Morse Potential, Contour Integrals, and Asian Options

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Abstract

Completeness of the eigenfunctions of a quantum mechanical system is crucial for its probability interpretation. By using the method of contour integral we give properly normalized eigenfunctions for both discrete and continuum spectrum of the Morse potential, and explicitly prove the completeness relation. As an application we use our spectral decomposition formula to study the problem of the pricing of an Asian option traded in financial markets.

1   Introduction

Since its first introduction [1] to quantum physics in 1929, the morse potential as a solvable model has been studied extensively by many different approaches, see e.g. [2]-[8]. Most of these works focus only on the discrete spectrum. In this paper we will discuss the completeness issue of its eigenfunctions for both discrete and continuum spectrum. Completeness is a very important property of a quantum mechanical system. It is crucial for the probability interpretation. To establish the completeness of a set of eigenfunctions, it is necessary to properly choose their normalization constants, which will give the correct integral measure in the spectral decomposition of a square-integrable wave function. The main purpose of the present paper is to do so for the Morse potential, especially for its continuum eigenstates.

There is a very powerful method [9], due to E. C. Titchmarsh, to study the above-mentioned spectral expansion problem for a given self-adjoint Sturm-Liouville problem. The advantage of this method is the fact that it uses only classical complex analysis, which is rather familiar to physicists, and does not require the knowledge of the abstract operator theory in functional analysis. By using Cauchy contour integrals, the Titchmarsh method will automatically gives a complete system of properly normalized eigenfunctions. The poles inside the contour correspond to the discrete spectrum, while the contributions from the cuts represent the continuum spectrum. It can even deal with subtleties related with self-adjoint extensions of the Hamiltonian operator.

In this paper we will apply the Titchmarsh method to establish the completeness of the eigenfunctions of the Hamiltonian with Morse potential. In section 2 we quickly review some main points of the Titchmarsh expansion theory. In section 3 we apply this method to the Morse potential. We give the proper normalization for both discrete and continuum eigenfunctions, and prove the completeness relation explicitly. In section 4 we discuss an interesting application of our result to the pricing of an Asian option in the financial market. Section 5 is for conclusions.

2   Review of the Titchmarch expansion theory

In this section we will review some main points of the Titchmarsh expansion theory [9]. For a stationary Schrödinger-type equation with a real potential function $V(x)$

$$-\frac{d^2\psi}{dx^2} + V(x)\,\psi(x) = \lambda\,\psi(x)\,, \tag{2.1}$$

we want to get the eigenvalue λ and the corresponding eigenfunction $\psi_\lambda(x)$ satisfying some proper boundary conditions. The discrete eigenvalues are interpreted as energy levels of possible bound states, while the continuum part corresponds to scattering states. For the consistency of the probability interpretation of quantum mechanics, it is crucial that the set of all eigenfunctions forms a complete basis of the Hilbert space. Without this property, probability is not conserved and unitarity is lost. If $x \in (a, b)$ with finite a, b and the potential function $V(x)$ is regular at both endpoints, this problem is called regular. However if one of or both of two points tend to infinity, or the potential becomes singular there, the problem is called singular. Since we want to apply this method to the Morse potential, we assume that $x \in (-\infty, +\infty)$.

Define functions $\phi(x, \lambda)$ and $\theta(x, \lambda)$ to be solutions of (2.1) with boundary conditions

$$\phi(0, \lambda) = 0, \quad \phi'(0, \lambda) = -1 ; \tag{2.2}$$

$$\theta(0, \lambda) = 1, \quad \theta'(0, \lambda) = 0 . \tag{2.3}$$

We introduce a finite cutoff $b > 0$ instead of $+\infty$, and consider the following mapping

$$\mu_+(z; b, \lambda) := -\frac{\theta(b, \lambda)\, z + \theta'(b, \lambda)}{\phi(b, \lambda)\, z + \phi'(b, \lambda)} . \tag{2.4}$$

