ERROR BOUNDS FOR SMALL JUMPS OF LÉVY PROCESSES AND FINANCIAL APPLICATIONS

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Abstract. The pricing of exotic options in exponential Lévy models amounts to the computation of expectations of functionals of the whole path of a Lévy process. In many situations, Monte-Carlo methods are used. However, the simulation of a Lévy process with infinite Lévy measure generally requires either to truncate small jumps or to replace them by a Brownian motion with the same variance. We derive bounds for the errors generated by these two types of approximation. These bounds can be applied to a number of exotic options (barriers, lookback, American, Asian).

Key words. Approximation of small jumps, Lévy processes, Skorokhod embedding, Spitzer identity

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1. Introduction. In the recent years, the use of general Lévy processes in financial models has grown extensively (see [2, 4, 7]). A variety of numerical methods have been subsequently developped, in particular methods based on Fourier analysis (see [3, 8, 9, 10]). Nonetheless, in many situations, Monte-Carlo methods have to be used. Yet, the simulation of a Lévy process with infinite Lévy measure is not straightforward, except in some special cases like the Gamma or Inverse Gaussian models. In practise, the small jumps of the Lévy process are either just truncated or replaced by a Brownian motion with the same variance. The latter approach was introduced by Asmussen and Rosinsky [1], who showed that, under suitable conditions, the normalized cumulated small jumps asymptotically behave like Brownian motion.

The purpose of this article is to derive bounds for the errors generated by these two methods of approximation in the computation of functions of the Lévy process at a fixed time or functions of the supremum process. Error bounds are also derived for the cumulative distribution functions. Our bounds can be applied to derive approximation errors for lookback, barrier, American or Asian options.

The paper is organized as follows. In the next section, we recall some basic facts about real Lévy processes. In section 3 we will study the errors resulting from the small jump truncation. The results of this section are based on estimates for the difference $X - X^{\epsilon}$, where X^{ϵ} is obtained by removing jumps with absolute value smaller than a given positive ϵ, and on Taylor's formula. We also derive an estimate for the expectation of the difference of the supremum processes, by using Sptzer's identity. The errors resulting from Brownian approximation are studied in section 4. The main result of this section is Theorem 4.5, which states an error bound for the computation of a the expectation of a function of the supremum. The proof of this results relies on the Skorohod embedding theorem.

2. Preliminaries. Recall that a Lévy process X is defined by its generating triplet (γ, σ, ν), where $(\gamma, \sigma) \in \mathbb{R} \times \mathbb{R}^{+}$, and ν is a Radon measure on $\mathbb{R}\backslash\{0\}$ satisfying

$$\int_{\mathbb{R}} \left(1 \wedge x^2\right) \nu(dx) < \infty.$$

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The process X has finite activity if $\nu(\mathbb{R}) < \infty$. If $\nu(\mathbb{R}) = +\infty$, the process X is called an infinite activity Lévy process. By the Lévy-Itô decomposition, X can be written in the form

$$X_t = \gamma t + \sigma B_t + \int_{|x|>1, s\in[0,t]} x J_X(dx \times ds) + \lim_{\delta\downarrow 0} \int_{\delta\leq|x|\leq 1, s\in[0,t]} x \widetilde{J}_X(dx \times ds) \quad (2.1)$$

Here J is a Poisson measure on $\mathbb{R} \times [0, \infty)$ with intensity $\nu(dx)dt$, $\widetilde{J}_X(dx \times ds) = J_X(dx \times ds) - \nu(dx)ds$ and B is a standard Brownian motion. Given $\epsilon > 0$, we define the process R^ϵ by

$$R_t^\epsilon = \int_{0\leq|x|\leq\epsilon, s\in[0,t]} x \widetilde{J}_X(dx \times ds), \ t \geq 0. \quad (2.2)$$

Note that

$$\mathbb{E} R_t^\epsilon = 0$$
$$\mathrm{Var}\left(R_t^\epsilon\right) = t \int_{|x|\leq\epsilon} x^2 \nu(dx) = \sigma(\epsilon)^2 t.$$

The process X^ϵ is then defined by

$$X_t^\epsilon = X_t - R_t^\epsilon, \ t \geq 0. \quad (2.3)$$

We also define the processes $\hat{X}^\epsilon$ by

$$\hat{X}_t^\epsilon = X_t^\epsilon + \sigma(\epsilon)\hat{W}_t, \ t \geq 0, \quad (2.4)$$

where $\hat{W}$ is a standard Brownian motion independent of X. We aim to study the behavior of the errors made by replacing X by X^ϵ or $\hat{X}^\epsilon$, with respect to the level ϵ. These error are studied for the process X at a fixed date and for its the running supremum. Set, for any $t \geq 0$

$$M_t^X = \sup_{0\leq s\leq t} X_s, \ M_t^{\epsilon,X} = \sup_{0\leq s\leq t} X_s^\epsilon, \ \hat{M}_t^{\epsilon,X} = \sup_{0\leq s\leq T} \hat{X}_s^\epsilon. \quad (2.5)$$

When there is no ambiguity we can remove the super index X.

3. Truncation of small jumps. In this section, we will study the errors resulting from the truncation of small jumps. The errors resulting from small jumps approximation are related to the moments of small jumps. Define

$$\sigma_0(\epsilon) = \max\left(\sigma(\epsilon), \epsilon\right). \quad (3.1)$$

The next result will be usefull for many proofs in this article.

PROPOSITION 3.1. *Let X be a Lévy process with generating triplet (γ, σ^2, ν), $\epsilon \in (0, 1]$ and R^ϵ defined in* (2.2). *Then*

$$\mathbb{E}\left|R_t^\epsilon\right|^4 = t \int_{-\epsilon}^{\epsilon} x^4 \nu(dx) + 3\left(t\sigma(\epsilon)^2\right)^2,$$

and for any real $q > 0$

$$\mathbb{E}\left|R_t^\epsilon\right|^q \leq K_{q,t}\sigma_0(\epsilon)^q,$$

where $K_{q,t}$ is a positive constant which depends on q and t.

