z,\overline{z}) = z\frac{1}{N+T}\left(\mathrm{Tr}\left(\mathbf{G}^{NN}\right) + \mathrm{Tr}\left(\mathbf{G}^{TT}\right)\right) - 1$. Denote: $a_1 \equiv \mathrm{Tr}(\mathbf{D}\mathbf{G}^{TT})$, $a_2 \equiv \mathrm{Tr}(\mathbf{D}^\dagger\mathbf{D}\mathbf{G}^{T\overline{T}})$, $a_3 \equiv \mathrm{Tr}(\mathbf{G}^{\overline{T}T})$, $a_4 \equiv \mathrm{Tr}(\mathbf{D}^\dagger\mathbf{G}^{\overline{T}\overline{T}})$.

On the r.h.s. of (A7a), each block is proportional to the unit matrix, hence it is inverted as a 2×2 matrix,

$$\mathbf{G}^{NN} = \frac{1}{W_T}\left(\overline{z} - \frac{\sigma^2}{T}a_4\right)\mathbf{1}_N, \tag{A8a}$$

$$\mathbf{G}^{N\overline{N}} = \frac{1}{W_T}\frac{\sigma^2}{T^2}a_2\mathbf{1}_N, \tag{A8b}$$

$$\mathbf{G}^{\overline{N}N} = \frac{1}{W_T}\sigma^2 a_3\mathbf{1}_N, \tag{A8c}$$

$$\mathbf{G}^{\overline{N}\overline{N}} = \frac{1}{W_T}\left(z - \frac{\sigma^2}{T}a_1\right)\mathbf{1}_N, \tag{A8d}$$

where $W_T \equiv (z - \frac{\sigma^2}{T}a_1)(\overline{z} - \frac{\sigma^2}{T}a_4) - \frac{\sigma^4}{T^2}a_2a_3$.

Substituting this to (A7b) and noting that $\mathbf{D}$ is unitary allows to easily invert the matrix on the r.h.s.,

$$\mathbf{G}^{TT} = \left(\overline{z} - N\frac{\sigma^2}{T}\frac{1}{W_T}\left(z - \frac{\sigma^2}{T}a_1\right)\mathbf{D}^\dagger\right)\mathbf{W}_N^{-1}, \tag{A9a}$$

$$\mathbf{G}^{T\overline{T}} = N\frac{\sigma^4}{T^2}\frac{1}{W_T}a_2\mathbf{W}_N^{-1}, \tag{A9b}$$

$$\mathbf{G}^{\overline{T}T} = N\frac{\sigma^4}{T^2}\frac{1}{W_T}a_3\mathbf{W}_N^{-1}, \tag{A9c}$$

$$\mathbf{G}^{\overline{T}\overline{T}} = \left(z - N\frac{\sigma^2}{T}\frac{1}{W_T}\left(\overline{z} - \frac{\sigma^2}{T}a_4\right)\mathbf{D}\right)\mathbf{W}_N^{-1}, \tag{A9d}$$

where $[\mathbf{W}_N]_{ab} \equiv A\delta_{ab} + B\delta_{a+1,b} + C\delta_{a,b+1}$ (the indices understood modulo T), with $A \equiv |z|^2 + \sigma^4 r^2\frac{1}{W_T}$, $B \equiv -\sigma^2 r\frac{1}{W_T}\overline{z}(\overline{z} - \frac{\sigma^2}{T}a_4)$, $C \equiv -\sigma^2 r\frac{1}{W_T}z(z - \frac{\sigma^2}{T}a_1)$.

Taking the appropriate traces of (A9a)–(A9d) gives four equations for four complex unknowns $a_{1,2,3,4}$,

$$a_1 = \overline{z}Tt_2 - N\sigma^2\frac{1}{W_T}\left(z - \frac{\sigma^2}{T}a_1\right)t_1, \tag{A10a}$$

$$a_2 = N\frac{\sigma^4}{T}\frac{1}{W_T}a_2t_1, \tag{A10b}$$

$$a_3 = N\frac{\sigma^4}{T}\frac{1}{W_T}a_3t_1, \tag{A10c}$$

$$a_4 = zTt_3 - N\sigma^2\frac{1}{W_T}\left(\overline{z} - \frac{\sigma^2}{T}a_4\right)t_1, \tag{A10d}$$

where for short $t_1 \equiv \frac{1}{T}\mathrm{Tr}(\mathbf{W}_N^{-1})$, $t_2 \equiv \frac{1}{T}\mathrm{Tr}(\mathbf{D}\mathbf{W}_N^{-1})$, $t_3 \equiv \frac{1}{T}\mathrm{Tr}(\mathbf{D}^\dagger\mathbf{W}_N^{-1})$.