This fractional linear transformation maps the real axis $\mathrm{Im}\, z = 0$ to a circle. It is proved by H. Weyl that, under the limit $b \to +\infty$, the image of real axis in the z-plane either contracts to a limit point or converges to a limit circle. The Weyl-Titchmarsh m-function $m_+(\lambda)$ to value of the limit point or any point on the limit circle. Then it follows that

$$\psi_+(x, \lambda) = \theta(x, \lambda) + m_+(\lambda)\, \phi(x, \lambda) \tag{2.5}$$

is a solution of (2.1) and normalizable at $+\infty$. Similarly, there is another m-function $m_-(\lambda)$ for the left endpoint such that

$$\psi_-(x, \lambda) = \theta(x, \lambda) + m_-(\lambda)\, \phi(x, \lambda) \tag{2.6}$$

is normalizable at $-\infty$. It can also be shown that, if in the limit point case, these are the only normalizable solutions for each endpoint. Define the resolvent kernel $G(x, x'; \lambda)$ as

$$
\begin{aligned}
G(x, x'; \lambda) &= \frac{\psi_+(x, \lambda)\, \psi_-(x, \lambda)}{m_+(\lambda) - m_-(\lambda)}, \quad (x \geq x') ; \\
&= \frac{\psi_-(x, \lambda)\, \psi_+(x, \lambda)}{m_+(\lambda) - m_-(\lambda)}, \quad (x < x') .
\end{aligned}
\tag{2.7}
$$

It is the unique solution of

$$-\frac{d^2 G}{dx^2} + (V(x) - \lambda)\, G = \delta(x - x') , \tag{2.8}$$
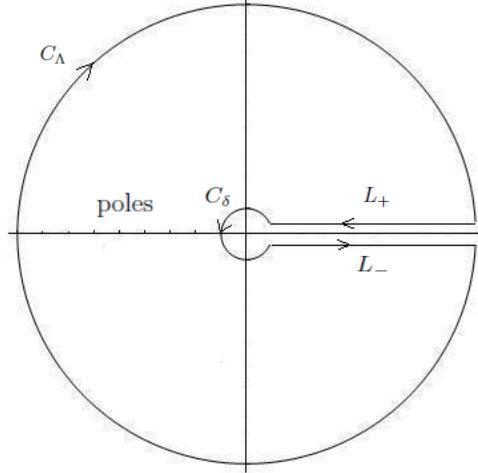
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Figure 1: The integration contour $C := C_\Lambda + L_+ + C_\delta + L_-$ which is used in (2.11) and (3.20). We assume that $\lambda = 0$ is a branch point, and on the negative real axis $\lambda < 0$ there are some poles. We use the convention about the argument: $\arg \lambda = 0$ for $\lambda \in L_-$.

which is symmetric between x and x' and square-integrable with respect to each variable. For any wave function $f(x)$ to be expanded, let

$$\Phi_f(x; \lambda) = \int_{-\infty}^{\infty} G(x, x'; \lambda) f(x') \, dx' \,, \tag{2.9}$$

which is the solution of the inhomogeneous equation

$$-\frac{d^2 \Phi}{dx^2} + (V(x) - \lambda) \Phi = f(x) \,. \tag{2.10}$$

The function $\Phi_f(x; \lambda)$ can be analytically continued to complex λ with possible poles and branch points. For the sake of simplicity and further applications in the next section, we assume the function $\Phi_f(x; \lambda)$ is meromorphic in the complex λ-plane cut open along the positive real axis $\lambda \geq 0$. By Cauchy's theorem we have

$$\frac{1}{2\pi i} \oint_C \Phi_f(x; \lambda) \, d\lambda + \sum_{\text{poles}} \text{Res} \, \Phi_f = 0 \,, \tag{2.11}$$

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where the contour C is chosen as shown in Figure 1. For reasonable potential functions it can be proved that

$$\frac{1}{2\pi i} \oint_{C_\Lambda} \Phi_f(x;\lambda)\,d\lambda \to f(x), \qquad \Lambda \to \infty. \tag{2.12}$$

Therefore we obtain finally an equation as

$$f(x) = -\sum_{\text{poles}} \text{Res}\,\Phi_f + \frac{1}{2\pi i}\int_0^\infty \left(\Phi_f(x;\lambda\,e^{-2\pi i}) - \Phi_f(x;\lambda)\right)d\lambda. \tag{2.13}$$

This is actually the desired expansion formula for wave function $f(x)$, and the completeness follows immediately. From this quick review we can see the advantage of this method. It does not require any abstract operator theories, and everything can be obtained just by the tools of classical complex analysis. So it is especially convenient for physicists.