Proof. Set

$$\Phi_\epsilon(u) = \mathbb{E}e^{iuR_t^\epsilon}.$$

We have

$$\Phi_\epsilon(u) = e^{\Psi_\epsilon(u)},$$

where $\Psi_\epsilon(u) = t\int_{|y|\leq\epsilon}\left(e^{iuy} - 1 - iuy\right)\nu(dy)$. Set

$$c_n(\epsilon) = \frac{1}{i^n}\frac{\partial^n\Psi_\epsilon}{\partial u^n}(0).$$

Using *[Cont-Tankov(2004), proposition 3.13]*, we get

$$c_n(\epsilon) = t\int_{|x|\leq\epsilon} x^n\nu(dx),\ \forall n \geq 2.$$

Furthermore

$$\mathbb{E}\left(R_t^\epsilon\right)^n = \frac{1}{i^n}\frac{\partial^n\Phi_\epsilon}{\partial u^n}(0).$$

By differentiating, we get

$$\frac{\partial^4\Phi_\epsilon}{\partial u^4}(u) = \left(\left(\frac{\partial\Psi_\epsilon}{\partial u}(u)\right)^4 + 6\left(\frac{\partial\Psi_\epsilon}{\partial u}(u)\right)^2\frac{\partial^2\Psi_\epsilon}{\partial^2 u}(u) + 4\frac{\partial\Psi_\epsilon}{\partial u}(u)\frac{\partial^3\Psi_\epsilon}{\partial^3 u}(u)\right.$$
$$\left. +3\left(\frac{\partial^2\Psi_\epsilon}{\partial^2 u}(u)\right)^2 + \frac{\partial^4\Psi_\epsilon}{\partial u^4}(u)\right)\Phi_\epsilon(u).$$

But, as $\mathbb{E}R_t^\epsilon = 0$, we have $\frac{\partial\Psi_\epsilon}{\partial u}(0) = 0$. Hence

$$\frac{\partial^4\Phi_\epsilon}{\partial u^4}(0) = 3\left(\frac{\partial^2\Psi_\epsilon}{\partial^2 u}(0)\right)^2 + \frac{\partial^4\Psi_\epsilon}{\partial u^4}(0).$$

Therefore

$$\begin{aligned}\mathbb{E}\left(R_t^\epsilon\right)^4 &= 3\left(\frac{1}{i^2}\frac{\partial^2\Psi_\epsilon}{\partial^2 u}(0)\right)^2 + \frac{1}{i^4}\frac{\partial^4\Psi_\epsilon}{\partial u^4}(0)\\ &= 3\left(c_2(\epsilon)\right)^2 + c_4(\epsilon)\\ &= 3t^2\sigma(\epsilon)^4 + t\int_{|x|\leq\epsilon}|x|^4\nu(dx).\end{aligned}$$

Hence the first result of the proposition. Besides, for an integer $n \geq 1$ we have

$$\Phi_\epsilon^{(2n)}(0) = \sum_{\{j,k,p,q\}\in E_n} a_{j,k,p,q}\left(\Psi_\epsilon^{(j)}(0)\right)^k\left(\Psi_\epsilon^{(p)}(0)\right)^q,$$

where E_n is a finite set, $\left(a_{j,k,p,q}\right)_{\{j,k,p,q\}\in E_n}$ are positive real which we can derive explicitly. Note that for any $\{j,k,p,q\} \in E_n$, we have

$$j > 1,\ p > 1,\ jk + pq = 2n$$

So

$$\left|\Phi_\epsilon^{(2n)}(0)\right| \leq \sum_{\{j,k,p,q\}\in E_n} a_{j,k,p,q}\left|\Psi_\epsilon^{(j)}(0)\right|^k \left|\Psi_\epsilon^{(p)}(0)\right|^q.$$

Therefore

$$\mathbb{E}\left|R_t^\epsilon\right|^{2n} = \left|\Phi_\epsilon^{(2n)}(0)\right| \leq \sum_{\{j,k,p,q\}\in E_n} a_{j,k,p,q}\left|c_j(\epsilon)\right|^k \left|c_p(\epsilon)\right|^q.$$

But for $j \geq 2$, we have

$$\begin{aligned}|c_j(\epsilon)| &\leq t\epsilon^{j-2}\int_{|x|\leq\epsilon}|x|^2\nu(dx)\\ &= t\epsilon^{j-2}\sigma(\epsilon)^2\\ &\leq \sigma_0(\epsilon)^j t.\end{aligned}$$

Hence

$$\begin{aligned}\mathbb{E}\left|R_t^\epsilon\right|^{2n} &\leq \sum_{\{j,k,p,q\}\in E_n, j>1, p>1} a_{j,k,p,q}\left|\sigma_0(\epsilon)^j t\right|^k \left|\sigma_0(\epsilon)^p t\right|^q\\ &= \sum_{\{j,k,p,q\}\in E_n} a_{j,k,p,q} t^{k+q}\sigma_0(\epsilon)^{jk+pq}\\ &= \sum_{\{j,k,p,q\}\in E_n} a_{j,k,p,q} t^{k+q}\sigma_0(\epsilon)^{2n}.\end{aligned}$$

Note $K_{n,t} = \sqrt{\sum_{\{j,k,p,q\}\in E_n} a_{j,k,p,q}t^{k+q}}$, we get

$$\mathbb{E}\left|R_t^\epsilon\right|^{2n} \leq K_{n,t}^2\sigma_0(\epsilon)^{2n}.$$

So

$$\begin{aligned}\mathbb{E}\left|R_t^\epsilon\right|^{n} &\leq \left(\mathbb{E}\left|R_t^\epsilon\right|^{2n}\right)^{\frac{1}{2}}\\ &\leq K_{n,t}\sigma_0(\epsilon)^n.\end{aligned}$$

The upper bound for $\mathbb{E}\left|R_t^\epsilon\right|^q$ where q is a positive real can be easily deduced. □

3.1. Estimates for smooth functions. Let X be a Lévy process with generating triplet (γ, σ, ν) and f a C-Lipschitz function where $C > 0$. Then

$$\begin{aligned}\mathbb{E}\left|f(X_t) - f(X_t^\epsilon)\right| &\leq C\mathbb{E}\left|X_t - X_t^\epsilon\right|\\ &= C\mathbb{E}\left|R_t^\epsilon\right|\\ &\leq C\sqrt{\mathbb{E}\left|R_t^\epsilon\right|^2}.\end{aligned}$$

Hence

$$\mathbb{E}\left|f(X_t) - f(X_t^\epsilon)\right| \leq C\sqrt{t}\sigma(\epsilon). \tag{3.2}$$

Note that we do not ask that $f(X_t)$ be integrable. If f is more regular, sharper estimates can be derived, as shown in the following Proposition.