To solve this system, knowledge of the traces $t_{1,2,3}$ is required. They are found by diagonalizing $\mathbf{W}_N = \mathbf{U}\mathbf{\Omega}\mathbf{U}^\dagger$ (which amounts to solving a Fibonacci recurrence with cyclic boundary conditions), with $U_{ab} = \frac{1}{\sqrt{T}}\mathrm{e}^{2\pi\mathrm{i}(a-1)(b-1)/T}$ and the eigenvalues $\omega_a = A + B\mathrm{e}^{2\pi\mathrm{i}(a-1)/T} + C\mathrm{e}^{-2\pi\mathrm{i}(a-1)/T}$, and approximating (3) the sums over a by contour integrals over $w \equiv \mathrm{e}^{2\pi\mathrm{i}(a-1)/T}$, which runs counterclockwise along the centered unit circle $C(0,1)$,

$$t_1 = \frac{1}{T}\sum_{a=1}^{T}\frac{1}{\omega_a} \approx \frac{1}{2\pi\mathrm{i}}\oint\frac{\mathrm{d}w}{Bw^2+Aw+C}, \tag{A11a}$$

$$t_2 = \frac{1}{T}\sum_{a=1}^{T}\frac{\mathrm{e}^{2\pi\mathrm{i}(a-1)/T}}{\omega_a} \approx \frac{1}{2\pi\mathrm{i}}\oint\frac{w\mathrm{d}w}{Bw^2+Aw+C}, \tag{A11b}$$

$$t_3 = \frac{1}{T}\sum_{a=1}^{T}\frac{\mathrm{e}^{-2\pi\mathrm{i}(a-1)/T}}{\omega_a} \approx \frac{1}{2\pi\mathrm{i}}\oint\frac{\mathrm{d}w}{w(Bw^2+Aw+C)}. \tag{A11c}$$

These integrals are computed by the method of residues. A meaningful solution to (A10a)–(A10d) is checked to be permitted only when one zero of $Bw^2 + Aw + C$ lies within $C(0,1)$, and one outside. Then, for some $s = \pm 1$,

$$t_1 = \frac{s}{\sqrt{A^2-4BC}}, \tag{A12a}$$

$$t_2 = \frac{1}{2B}\left(1 - \frac{sA}{\sqrt{A^2-4BC}}\right), \tag{A12b}$$

$$t_3 = \frac{1}{2C}\left(1 - \frac{sA}{\sqrt{A^2-4BC}}\right). \tag{A12c}$$

One sector of solutions to (A10a)–(A10d) is defined by $a_{2,3} = 0$, as follows from (A10b), (A10c). Then one verifies that also $a_{1,4} = 0$, and consequently $M_{\mathbf{c}}(z,\overline{z}) = 0$. This "holomorphic solution" represent the outside of the eigenvalues domain $\mathcal{D}$.

Assuming $a_{2,3} \neq 0$ (the "non–holomorphic solution"), (A10a)–(A10d) reduce to $t_1 = \frac{1}{\sigma^4 r}W_T$, $t_2 = \frac{1}{\sigma^2}\frac{z}{\overline{z}}$, $t_3 = \frac{1}{\sigma^2}\frac{\overline{z}}{z}$, while $M_{\tilde{\mathbf{c}}}(z,\overline{z}) = \frac{|z|^2}{\sigma^4 r(1+r)}W_T - 1$. This straightforwardly leads to the final result (9).

The matching condition between holomorphic and non–holomorphic solutions gives the borderline of the eigenvalues domain $\mathcal{D}$: Setting $M = 0$ in (9) implies $|z| = \sigma^2\sqrt{r(1+r)}$, which means the eigenvalues of $\mathbf{c}$ fill the centered circle of this radius.