3   Spectral decomposition of the Morse potential

Suppose $\kappa > 0$, the Morse potential is [1]

$$V_{\text{M}}(x) = \kappa^2 \left(e^{-2(x-x_0)} - 2\,e^{-(x-x_0)} \right). \tag{3.1}$$

The corresponding Schrödinger equation is

$$-\frac{d^2\psi}{dx^2} + V_{\text{M}}(x)\,\psi(x) = \lambda\,\psi(x). \tag{3.2}$$

Define

$$u = 2\kappa\,e^{-(x-x_0)}, \qquad w(u) = \sqrt{u}\,\psi(x(u)), \tag{3.3}$$

we have

$$\frac{d^2 w}{du^2} + \left\{ -\frac{1}{4} + \frac{\kappa}{u} + \frac{1/4 - (-\lambda)}{u^2} \right\} w = 0. \tag{3.4}$$

This is the Whittaker equation, so we have

$$w(u) = c_1 M_{\kappa,\,i\sqrt{\lambda}}(u) + c_2 W_{\kappa,\,i\sqrt{\lambda}}(u). \tag{3.5}$$

Therefore the solution of ψ is

$$\begin{aligned}
\psi(x) &= c_1\,\psi_1(x) + c_2\,\psi_2(x) \\
&\equiv \frac{1}{\sqrt{2\kappa}}\,e^{(x-x_0)/2}\left[c_1 M_{\kappa,\,i\sqrt{\lambda}}\left(2\kappa\,e^{-(x-x_0)}\right) + c_2 W_{\kappa,\,i\sqrt{\lambda}}\left(2\kappa\,e^{-(x-x_0)}\right) \right].
\end{aligned} \tag{3.6}$$

The Wronskian $W(\psi_1, \psi_2)$ is

$$W(\psi_1, \psi_2) \equiv \psi_1(x)\,\psi_2'(x) - \psi_1'(x)\,\psi_2(x) = \frac{\Gamma(1 + 2\,i\sqrt{\lambda}\,)}{\Gamma(\frac{1}{2} - \kappa + i\sqrt{\lambda}\,)}\ . \tag{3.7}$$

When $x \to -\infty$, i.e. $u \to +\infty$, we have

$$\psi_1 \;=\; \frac{1}{\sqrt{u}}\,M_{\kappa,\,i\sqrt{\lambda}}(u) \;\sim\; \left(\alpha_+ e^{u/2} u^{-\kappa} + \alpha_- e^{-u/2} u^{\kappa}\right) , \tag{3.8}$$

$$\psi_2 \;=\; \frac{1}{\sqrt{u}}\,W_{\kappa,\,i\sqrt{\lambda}}(u) \;\sim\; e^{-u/2} u^{\kappa} . \tag{3.9}$$

When $x \to +\infty$, i.e. $u \to 0$, we have

$$\psi_1 \;=\; \frac{1}{\sqrt{u}}\,M_{\kappa,\,i\sqrt{\lambda}}(u) \;\sim\; u^{i\sqrt{\lambda}}, \tag{3.10}$$

$$\psi_2 \;=\; \frac{1}{\sqrt{u}}\,W_{\kappa,\,i\sqrt{\lambda}}(u) \;\sim\; \left(\beta_+ u^{i\sqrt{\lambda}} + \beta_- u^{-i\sqrt{\lambda}}\right) . \tag{3.11}$$

If $-2\pi < \arg\lambda < 0$, then $\mathrm{Im}\sqrt{\lambda} < 0$, so the solution normalizable at $-\infty$ is $\psi_2(x)$, while the solution normalizable at $+\infty$ is $\psi_1(x)$. Therefore both singular points are limit points. The resolvent kernel $G(x, x'; \lambda)$ is the normalizable solution of

$$-\frac{d^2 G}{dx^2} + (V(x) - \lambda)\,G = \delta(x - x') , \tag{3.12}$$

which is continuous at $x = x'$, but

$$\left.\frac{\partial G}{\partial x}\right|_{x = x' + \varepsilon} - \left.\frac{\partial G}{\partial x}\right|_{x = x' - \varepsilon} = -1 , \qquad \varepsilon \to 0^+ . \tag{3.13}$$