PROPOSITION 3.2. *Let X be an infinite activity Lévy process with generating triplet $\left(\gamma, \sigma^2, \nu\right)$, $\epsilon \in (0, 1]$ and $t > 0$.*

1. *If* $f \in C^1(\mathbb{R})$ *and satisfies* $\mathbb{E}\left|f^{'}\left(X_t^\epsilon\right)\right| < \infty$, *and if there exists* $\beta > 1$ *such that* $\left(\sup_{\delta\in[0,1]} \mathbb{E}\left|f^{'}\left(X_t^\delta + \theta R_t^\delta\right) - f^{'}\left(X_t^\delta\right)\right|^\beta\right)^{\frac{1}{\beta}}$ *is finite and integrable with respect to* θ *on* $[0,1]$, *then*
$$\mathbb{E}\left(f\left(X_t\right) - f\left(X_t^\epsilon\right)\right) = o\left(\sigma_0(\epsilon)\right).$$
2. *If* $f \in C^2(\mathbb{R})$ *and satisfies* $\mathbb{E}\left|f^{'}\left(X_t^\epsilon\right)\right| + \mathbb{E}\left|f^{''}\left(X_t^\epsilon\right)\right| < \infty$, *and if there exists* $\beta > 1$ *such that* $\left(\sup_{\delta\in[0,1]} \mathbb{E}\left|f^{''}\left(X_t^\delta + \theta R_t^\delta\right) - f^{''}\left(X_t^\delta\right)\right|^\beta\right)^{\frac{1}{\beta}}$ *is finite and integrable with respect to* θ *on* $[0,1]$, *then*
$$\mathbb{E}\left(f\left(X_t\right) - f\left(X_t^\epsilon\right)\right) = \frac{\sigma(\epsilon)^2 t}{2}\mathbb{E}f^{''}\left(X_t^\epsilon\right) + o\left(\sigma_0(\epsilon)^2\right).$$

Note that, if f has bounded derivatives or f is the exponetial function and $e^{\beta X_t}$ is integrable, where $\beta > 1$, the conditions in the above proposition are satisfied. Recall that the truncation of small jumps is used when $\nu(\mathbb{R}) = \infty$. In typical applications, we have $\liminf \sigma(\epsilon)/\epsilon > 0$, so that $o\left(\sigma_0(\epsilon)^2\right)$ is in fact $o\left(\sigma(\epsilon)^2\right)$. This is the case for Lévy processes with a Lévy density that behaves like $x^{-\alpha}$ in the neighborhood of 0 with $\alpha \geq 1$. Examples in finance are VG, NIG and CGMY processes.

Proof. We have
$$f\left(X_t\right) - f\left(X_t^\epsilon\right) = \int_0^1 f^{'}\left(X_t^\epsilon + \theta R_t^\epsilon\right) R_t^\epsilon d\theta.$$
The last equality is obtained by the substitution $\theta = \frac{x - X_t^\epsilon}{X_t - X_t^\epsilon}$ and moreover we have by theorem 27.4 of [11]
$$\mathbb{P}\left[X_t - X_t^\epsilon = 0\right] = \mathbb{P}\left[R_t^\epsilon = 0\right] = 0.$$
So
$$\begin{aligned}\frac{1}{\sigma(\epsilon)}\mathbb{E}\left(f\left(X_t\right) - f\left(X_t^\epsilon\right)\right) &= \int_0^1 \mathbb{E}\left(f^{'}\left(X_t^\epsilon + \theta R_t^\epsilon\right) - f^{'}\left(X_t^\epsilon\right)\right)\frac{R_t^\epsilon}{\sigma(\epsilon)} d\theta \\ &\quad + \int_0^1 \mathbb{E}f^{'}\left(X_t^\epsilon\right)\frac{R_t^\epsilon}{\sigma(\epsilon)} d\theta \\ &= \int_0^1 \mathbb{E}\left(f^{'}\left(X_t^\epsilon + \theta R_t^\epsilon\right) - f^{'}\left(X_t^\epsilon\right)\right)\frac{R_t^\epsilon}{\sigma(\epsilon)} d\theta.\end{aligned}$$
Because R_t^ϵ and X_t^ϵ are independent and $\mathbb{E}R_t^\epsilon = 0$. Let $1 < \alpha < \beta$, by Hölder's inequality we have
$$\begin{aligned}\mathbb{E}\left|\left(f^{'}\left(X_t^\epsilon + \theta R_t^\epsilon\right) - f^{'}\left(X_t^\epsilon\right)\right)\frac{R_t^\epsilon}{\sigma(\epsilon)}\right| &\leq \left(\mathbb{E}\left|f^{'}\left(X_t^\epsilon + \theta R_t^\epsilon\right) - f^{'}\left(X_t^\epsilon\right)\right|^\alpha\right)^{\frac{1}{\alpha}} \\ &\quad \times \left(\mathbb{E}\left(\frac{R_t^\epsilon}{\sigma(\epsilon)}\right)^{\frac{\alpha}{\alpha-1}}\right)^{\frac{\alpha-1}{\alpha}}.\end{aligned}$$
Hence using the assumption of uniform integrability and proposition 3.1, we get
$$\mathbb{E}\left(f\left(X_t\right) - f\left(X_t^\epsilon\right)\right) = o\left(\sigma_0(\epsilon)\right).$$

On the other hand, using Taylor's formula we get

$$\begin{aligned}\mathbb{E}\left(f\left(X_t\right)-f\left(X_t^{\epsilon}\right)\right) &= \mathbb{E}f^{'}\left(X_t^{\epsilon}\right)\left(X_t-X_t^{\epsilon}\right)+\int_{X_t^{\epsilon}}^{X_t} f^{''}\left(x\right)\left(X_t-x\right)dx \\ &= \mathbb{E}f^{'}\left(X_t^{\epsilon}\right)R_t^{\epsilon}+\int_0^1 f^{''}\left(X_t^{\epsilon}+\theta R_t^{\epsilon}\right)\left(1-\theta\right)\left(R_t^{\epsilon}\right)^2 d\theta \\ &= \int_0^1 f^{''}\left(X_t^{\epsilon}+\theta R_t^{\epsilon}\right)\left(1-\theta\right)\left(R_t^{\epsilon}\right)^2 d\theta \\ &= \int_0^1 f^{''}\left(X_t^{\epsilon}\right)\left(1-\theta\right)\left(R_t^{\epsilon}\right)^2 d\theta \\ &\quad +\int_0^1 \Big(f^{''}\left(X_t^{\epsilon}+\theta R_t^{\epsilon}\right)-f^{''}\left(X_t^{\epsilon}\right)\Big)\left(1-\theta\right)\left(R_t^{\epsilon}\right)^2 d\theta.\end{aligned}$$

So

$$\begin{aligned}\mathbb{E}\left(f\left(X_t\right)-f\left(X_t^{\epsilon}\right)\right) &= \frac{t}{2}\mathbb{E}f^{''}\left(X_t^{\epsilon}\right)\sigma(\epsilon)^2 \\ &\quad +\int_0^1 \mathbb{E}\Big(f^{''}\left(X_t^{\epsilon}+\theta R_t^{\epsilon}\right)-f^{''}\left(X_t^{\epsilon}\right)\Big)\left(1-\theta\right)\left(R_t^{\epsilon}\right)^2 d\theta.\end{aligned}$$