Remark: For real instead of complex assets, one has six additional non–zero propagators, $\langle[\tilde{\mathbf{c}}^{\mathrm{D}}]_{ia}[\tilde{\mathbf{c}}^{\mathrm{D}}]_{\overline{j}\overline{b}}\rangle = \frac{\sigma^2}{T_2}\delta_{i\overline{j}}D_{\overline{b}a}$, $\langle[\tilde{\mathbf{c}}^{\mathrm{D}}]_{ai}[\tilde{\mathbf{c}}^{\mathrm{D}}]_{\overline{b}\overline{j}}\rangle = \frac{\sigma^2}{T}\delta_{i\overline{j}}D_{a\overline{b}}$, $\langle[\tilde{\mathbf{c}}^{\mathrm{D}}]_{ia}[\tilde{\mathbf{c}}^{\mathrm{D}}]_{jb}\rangle = \frac{\sigma^2}{T^2}\delta_{ij}\delta_{ab}$, $\langle[\tilde{\mathbf{c}}^{\mathrm{D}}]_{\overline{i}\overline{a}}[\tilde{\mathbf{c}}^{\mathrm{D}}]_{\overline{j}\overline{b}}\rangle = \sigma^2\delta_{\overline{i}\overline{j}}\delta_{\overline{a}\overline{b}}$, $\langle[\tilde{\mathbf{c}}^{\mathrm{D}}]_{ai}[\tilde{\mathbf{c}}^{\mathrm{D}}]_{bj}\rangle = \sigma^2\delta_{ij}\delta_{ab}$ and $\langle[\tilde{\mathbf{c}}^{\mathrm{D}}]_{\overline{a}\overline{i}}[\tilde{\mathbf{c}}^{\mathrm{D}}]_{\overline{b}\overline{j}}\rangle = \frac{\sigma^2}{T^2}\delta_{\overline{i}\overline{j}}\delta_{\overline{a}\overline{b}}$. This leads to eight additional blocks in the self–energy matrix, $\mathbf{\Sigma}^{NT} = \frac{\sigma^2}{T^2}(\mathbf{G}^{TN})^{\mathrm{T}}$, $\mathbf{\Sigma}^{TN} = \sigma^2(\mathbf{G}^{NT})^{\mathrm{T}}$, $\mathbf{\Sigma}^{\overline{N}\overline{T}} = \sigma^2(\mathbf{G}^{\overline{T}\overline{N}})^{\mathrm{T}}$, $\mathbf{\Sigma}^{\overline{T}\overline{N}} = \frac{\sigma^2}{T^2}(\mathbf{G}^{\overline{N}\overline{T}})^{\mathrm{T}}$, $\mathbf{\Sigma}^{N\overline{T}} = \frac{\sigma^2}{T}(\mathbf{G}^{T\overline{N}})^{\mathrm{T}}\mathbf{D}^{\mathrm{T}}$, $\mathbf{\Sigma}^{\overline{N}T} = \frac{\sigma^2}{T}(\mathbf{G}^{\overline{T}N})^{\mathrm{T}}\mathbf{D}$, $\mathbf{\Sigma}^{T\overline{N}} = \frac{\sigma^2}{T}\mathbf{D}(\mathbf{G}^{N\overline{T}})^{\mathrm{T}}$ and $\mathbf{\Sigma}^{\overline{T}N} = \frac{\sigma^2}{T}\mathbf{D}^{\mathrm{T}}(\mathbf{G}^{\overline{N}T})^{\mathrm{T}}$. They have no traces, contrary to (A5a)–(A5h), hence these new entries are smaller by factor N. The large–N leading–order result (9) is therefore the same for complex and real assets.

Appendix B: Free probability derivation

A much simpler solution is provided by Voiculescu free probability theory [8] and the conjecture (11). It is easier to work with the matrix $\check{\mathbf{c}} \equiv \frac{1}{T}\mathbf{X}^\dagger\mathbf{X}\mathbf{D}$, which differs from $\mathbf{c}$ only by the order of terms and dimension, hence $M_{\check{\mathbf{c}}}(z,\overline{z}) = rM_{\mathbf{c}}(z,\overline{z})$.