It can be checked that the solution is[1]

$$\begin{aligned}
G(x, x'; \lambda) &= G_+(x, x'; \lambda) \equiv \frac{\Gamma(\frac{1}{2} - \kappa + i\sqrt{\lambda}\,)}{\Gamma(1 + 2\,i\sqrt{\lambda}\,)}\,\frac{M_{\kappa,\,i\sqrt{\lambda}}(u)}{\sqrt{u}}\,\frac{W_{\kappa,\,i\sqrt{\lambda}}(u')}{\sqrt{u'}}\,, \quad x \geq x' ; \\[2mm]
&= G_-(x, x'; \lambda) \equiv \frac{\Gamma(\frac{1}{2} - \kappa + i\sqrt{\lambda}\,)}{\Gamma(1 + 2\,i\sqrt{\lambda}\,)}\,\frac{W_{\kappa,\,i\sqrt{\lambda}}(u)}{\sqrt{u}}\,\frac{M_{\kappa,\,i\sqrt{\lambda}}(u')}{\sqrt{u'}}\,, \quad x < x' .
\end{aligned}$$

This resolvent kernel $G(x, x'; \lambda)$ has simple poles at

$$\lambda_n = -\,(\kappa - n - 1/2)^2\,, \qquad n = 0, 1, \cdots, [\kappa - 1/2]\,. \tag{3.14}$$

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[1]In this paper we choose $-2\pi < \arg\lambda < 0$. If we use the usual convention $0 < \arg\lambda < 2\pi$, then $i\sqrt{\lambda}$ should be replaced by $-i\sqrt{\lambda}$ everywhere. These two different choices will give us the same normalized eigenfunctions as in (3.28) and (3.29).

which come form the factor $\Gamma(1/2 - \kappa + i\sqrt{\lambda})$ since[2]

$$\Gamma(1/2 - \kappa + i\sqrt{\lambda}) = \frac{(-1)^{n+1}}{n!} \frac{2\,i\sqrt{\lambda_n}}{\lambda - \lambda_n} + O(1)\,. \tag{3.15}$$

Note that the upper limit $\kappa - 1/2$ of n follows from the requirement $\mathrm{Im}\sqrt{\lambda} < 0$, which is a direct sequence of $-2\pi < \arg\lambda < 0$.[3] At these poles we have

$$M_{\kappa,\,i\sqrt{\lambda_n}}(u) = \frac{n!\,\Gamma(2\kappa - 2n)}{\Gamma(2\kappa - n)}\,e^{-u/2}u^{\kappa-n}L_n^{2\kappa-2n-1}(u)\,, \tag{3.16}$$

$$W_{\kappa,\,i\sqrt{\lambda_n}}(u) = (-1)^n\,n!\,e^{-u/2}u^{\kappa-n}L_n^{2\kappa-2n-1}(u)\,, \tag{3.17}$$

where $L_n^\alpha(u)$ is the generalized Laguerre polynomial. Therefore the residues of $G(x, x'; \lambda)$ at these poles are

$$-\frac{n!\,(2\kappa - 2n - 1)}{\Gamma(2\kappa - n)}\,e^{-u/2}u^{\kappa-n-1/2}L_n^{2\kappa-2n-1}(u)\,e^{-u'/2}u'^{\kappa-n-1/2}L_n^{2\kappa-2n-1}(u')\,. \tag{3.18}$$

For any function $f(x)$, the solution $\Phi_f(x; \lambda)$ of $\Phi'' + (\lambda - V(x))\,\Phi = f(x)$ is

$$\begin{aligned}
\Phi_f(x; \lambda) &= \int_{-\infty}^{\infty} G(x, x'; \lambda)\,f(x')\,dx' \\
&= \int_{-\infty}^{x} G_+(x, x'; \lambda)\,f(x')\,dx' + \int_{x}^{\infty} G_-(x, x'; \lambda)\,f(x')\,dx'\,.
\end{aligned} \tag{3.19}$$