Again, using Hölder's inequality, the assumption of uniform integrabilty and proposition 3.1, we get

$$\mathbb{E}\left(f\left(X_t\right)-f\left(X_t^{\epsilon}\right)\right)=\frac{\sigma(\epsilon)^2 t}{2}\mathbb{E}f^{''}\left(X_t^{\epsilon}\right)+o\left(\sigma_0(\epsilon)^2\right).$$

□

Remark 3.3. *If X is an integrable infinite activity Lévy process with generating triplet (γ,σ,ν) and $t>0$. If we also assume that $f\in C^1(\mathbb{R})$ and $f^{'}$ is Lipschitz, then*

$$\mathbb{E}\left(f\left(X_t\right)-f\left(X_t^{\epsilon}\right)\right)=O\left(\sigma(\epsilon)^2\right).$$

The proof is similar to the proof of proposition 3.2. If $f^{'}$ is C-Lipschitz, and if we denote by $f^{''}$ its a.e. derivative, we get $\left|f^{''}\right|\leq C$. So we can prove that

$$\mathbb{E}\left|f^{''}\left(X_t^{\epsilon}+\theta R_t^{\epsilon}\right)-f^{''}\left(X_t^{\epsilon}\right)\right|\left(R_t^{\epsilon}\right)^2\leq 2Ct\sigma(\epsilon)^2.$$

This concludes the proof.

When we consider functions X-path-dependent, errors in Lipschitz case are similar to that obtained above. We will consider now the case of the supremum process.

Proposition 3.4. *Let X be a Lévy process with generating triplet (γ,σ,ν), f a K-Lipschitz function, $\epsilon\in(0,1]$ and $t>0$, then we have*

$$\mathbb{E}\left|f\left(M_t\right)-f\left(M_t^{\epsilon}\right)\right|\leq 2K\sqrt{t}\sigma(\epsilon).$$

Proof. Consider R^ϵ defined in (2.2). So R^ϵ is a martingale. Thus

$$\begin{aligned}\mathbb{E}\left|f\left(\sup_{0\le s\le t}X_s\right)-f\left(\sup_{0\le s\le t}X_s^\epsilon\right)\right| &\le K\mathbb{E}\left|\sup_{0\le s\le t}X_s-\sup_{0\le s\le t}X_s^\epsilon\right|\\ &\le K\mathbb{E}\sup_{0\le s\le t}|X_s-X_s^\epsilon|\\ &\le K\mathbb{E}\sup_{0\le s\le t}|R_s^\epsilon|\\ &\le K\sqrt{\mathbb{E}\left(\sup_{0\le s\le t}|R_s^\epsilon|\right)^2}.\end{aligned}$$

U Doob's inequality, we get

$$\begin{aligned}\mathbb{E}\left|f\left(\sup_{0\le s\le t}X_s\right)-f\left(\sup_{0\le s\le t}X_s^\epsilon\right)\right| &\le 2K\sqrt{\mathbb{E}\left|R_s^\epsilon\right|^2}\\ &= 2K\sqrt{t}\sigma(\epsilon).\end{aligned}$$

□

REMARK 3.5. *If X is a Lévy process with generating triplet (γ,σ^2,ν), $\epsilon\in(0,1]$, $t>0$ and f a function from $\mathbb{R}^+\times\mathbb{R}$ to $\mathbb{R}$, K-Lipschitz with respect to its second variable. Then*

$$\left|\sup_{\tau\in\mathcal{T}_{[0,t]}}\mathbb{E}f\left(\tau,X_\tau\right)-\sup_{\tau\in\mathcal{T}_{[0,t]}}\mathbb{E}f\left(\tau,X_\tau^\epsilon\right)\right|\le 2K\sqrt{t}\sigma(\epsilon),$$

where $\mathcal{T}_{[0,t]}$ denote the set of stopping times with values in $[0,t]$. The proof can be found in [6].

The bound in proposition 3.5 might not be optimal. This is what suggests the following result.

THEOREM 3.6. *Let X be an integrable infinite activity Lévy process with generating triplet (γ,σ^2,ν), $\epsilon\in(0,1]$ and $t>0$, then*

$$0\le\mathbb{E}\left(M_t-M_t^\epsilon\right)=o\left(\sigma(\epsilon)\right).$$

Proof. Using Spitzer's identity (see Proposition 3.2 of [6] for details), we have

$$\begin{aligned}\mathbb{E}\left(M_t-M_t^\epsilon\right) &= \int_0^t\frac{\mathbb{E}X_s^+}{s}ds-\int_0^t\frac{\mathbb{E}\left(X_s^\epsilon\right)^+}{s}ds\\ &= \int_0^t\mathbb{E}\left(X_s^+-\left(X_s^\epsilon\right)^+\right)\frac{ds}{s}\end{aligned}$$

Set $I_s^\epsilon = \mathbb{E}\left(X_s^+ - (X_s^\epsilon)^+\right)$. So

$$\begin{aligned}
I_s^\epsilon &= \mathbb{E}\left(X_s^+ - (X_s^\epsilon)^+\right)\\
&= \mathbb{E}\left((X_s^\epsilon + R_s^\epsilon)^+ - (X_s^\epsilon)^+\right)\\
&= \mathbb{E}X_s^\epsilon\left(\mathbb{1}_{X_s^\epsilon + R_s^\epsilon > 0} - \mathbb{1}_{X_s^\epsilon > 0}\right) + \mathbb{E}R_s^\epsilon \mathbb{1}_{X_s^\epsilon + R_s^\epsilon > 0}\\
&= \mathbb{E}X_s^\epsilon\left(\mathbb{1}_{-R_s^\epsilon < X_s^\epsilon \le 0} - \mathbb{1}_{0 < X_s^\epsilon \le -R_s^\epsilon}\right) + \mathbb{E}R_s^\epsilon \mathbb{1}_{X_s^\epsilon > -R_s^\epsilon}\\
&= \mathbb{E}X_s^\epsilon\left(\mathbb{1}_{-R_s^\epsilon < X_s^\epsilon \le 0} - \mathbb{1}_{0 < X_s^\epsilon \le -R_s^\epsilon}\right) + \mathbb{E}R_s^\epsilon \mathbb{1}_{-R_s^\epsilon < X_s^\epsilon \le 0}\\
&\quad + \mathbb{E}R_s^\epsilon \mathbb{1}_{X_s^\epsilon > -R_s^\epsilon, X_s^\epsilon > 0}\\
&= \mathbb{E}X_s^\epsilon\left(\mathbb{1}_{-R_s^\epsilon < X_s^\epsilon \le 0} - \mathbb{1}_{0 < X_s^\epsilon \le -R_s^\epsilon}\right) + \mathbb{E}R_s^\epsilon \mathbb{1}_{-R_s^\epsilon < X_s^\epsilon \le 0}\\
&\quad + \mathbb{E}R_s^\epsilon\left(\mathbb{1}_{X_s^\epsilon > 0} - \mathbb{1}_{0 < X_s^\epsilon \le -R_s^\epsilon}\right).
\end{aligned}$$