Now consider the Hermitian matrix $\check{\mathbf{c}}^\dagger\check{\mathbf{c}} = \frac{1}{T^2}\mathbf{D}^\dagger(\mathbf{X}^\dagger\mathbf{X})^2\mathbf{D}$. By again changing the order of the constituents, and using unitarity of $\mathbf{D}$, there is $M_{\check{\mathbf{c}}^\dagger\check{\mathbf{c}}}(z) = M_{\mathbf{H}^2}(z)$, where $\mathbf{H} \equiv \frac{1}{T}\mathbf{X}^\dagger\mathbf{X}$. Using the general formula $M_{\mathbf{H}^2}(z^2) = \frac{1}{2}(M_{\mathbf{H}}(z) + M_{\mathbf{H}}(-z))$, as well as the known result [9] (best derived using the free probability's multiplication law [8]) for the N–transform $N_{\mathbf{H}}(z) = \sigma^2(1+z)(r+z)/z$, one finds

$$N_{\check{\mathbf{c}}^\dagger\check{\mathbf{c}}}(z) = N_{\mathbf{H}^2}(z) = \frac{\sigma^4(1+z)(r+z)(1+r+2z)^2}{z(1+r+z)}. \tag{B1}$$

At this point, I assume that the average spectrum of $\check{\mathbf{c}}$ is rotationally–symmetric around zero (which will be verified *a posteriori*), and apply the conjecture (11) to get from (B1) the rotationally–symmetric non–holomorphic N–transform of $\check{\mathbf{c}}$,

$$\mathfrak{N}_{\check{\mathbf{c}}}(z) = \frac{\sigma^4(r+z)(1+r+2z)^2}{1+r+z}. \tag{B2}$$

Inverting it functionally, and translating into the M–transform of $\mathbf{c}$, reproduces the main equation (9).

Remark: If one permits an arbitrary time lag t in the estimator, the r.h.s. of (B2) will have an additional factor of $(1-\tau/(1+z))/(1-\tau)^2$, where $\tau \equiv t/T$. Hence for any $t \ll T$ there is $\tau = 0$, and equation (9) stays the same.

[1] C. Biely and S. Thurner, arXiv:physics/0609053v1 [physics.soc-ph]; S. Thurner and C. Biely, Acta Phys. Pol. B **38**, 4111 (2007).

[2] J. Kwapień, S. Drożdż and A. A. Ioannides, Phys. Rev. E **62**, 5557 (2000), arXiv:cond-mat/0002175v1 [cond-mat.stat-mech].

[3] K. B. K. Mayya and R. E. Amritkar, arXiv:cond-mat/0601279v1 [cond-mat.stat-mech].

[4] J. Ginibre, J. Math. Phys. **6**, 440 (1965);
V. L. Girko, Teor. Veroyatnost. i Primenen. **29**, 669 (1984) [Theory Probab. Appl. **29**, 694 (1985)];
V. L. Girko, Uspekhi Mat. Nauk **40**, 67 (1985) [Russian Mathematical Surveys **40**, 77 (1985)].

[5] Z. Burda, R. A. Janik and B. Wacław, Phys. Rev. E **81**, 041132 (2010), arXiv:0912.3422v2 [cond-mat.stat-mech].

[6] P. J. Forrester and G. Honner, J. Phys. A **32**, 2961 (1999);
E. Kanzieper, in *Frontiers in Field Theory* (Nova Science Publishers, Inc., 2005), edited by O. Kovras, p. 23;
B. A. Khoruzhenko and H.–J. Sommers, in *The Handbook of Random Matrix Theory* (Oxford University Press), edited by G. Akemann, J. Baik and P. Di Francesco, ch. 18, arXiv:0911.5645v1 [math-ph].

[7] Z. Burda, A. Jarosz, G. Livan, M. A. Nowak and A. Święch, accepted to Phys. Rev. E, arXiv:1007.3594v1 [cond-mat.stat-mech];
Z. Burda, A. Jarosz, G. Livan, M. A. Nowak and A. Święch, to be published in Acta Phys. Pol. B.

[8] D.–V. Voiculescu, K. J. Dykema and A. Nica, *Free Random Variables* (American Mathematical Society, Providence, 1992).

[9] Z. Burda, A. Jarosz, J. Jurkiewicz, M. A. Nowak, G. Papp and I. Zahed, accepted to Quant. Fin., arXiv:physics/0603024v2 [physics.soc-ph].