As reviewed in the previous section, to obtain the spectral decomposition of the function $f(x)$, we need to consider $\oint_C \Phi_f(x; \lambda)\,d\lambda$ with the contour C defined in Figure 1. The Cauchy theorem tells us

$$\oint_C \Phi_f(x; \lambda)\,d\lambda + 2\pi i \sum_{\text{poles}} \mathrm{Res}\,\Phi_f = 0\,. \tag{3.20}$$

The contributions from poles, as calculated above, are

$$\begin{aligned}
-2\pi i \sum_{n=0}^{[\kappa - \frac{1}{2}]} &\frac{n!\,(2\kappa - 2n - 1)}{\Gamma(2\kappa - n)}\,e^{-u/2}u^{\kappa-n-1/2}L_n^{2\kappa-2n-1}(u) \\
&\times \int_{-\infty}^{\infty} e^{-u'/2}u'^{\kappa-n-1/2}L_n^{2\kappa-2n-1}(u')\,f(x')\,dx'\,.
\end{aligned} \tag{3.21}$$

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[2]Note that the other factor $M_{\kappa,\,i\sqrt{\lambda}}(u)/\Gamma(1 + 2\,i\sqrt{\lambda})$ has no poles in the λ-plane.

[3]This upper limit does not depend on the convention about the argument range of λ. If we choose $0 < \lambda < 2\pi$, it still holds.

Next we consider the sum of two integrals along the positive axis

$$
\begin{aligned}
I_{L^+} + I_{L^-} &= \int_{L^+} \Phi_f(x;\lambda)\, d\lambda + \int_{L^-} \Phi_f(x;\lambda)\, d\lambda \\
&= \int_{-\infty}^{x} dx'\, f(x') \int_0^{\infty} d\lambda \left(G_+(x,x';\lambda) - G_+(x,x';\lambda\, e^{-2\pi i}) \right) \\
&\quad + \int_{x}^{\infty} dx'\, f(x') \int_0^{\infty} d\lambda \left(G_-(x,x';\lambda) - G_-(x,x';\lambda\, e^{-2\pi i}) \right) \\
&= \int_{-\infty}^{x} dx'\, f(x') \int_0^{\infty} d\lambda \left(\frac{\Gamma(\tfrac{1}{2}-\kappa+i\sqrt{\lambda}\,)}{\Gamma(1+2\,i\sqrt{\lambda}\,)} \frac{M_{\kappa,\,i\sqrt{\lambda}}(u)}{\sqrt{u}} \right. \\
&\quad \left. - \frac{\Gamma(\tfrac{1}{2}-\kappa-i\sqrt{\lambda}\,)}{\Gamma(1-2\,i\sqrt{\lambda}\,)} \frac{M_{\kappa,\,-i\sqrt{\lambda}}(u)}{\sqrt{u}} \right) \frac{W_{\kappa,\,i\sqrt{\lambda}}(u')}{\sqrt{u'}} \\
&\quad + \int_{x}^{\infty} dx'\, f(x') \int_0^{\infty} d\lambda \left(\frac{\Gamma(\tfrac{1}{2}-\kappa+i\sqrt{\lambda}\,)}{\Gamma(1+2\,i\sqrt{\lambda}\,)} \frac{M_{\kappa,\,i\sqrt{\lambda}}(u')}{\sqrt{u'}} \right. \\
&\quad \left. - \frac{\Gamma(\tfrac{1}{2}-\kappa-i\sqrt{\lambda}\,)}{\Gamma(1-2\,i\sqrt{\lambda}\,)} \frac{M_{\kappa,\,-i\sqrt{\lambda}}(u')}{\sqrt{u'}} \right) \frac{W_{\kappa,\,i\sqrt{\lambda}}(u)}{\sqrt{u}} \\
&= \frac{1}{i\pi} \int_0^{\infty} d\lambda\, \sinh(2\pi\sqrt{\lambda}\,) \left| \Gamma\!\left(\frac{1}{2}-\kappa+i\sqrt{\lambda}\, \right) \right|^2 \frac{W_{\kappa,\,i\sqrt{\lambda}}(u)}{\sqrt{u}} \\
&\quad \times \int_{-\infty}^{\infty} \frac{W_{\kappa,\,i\sqrt{\lambda}}(u')}{\sqrt{u'}}\, f(x')\, dx' \,.
\end{aligned}
\tag{3.22}
$$