Besides R^ϵ and X^ϵ are independent and $\mathbb{E}R^\epsilon = 0$, thus

$$\begin{aligned}
I_s^\epsilon &= \mathbb{E}X_s^\epsilon\left(\mathbb{1}_{-R_s^\epsilon < X_s^\epsilon \le 0} - \mathbb{1}_{0 < X_s^\epsilon \le -R_s^\epsilon}\right) + \mathbb{E}R_s^\epsilon \mathbb{1}_{-R_s^\epsilon < X_s^\epsilon \le 0}\\
&\quad + \mathbb{E}R_s^\epsilon \mathbb{1}_{0 < X_s^\epsilon \le -R_s^\epsilon}\\
&= \mathbb{E}\left(X_s^\epsilon + R_s^\epsilon\right)\mathbb{1}_{-R_s^\epsilon < X_s^\epsilon \le 0} - \mathbb{E}\left(X_s^\epsilon + R_s^\epsilon\right)\mathbb{1}_{0 < X_s^\epsilon \le -R_s^\epsilon}\\
&= \mathbb{E}\left(|R_s^\epsilon| - |X_s^\epsilon|\right)\mathbb{1}_{-R_s^\epsilon < X_s^\epsilon \le 0} - \mathbb{E}\left(|X_s^\epsilon| - |R_s^\epsilon|\right)\mathbb{1}_{0 < X_s^\epsilon \le -R_s^\epsilon}\\
&= \mathbb{E}\left(|R_s^\epsilon| - |X_s^\epsilon|\right)\left(\mathbb{1}_{-R_s^\epsilon < X_s^\epsilon \le 0} + \mathbb{1}_{0 < X_s^\epsilon \le -R_s^\epsilon}\right)\\
&= \mathbb{E}\left(|R_s^\epsilon| - |X_s^\epsilon|\right)^+\left(\mathbb{1}_{-R_s^\epsilon < X_s^\epsilon \le 0} + \mathbb{1}_{0 < X_s^\epsilon \le -R_s^\epsilon}\right).
\end{aligned}$$

But

$$\mathbb{1}_{-R_s^\epsilon < X_s^\epsilon \le 0} + \mathbb{1}_{0 < X_s^\epsilon \le -R_s^\epsilon} = \mathbb{1}_{R_s^\epsilon X_s^\epsilon < 0} + \mathbb{1}_{X_s^\epsilon = 0, R_s^\epsilon < 0}.$$

Hence

$$\begin{aligned}
I_s^\epsilon &= \mathbb{E}\left(|R_s^\epsilon| - |X_s^\epsilon|\right)^+\left(\mathbb{1}_{R_s^\epsilon X_s^\epsilon < 0} + \mathbb{1}_{X_s^\epsilon = 0, R_s^\epsilon < 0}\right)\\
&= \mathbb{E}\left(|R_s^\epsilon| - |X_s^\epsilon|\right)^+ \mathbb{1}_{R_s^\epsilon X_s^\epsilon < 0} + \mathbb{E}\left|R_s^\epsilon\right| \mathbb{1}_{X_s^\epsilon = 0, R_s^\epsilon < 0}\\
&= \mathbb{E}\left(|R_s^\epsilon| - |X_s^\epsilon|\right)^+ \mathbb{1}_{R_s^\epsilon X_s^\epsilon < 0} + \mathbb{E}\left|R_s^\epsilon\right| \mathbb{1}_{R_s^\epsilon < 0}\mathbb{P}\left[X_s^\epsilon = 0\right]\\
&= \mathbb{E}\left(|R_s^\epsilon| - |X_s^\epsilon|\right)^+ \mathbb{1}_{R_s^\epsilon X_s^\epsilon < 0} + \mathbb{E}\left(R_s^\epsilon\right)^- \mathbb{P}\left[X_s^\epsilon = 0\right].
\end{aligned}$$

But $I_s^\epsilon \ge 0$, therefore

$$\mathbb{E}\left(M_t - M_t^\epsilon\right) \ge 0.$$

On the other hand, using Cauchy-Scwarz inequality

$$\begin{aligned}
I_s^\epsilon &\le \left(\mathbb{E}\left|R_s^\epsilon\right|^2\right)^{\frac{1}{2}}\left(\mathbb{E}\left(\left(1 - \frac{|X_s^\epsilon|}{|R_s^\epsilon|}\right)^+\right)^2\right)^{\frac{1}{2}} + \left(\mathbb{E}\left|R_s^\epsilon\right|^2\right)^{\frac{1}{2}}\mathbb{P}\left[X_s^\epsilon = 0\right]\\
&\le \sigma(\epsilon)\sqrt{s}\left(\left(\mathbb{E}\left(\left(1 - \frac{|X_s^\epsilon|}{|R_s^\epsilon|}\right)^+\right)^2\right)^{\frac{1}{2}} + \mathbb{P}\left[X_s^\epsilon = 0\right]\right).
\end{aligned}$$

Note that $\nu(\mathbb{R}) = +\infty$, so $R_s^\epsilon \neq 0$ $a.s.$ Then we get

$$\mathbb{E}\left(M_t - M_t^\epsilon\right) \leq \sigma(\epsilon) \int_0^t \left(\left(\mathbb{E}\left(\left(1 - \frac{|X_s^\epsilon|}{|R_s^\epsilon|} \right)^+ \right)^2 \right)^{\frac{1}{2}} + \mathbb{P}\left[X_s^\epsilon = 0\right] \right) \frac{ds}{\sqrt{s}}.$$

Furthermore by dominated convergence, we have

$$\lim_{\epsilon \to 0} \mathbb{E}\left(\left(1 - \frac{|X_s^\epsilon|}{|R_s^\epsilon|} \right)^+ \right)^2 = 0.$$

Indeed $R_s^\epsilon \to 0$ $a.s.$ and $X_s^\epsilon \to X_s$ $a.s.$ with $X_s \neq 0$ $a.s.$ We also have

$$\begin{aligned} \lim_{\epsilon \to 0} \mathbb{P}\left[X_s^\epsilon = 0\right] &= \mathbb{P}\left[X_s = 0\right] \\ &= 0. \end{aligned}$$

On the other hand

$$\left(\mathbb{E}\left(\left(1 - \frac{|X_s^\epsilon|}{|R_s^\epsilon|} \right)^+ \right)^2 \right)^{\frac{1}{2}} + \mathbb{P}\left[X_s^\epsilon = 0\right] \leq 2.$$

Therefore by dominated convergence

$$\lim_{\epsilon \to 0} \int_0^t \left(\left(\mathbb{E}\left(\left(1 - \frac{|X_s^\epsilon|}{|R_s^\epsilon|} \right)^+ \right)^2 \right)^{\frac{1}{2}} + \mathbb{P}\left[X_s^\epsilon = 0\right] \right) \frac{ds}{\sqrt{s}} = 0.$$

Hence

$$\mathbb{E}\left(M_t - M_t^\epsilon\right) = o\left(\sigma(\epsilon)\right)$$

□

Remark 3.7. *The result of theorem 3.6 is optimal, in the sense that we cannot have a better power for $\sigma(\epsilon)$.*

Indeed, in the previous proof we have shown that

$$\mathbb{E}\left(X_s\right)^+ - \mathbb{E}\left(X_s^\epsilon\right)^+ = \mathbb{E}\left(|R_s^\epsilon| - |X_s^\epsilon|\right)^+ \mathbb{1}_{R_s^\epsilon X_s^\epsilon < 0} + \mathbb{E}\left(R_s^\epsilon\right)^- \mathbb{P}\left[X_s^\epsilon = 0\right].$$

So

$$\begin{aligned} \mathbb{E}\left(X_s\right)^+ - \mathbb{E}\left(X_s^\epsilon\right)^+ &\geq \mathbb{E}\left(R_s^\epsilon\right)^- \mathbb{P}\left[X_s^\epsilon = 0\right] \\ &= \frac{1}{2}\mathbb{E}\left|R_s^\epsilon\right| \mathbb{P}\left[X_s^\epsilon = 0\right], \end{aligned}$$

because $\mathbb{E}R_s^\epsilon = 0$. Let X be a Lévy process with generating triplet $(0, 0, \nu)$ where

$$\nu(dx) = \alpha \mathbb{1}_{0<|x|<1} \frac{dx}{|x|^{1+\alpha}},$$

and $\alpha \in]0, 2[$. We have

$$\sigma(\epsilon) = \left(\frac{2\alpha}{2-\alpha} \right)^{\frac{1}{2}} \epsilon^{1-\frac{\alpha}{2}}.$$

We can prove that there exists a positive constant $C_\alpha > 0$ such that

$$\mathbb{E}\left(M_t - M_t^\epsilon\right) \geq C_\alpha \sigma(\epsilon)^{\frac{2}{2-\alpha}}.$$

This justifies the above remark.

In financial applications, the function f defined in proposition 3.4 is not always Lipschitz, as for call lookback option where the function is exponential. Hence the following proposition.

PROPOSITION 3.8. *Let X be a Lévy process with generating triplet (γ, σ^2, ν), $p > 1$, $\epsilon \in (0,1]$ and $t > 0$. We suppose that $\mathbb{E}e^{pM_t} < \infty$, then*

$$\mathbb{E}\left|e^{M_t} - e^{M_t^\epsilon}\right| \leq C_p \sigma_0(\epsilon),$$

where C_p is a positive constant independent of ϵ.

In fact we have to ensure that $\mathbb{E}e^{pM_t^\epsilon}$ is bounded by a constant which is independent of ϵ.

LEMMA 3.9. *Let $p > 0$ and $\epsilon \in (0,1]$. If $\mathbb{E}e^{pM_t} < \infty$, then*

$$\sup_{0 \leq \delta \leq 1} \mathbb{E}e^{pM_t^\delta} < \infty$$

REMARK 3.10. *For any $p > 0$, $\mathbb{E}e^{pM_t} < \infty$ if only if $\int_{x>1} e^{px}\nu(dx) < \infty$.*

Proof of lemma 3.9. Let $\delta \in (0,1]$,

$$X_s^1 = \gamma s + \sigma B_s + \int_{0 \leq \tau \leq s, |x|>1} J_X(ds \times dx)$$
$$\bar{R}_s^\delta = \int_{0 \leq \tau \leq s, \delta < |x| \leq 1} \tilde{J}_X(ds \times dx)$$

So

$$\begin{aligned}\mathbb{E}e^{pM_t^\delta} &\leq \mathbb{E}e^{p\sup_{0\leq s\leq t} X_s^1 + \sup_{0\leq s\leq t} \bar{R}_s^\delta} \\ &\leq \mathbb{E}e^{p\sup_{0\leq s\leq t} X_s^1}\mathbb{E}e^{\sup_{0\leq s\leq t}\left|\bar{R}_s^\delta\right|}\end{aligned}$$

By hypothesis and remark 3.10, $\mathbb{E}e^{p\sup_{0\leq s\leq t} X_s^1} < \infty$. We need to bound $\mathbb{E}e^{p\sup_{0\leq s\leq t}\left|\bar{R}_s^\delta\right|}$ independently of δ. We have

$$\begin{aligned}\mathbb{E}e^{p\sup_{0\leq s\leq t}\left|\bar{R}_s^\delta\right|} &= \mathbb{E}\sum_{n=0}^{+\infty} \frac{\left(p\sup_{0\leq s\leq t}\left|\bar{R}_s^\delta\right|\right)^n}{n!} \\ &= \mathbb{E}\sum_{n=0}^{+\infty} \frac{p^n}{n!} \sup_{0\leq s\leq t}\left|\bar{R}_s^\delta\right|^n \\ &= \sum_{n=0}^{+\infty} \frac{p^n}{n!}\mathbb{E}\sup_{0\leq s\leq t}\left|\bar{R}_s^\delta\right|^n \\ &= 1 + p\mathbb{E}\sup_{0\leq s\leq t}\left|\bar{R}_s^\delta\right| + \sum_{n=2}^{+\infty} \frac{p^n}{n!}\mathbb{E}\sup_{0\leq s\leq t}\left|\bar{R}_s^\delta\right|^n\end{aligned}$$