Here we have used the relation $W_{\kappa,-\mu}(u) = W_{\kappa,\mu}(u)$ and

$$
W_{\kappa,\mu}(z) = \frac{\Gamma(-2\mu)}{\Gamma\left(\tfrac{1}{2}-\kappa-\mu\right)}\, M_{\kappa,\mu}(z) + \frac{\Gamma(2\mu)}{\Gamma\left(\tfrac{1}{2}-\kappa+\mu\right)}\, M_{\kappa,-\mu}(z)\,.
\tag{3.23}
$$

By (2.15.2) of [9], for square-integrable function $f(x)$, we have

$$
\Phi_f(x;\lambda) = -\frac{f(x)}{\lambda} + O\left(\frac{1}{|\lambda|^{\frac{3}{4}} |\operatorname{Im}\lambda|} \right), \qquad |\lambda| \to \infty\,.
\tag{3.24}
$$

So we have

$$
\lim_{\Lambda \to \infty} \int_{C_\Lambda} \Phi_f(x;\lambda)\, d\lambda = 2\pi i f(x)\,.
\tag{3.25}
$$

It is easy to check that

$$
\lim_{\delta \to 0} \int_{C_\delta} \Phi_f(x;\lambda)\, d\lambda = 0\,.
\tag{3.26}
$$

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Therefore from the Cauchy's theorem (3.20) we have

$$
\begin{aligned}
f(x) \;=\; & \sum_{n=0}^{[\kappa-\frac{1}{2}]} \frac{n!\,(2\kappa-2n-1)}{\Gamma(2\kappa-n)}\; e^{-u/2}u^{\kappa-n-1/2}L_n^{2\kappa-2n-1}(u) \\
& \times \int_{-\infty}^{\infty} e^{-u'/2}u'^{\kappa-n-1/2}L_n^{2\kappa-2n-1}(u')\,f(x')\,dx' \\
& + \frac{1}{2\pi^2}\int_0^{\infty} d\lambda\,\sinh(2\pi\sqrt{\lambda})\,\left|\Gamma\!\left(\frac{1}{2}-\kappa+i\sqrt{\lambda}\right)\right|^2 u^{-1/2}\,W_{\kappa,\,i\sqrt{\lambda}}(u) \\
& \times \int_{-\infty}^{\infty} u'^{-1/2}\,W_{\kappa,\,i\sqrt{\lambda}}(u')\,f(x')\,dx' \,, \qquad (3.27)
\end{aligned}
$$

with $u = 2\kappa\,e^{-(x-x_0)}$ and similarly for u'. We define the normalized eigenfunctions as[4]

$$
\psi_n(x) \;=\; \sqrt{\frac{n!\,(2\kappa-2n-1)}{\Gamma(2\kappa-n)}}\; e^{-u/2}u^{\kappa-n-1/2}L_n^{2\kappa-2n-1}(u)\,, \qquad (3.28)
$$

$$
\psi_\lambda(x) \;=\; \frac{1}{\sqrt{2}\,\pi}\,\sinh^{1/2}(2\pi\sqrt{\lambda})\,\left|\Gamma\!\left(\frac{1}{2}-\kappa+i\sqrt{\lambda}\right)\right|\,u^{-1/2}\,W_{\kappa,\,i\sqrt{\lambda}}(u)\,, \qquad (3.29)
$$

with $n = 0,1,\cdots,[\,\kappa-1/2\,]$ and $\lambda > 0$. Then the expansion formula (3.27) leads to the completeness relation

$$
\sum_{n=0}^{[\kappa-\frac{1}{2}]} \psi_n^*(x)\,\psi_n(x') \;+\; \int_0^{\infty} \psi_\lambda^*(x)\,\psi_\lambda(x')\,d\lambda \;=\; \delta(x-x')\,. \qquad (3.30)
$$

This is the main result of the present paper, which depends crucially on the correct normalization (3.28) and (3.29) of the eigenfunctions. The normalization of continuum states seems not be discussed in the literature so far. In this section we have derived it by the method of contour integral.