By Doob's inequality ($\bar{R}^\delta$ is a martingale)

$$\begin{aligned}
\mathbb{E}e^{p\sup_{0\le s\le t}\left|\bar{R}_s^\delta\right|} &\le 1 + p\sqrt{\mathbb{E}\sup_{0\le s\le t}\left|\bar{R}_s^\delta\right|^2} + \sum_{n=2}^{+\infty}\frac{p^n}{n!}\left(\frac{n}{n-1}\right)^n \mathbb{E}\left|\bar{R}_t^\delta\right|^n \\
&\le 1 + 2p\sqrt{\mathbb{E}\left|\bar{R}_t^\delta\right|^2} + \sum_{n=2}^{+\infty}\frac{p^n}{n!}2^n\mathbb{E}\left|\bar{R}_t^\delta\right|^n \\
&= 2p\left(\sqrt{\mathbb{E}\left|\bar{R}_t^\delta\right|^2} - \mathbb{E}\left|\bar{R}_t^\delta\right|\right) + \sum_{n=0}^{+\infty}\frac{2^n p^n}{n!}\mathbb{E}\left|\bar{R}_t^\delta\right|^n
\end{aligned}$$

Hence

$$\begin{aligned}
\mathbb{E}e^{p\sup_{0\le s\le t}\left|\bar{R}_s^\delta\right|} &\le 2p\left(\sqrt{\mathbb{E}\left|\bar{R}_t^\delta\right|^2} - \mathbb{E}\left|\bar{R}_t^\delta\right|\right) + \mathbb{E}\sum_{n=0}^{+\infty}\frac{2^n p^n}{n!}\left|\bar{R}_t^\delta\right|^n \\
&= 2p\left(\sqrt{\mathbb{E}\left|\bar{R}_t^\delta\right|^2} - \mathbb{E}\left|\bar{R}_t^\delta\right|\right) + \mathbb{E}e^{2p\left|\bar{R}_t^\delta\right|} \\
&\le 2p\left(\sqrt{\mathbb{E}\left|\bar{R}_t^\delta\right|^2} - \mathbb{E}\left|\bar{R}_t^\delta\right|\right) + \mathbb{E}e^{2p\bar{R}_t^\delta} + \mathbb{E}e^{-2p\bar{R}_t^\delta}
\end{aligned}$$

But

$$\begin{aligned}
\mathbb{E}\left|\bar{R}_t^\delta\right| &\le \sqrt{\mathbb{E}\left|\bar{R}_t^\delta\right|^2} \\
&\le \sqrt{t\int_{\delta<|x|\le 1} x^2\nu(dx)} \\
&\le \sqrt{t\int_{|x|\le 1} x^2\nu(dx)}
\end{aligned}$$

And we can prove that for any $\beta\in\mathbb{R}$, $\left(e^{\beta\bar{R}_t^\delta}\right)_{0\le\delta\le 1}$ is uniformly integrable. Hence

$$\sup_{0\le\delta\le 1}\mathbb{E}e^{p\sup_{0\le s\le t}\left|\bar{R}_s^\delta\right|} < \infty$$

□

Proof of proposition 3.8. By the mean value theorem, we have

$$e^{M_t} - e^{M_t^\epsilon} = \left(M_t - M_t^\epsilon\right)e^{\bar{M}_t^\epsilon}$$

where $\bar{M}_t^\epsilon$ is between M_t and M_t^ϵ. Let q be defined such that $\frac{1}{p}+\frac{1}{q}=1$. In the sequel C_p will denote a constant depending on p.

$$\begin{aligned}
\mathbb{E}\left|e^{M_t} - e^{M_t^\epsilon}\right| &\le \mathbb{E}\left|M_t - M_t^\epsilon\right| e^{\bar{M}_t^\epsilon} \\
&\le \mathbb{E}\sup_{0\le s\le t}\left|R_t^\epsilon\right| e^{\bar{M}_t^\epsilon} \\
&\le \left(\mathbb{E}\sup_{0\le s\le t}\left|R_t^\epsilon\right|^q\right)^{\frac{1}{q}}\left(\mathbb{E}e^{p\bar{M}_t^\epsilon}\right)^{\frac{1}{p}}
\end{aligned}$$

Hence, using Doob's inequality and then proposition 3.1, we get

$$\mathbb{E}\left|e^{M_t}-e^{M_t^\epsilon}\right| \leq \frac{q}{q-1}\left(\mathbb{E}\left|R_t^\epsilon\right|^q\right)^{\frac{1}{q}}\left(\mathbb{E}e^{p\bar{M}_t^\epsilon}\right)^{\frac{1}{p}}$$
$$\leq C_p\sigma_0(\epsilon)\left(\mathbb{E}\left(e^{pM_t}+e^{pM_t^\epsilon}\right)\right)^{\frac{1}{p}}.$$

We conclude the proof by lemma 3.9. ∎

3.2. Estimates for cumulative distribution functions. For cumulative distribution functions bounds are expected to be bigger. However, in some cases we can get similar results as in Lipschitz case. Consider the following result which will be useful for the next proofs.

PROPOSITION 3.11. *Let X and Y be two r.v. We assume that X has a bounded density in a neighborhood of $x \in \mathbb{R}$, and there exists $p \geq 1$ such that $\mathbb{E}\left|X-Y\right|^p$ is finite. Then there exists a constant $K_x > 0$, such that for any $\delta > 0$*

$$\left|\mathbb{P}\left[X \geq x\right]-\mathbb{P}\left[Y \geq x\right]\right| \leq K_x\delta+\frac{\mathbb{E}\left|X-Y\right|^p}{\delta^p}.$$

Proof. We have

$$\left|\mathbb{P}\left[X \geq x\right]-\mathbb{P}\left[Y \geq x\right]\right| = \left|\mathbb{P}\left[X \geq x, Y < x\right]-\mathbb{P}\left[X < x, Y \geq x\right]\right|.$$

We will study the above terms on the right of the equality.

$$\begin{aligned}\mathbb{P}\left[X \geq x, Y < x\right] &= \mathbb{P}\left[x \leq X < x+(X-Y)\right]\\ &= \mathbb{P}\left[x \leq X < x+(X-Y), |X-Y| \leq \delta\right]\\ &\quad +\mathbb{P}\left[x \leq X < x+(X-Y), |X-Y| > \delta\right]\\ &\leq \mathbb{P}\left[x \leq X < x+\delta\right]+\mathbb{P}\left[|X-Y| > \delta\right].\end{aligned}$$

But the probability density function of X is bounded in the neighborhood of x. So there exists a constant $K_x^1 > 0$ such that

$$\mathbb{P}\left[x \leq X < x+\delta\right] \leq K_x^1\delta.$$

Hence

$$\begin{aligned}\mathbb{P}\left[X \geq x, Y < x\right] &\leq K_x^1\delta+\mathbb{P}\left[|X-Y| > \delta\right]\\ &\leq K_x^1\delta+\frac{\mathbb{E}\left|X-Y\right|^p}{\delta^p}, \text{ by Markov's inequality.}\end{aligned}$$