4   Application to the Asian option pricing

In this section we will apply the spectral decomposition formula (3.27) or (3.30) to the problem of Asian option pricing. Let $S(t)$ denote the price of a stock and $A(t) = \frac{1}{t}\int_0^t S(t)\,dt$ the time average up to the time t. An Asian put option with the strike price K is a contract which gives you the right to sale this stock at the expiration time $t = T$ and obtain the payoff $K - A(T)$ if $K > A(T)$, while continue to hold the stock

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[4]These eigenfunctions are actually real functions.

if $K < A(T)$. Therefore in either case the payoff is $P(T) = (K - A(T))^{+}$.[5] Obviously you should pay some money when signing this option contract at $t = 0$. Our aim is to determine its fair price.

Usually it is assumed that the stock price $S(t)$ follows the law of geometric Brownian motion, i.e. $S(t) = S_0 e^{\sigma W(t) + (r - \sigma^2/2)t}$, where $W(t)$ is a Brownian motion with constant volatility σ, and r is the risk-free interest rate. After some rescalings the value of an Asian put option can be written as [10]

$$e^{-rT} \left(\frac{4S_0}{\sigma^2 T} \right) \left\langle (k - a(\tau))^{+} \right\rangle, \tag{4.1}$$

where we have introduced the exponential functional $a(t)$ of the Brownian motion $W(t)$ as $a(t) = \int_0^t e^{2(W(t) + \nu t)} dt$, together with dimensionless parameters $\tau = \sigma^2 T/4$, $\nu = 2r/\sigma^2 - 1$ and $k = \tau K/S_0$. The probability density function $f(a, t)$ of the stochastic process $a(t)$ satisfies the following diffusion-type equation [11, 12]

$$\frac{\partial f}{\partial t} = 2a^2 \frac{\partial^2 f}{\partial a^2} + (2(\nu + 1) a + 1) \frac{\partial f}{\partial a}, \quad a > 0. \tag{4.2}$$

The heat kernel $K(a, a'; t)$ is the solution of the above equation with the initial condition $K(a, a'; 0) = \delta(a - a')$. Then the option value can be written as

$$e^{-rT} \left(\frac{4S_0}{\sigma^2 T} \right) \int_0^{\infty} (k - a)^{+} K(a, 0; \tau) \, da. \tag{4.3}$$

Suppose f_λ satisfies

$$- 2a^2 \frac{d^2 f_\lambda}{da^2} - (2(\nu + 1) a + 1) \frac{d f_\lambda}{da} = 2\lambda f_\lambda, \tag{4.4}$$

which can be rewritten in the form $-(pf')' = \lambda w f$ by defining $p(a) = a^{\nu+1} e^{1/(2a)}$ and $w(a) = a^{\nu-1} e^{-1/(2a)}$. The heat kernel $K(a, a'; t)$ has the following spectral decomposition

$$K(a, a'; t) = w(a) \sum_{\lambda} e^{-2\lambda t} f_\lambda^*(a) f_\lambda(a'). \tag{4.5}$$

Introduce the new variable x and new function $\psi_\lambda(x)$ by

$$x = \log a, \quad \psi_\lambda(x) = a^{\nu/2} e^{-1/(4a)} f_\lambda(a), \tag{4.6}$$

then the equation (4.4) can be reduced to

$$- \frac{d^2 \psi_\lambda}{dx^2} + \left(\frac{1}{16} e^{-2x} - \frac{1 - \nu}{4} e^{-x} + \frac{\nu^2}{4} \right) \psi_\lambda = \lambda \psi_\lambda. \tag{4.7}$$

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[5]We use the definition $X^{+} \equiv X \cdot \theta(X)$, where $\theta(\cdot)$ is the Heaviside step function.

Actually, up to a additional constant $\nu^2/4$, this is the static Schrödinger equation with the Morse potential. Compared with it standard form (3.1), we have

$$\kappa = \frac{1-\nu}{2}, \quad e^{x_0} = \frac{1}{2(1-\nu)}. \tag{4.8}$$

Here we restrict to the case $\nu < 1$. By use of our results (3.28) and (3.29) in the previous section, we can obtain the complete system of eigenfunctions of (4.4) as follows

$$f_n(a) = \sqrt{\frac{n!\,(-\nu-2n)}{2^{-\nu-2n}\,\Gamma(1-\nu-n)}}\; a^n L_n^{-\nu-2n}\left(\frac{1}{2a}\right), \tag{4.9}$$

$$f_\lambda(a) = \frac{1}{\pi}\;\sinh^{1/2}\left(2\pi\sqrt{\lambda-\nu^2/4}\right)\left|\Gamma\!\left(\nu/2+i\sqrt{\lambda-\nu^2/4}\right)\right|$$

$$\times a^{(1-\nu)/2}\,e^{1/(4a)}\,W_{(1-\nu)/2,\,i\sqrt{\lambda-\nu^2/4}}\left(\frac{1}{2a}\right). \tag{4.10}$$

The corresponding eigenvalues are $\lambda_n = n(-\nu-n)$ with $n = 0, 1, \cdots, [-\nu/2]$, together with the continuous spectrum $\lambda > \nu^2/4$. Therefore, by redefining $p = 2\sqrt{\lambda-\nu^2/4}$, we have

$$K(a,0;\tau) = \sum_{n=0}^{[-\frac{\nu}{2}]} \frac{(-1)^n 2(-\nu-2n)}{\Gamma(1-\nu-n)}\; e^{-2n(-\nu-n)\tau}(2a)^{\nu+n-1}e^{-1/(2a)}L_n^{-\nu-2n}\left(\frac{1}{2a}\right)$$

$$+ \frac{1}{2\pi^2}\int_0^\infty e^{-(p^2+\nu^2)\tau/2}(2a)^{(\nu-1)/2}\,e^{-1/(4a)}$$

$$\times W_{(1-\nu)/2,\,ip/2}\left(\frac{1}{2a}\right)\left|\Gamma\!\left(\frac{\nu+ip}{2}\right)\right|^2 \sinh(\pi p)\,p\,dp. \tag{4.11}$$

We have used the asymptotic relation $L_n^\alpha(z) \sim \frac{(-1)^n}{n!}\,z^n$ and $W_{\kappa,\mu}(z) \sim e^{-z/2}z^\kappa$ as $z \to \infty$. Therefore the value of an Asian put option is

$$e^{-rT}\left(\frac{4S_0}{\sigma^2 T}\right)\left\{\sum_{n=0}^{[-\nu/2]} \frac{(-1)^n(-\nu-2n)}{2n!\,\Gamma(1-\nu-n)}\; e^{-2n(-\nu-n)\tau}(2k)^{\nu+n+3}\,e^{-1/(2k)}L_n^{-\nu-2n}\left(\frac{1}{2k}\right)\right.$$

$$+ \frac{1}{8\pi^2}\int_0^\infty e^{-(p^2+\nu^2)\tau/2}(2k)^{\nu+3}\,e^{-1/(4k)}$$

$$\left.\times W_{-\frac{\nu+3}{2},\frac{ip}{2}}\left(\frac{1}{2k}\right)\left|\Gamma\!\left(\frac{\nu+ip}{2}\right)\right|^2 \sinh(\pi p)\,p\,dp\right\}. \tag{4.12}$$

Here we have used the formula (7.623.7) of [13] to work out the integration over a in (4.3). This is completely equivalent to the result of [12] which is obtained by a different method.

5 Conclusions

In this paper we obtain the properly normalized eigenfunctions of the Morse potential and explicitly prove the completeness relation by use of the method of contour integrals. In this method the discrete spectrum corresponds the pole of the resolvent kernel, while the continuum spectrum comes from the contributions of the branch cut. We also apply this spectral decomposition to the problem of the pricing of an Asian option. We expect this method can be applied to more subtle spectral problems, e.g. issues about self-adjoint extensions, in quantum mechanics and its related fields.

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