On the other hand,

$$\begin{aligned}\mathbb{P}\left[X < x, Y \geq x\right] &= \mathbb{P}\left[x-(X-Y) \leq X < x\right]\\ &= \mathbb{P}\left[x-(X-Y) \leq X < x, |X-Y| \leq \delta\right]\\ &\quad +\mathbb{P}\left[x-(X-Y) \leq X < x, |X-Y| > \delta\right]\\ &\leq \mathbb{P}\left[x-\delta \leq X < x\right]+\mathbb{P}\left[|X-Y| > \delta\right].\end{aligned}$$

The boundedness of the probability density function of X in the neighborhood of x, also yields that there exists a constant $K_x^2 > 0$ such that

$$\mathbb{P}\left[x-\delta \leq X < x\right] \leq K_x^2\delta.$$

Hence

$$\mathbb{P}\left[X < x, Y \geq x\right] \leq K_x^2 \delta + \frac{\mathbb{E}\left|X-Y\right|^p}{\delta^p}.$$

Therefore

$$\left|\mathbb{P}\left[X \geq x\right] - \mathbb{P}\left[Y \geq x\right]\right| \leq \max\left(K_x^1, K_x^2\right)\delta + \frac{\mathbb{E}\left|X-Y\right|^p}{\delta^p}.$$

□

In the first result below, we assume local boundedness of the probability density function of the Lévy process X and its supremum process M at fixed time t.

PROPOSITION 3.12. *Let X be a Lévy process with generating triplet (γ, σ^2, ν), $\epsilon \in (0,1]$ and $t > 0$.*

1. *If $\sigma > 0$, then*

$$\sup_{x\in\mathbb{R}} \left|\mathbb{P}\left[X_t \geq x\right] - \mathbb{P}\left[X_t^\epsilon \geq x\right]\right| \leq \frac{1}{\sqrt{2\pi}\sigma}\sigma(\epsilon).$$

2. *If X_t has a locally bounded probability density function, then for any $q \in (0,1)$,*

$$\left|\mathbb{P}\left[X_t \geq x\right] - \mathbb{P}\left[X_t^\epsilon \geq x\right]\right| \leq C_{x,t,q}\sigma_0(\epsilon)^{1-q}.$$

3. *If M_t has a locally bounded probability density function on $(0, +\infty)$, then for any $q \in (0,1)$,*

$$\left|\mathbb{P}\left[M_t \geq x\right] - \mathbb{P}\left[M_t^\epsilon \geq x\right]\right| \leq C_{x,t,q}\sigma_0(\epsilon)^{1-q},$$

where $C_{x,t,q}$ means a positive constant depending on x, q and t.

Proof. We have

$$\left|\mathbb{P}\left[X_t \geq x\right] - \mathbb{P}\left[X_t^\epsilon \geq x\right]\right| = \left|\mathbb{P}\left[X_t \geq x, X_t^\epsilon < x\right] - \mathbb{P}\left[X_t < x, X_t^\epsilon \geq x\right]\right|.$$

But, in the case $\sigma > 0$

$$\begin{aligned}\mathbb{P}\left[X_t \geq x, X_t^\epsilon < x\right] &= \mathbb{P}\left[x - (X_t - X_t^\epsilon) \leq X_t^\epsilon < x\right] \\ &= \mathbb{P}\left[x - R_t^\epsilon \leq \sigma B_t + (X_t^\epsilon - \sigma B_t) < x\right].\end{aligned}$$

Note that the r.v. σB_t, $(X_t^\epsilon - \sigma B_t)$ and R_t^ϵ are independent, and $\frac{1}{\sqrt{2\pi t}\sigma}$ is a bound of the probability density function of σB_t. Then

$$\begin{aligned}\mathbb{P}\left[X_t \geq x, X_t^\epsilon < x\right] &\leq \frac{1}{\sqrt{2\pi t}\sigma}\mathbb{E}\left|R_t^\epsilon\right| \\ &\leq \frac{1}{\sqrt{2\pi t}\sigma}\sqrt{\mathbb{E}\left|R_t^\epsilon\right|^2} \\ &= \frac{1}{\sqrt{2\pi}\sigma}\sigma(\epsilon).\end{aligned}$$

Similarly

$$\begin{aligned}\mathbb{P}\left[X_t < x, X_t^\epsilon \geq x\right] &= \mathbb{P}\left[x \leq X_t^\epsilon < x + (X_t - X_t^\epsilon)\right] \\ &= \mathbb{P}\left[x \leq \sigma B_t + (X_t^\epsilon - \sigma B_t) < x + R_t^\epsilon\right] \\ &\leq \frac{1}{\sqrt{2\pi}\sigma}\sigma(\epsilon).\end{aligned}$$

So

$$|\mathbb{P}[X_t \geq x] - \mathbb{P}[X_t^\epsilon \geq x]| \leq \frac{1}{\sqrt{2\pi}\sigma}\sigma(\epsilon).$$

Consider now the second part of the proposition. By proposition 3.11, there exists a constant $K_{x,t} > 0$ such that for any $p > 1$, we have

$$\begin{aligned}|\mathbb{P}[X_t \geq x] - \mathbb{P}[X_t^\epsilon \geq x]| &\leq K_{x,t}\delta + \frac{\mathbb{E}|X_t - X_t^\epsilon|^p}{\delta^p} \\ &= K_{x,t}\delta + \frac{\mathbb{E}|R_t^\epsilon|^p}{\delta^p}.\end{aligned}$$

But by proposition 3.1, there exists a constant $K_{p,t} > 0$ such that

$$\mathbb{E}|R_t^\epsilon|^p \leq K_{p,t}\sigma_0(\epsilon)^p.$$

So

$$\begin{aligned}|\mathbb{P}[X_t \geq x] - \mathbb{P}[X_t^\epsilon \geq x]| &\leq K_{x,t}\delta + K_{p,t}\frac{\sigma_0(\epsilon)^p}{\delta^p} \\ &\leq \max(K_{x,t}, K_{p,t})\left(\delta + \frac{\sigma_0(\epsilon)^p}{\delta^p}\right).\end{aligned}$$

Thus by choosing $\delta = \sigma(\epsilon)^{\frac{p}{p+1}}$, we get

$$|\mathbb{P}[X_t \geq x] - \mathbb{P}[X_t^\epsilon \geq x]| \leq \max(K_{x,t}, K_{p,t})\,\sigma_0(\epsilon)^{\frac{p}{p+1}}.$$

Therefore for any $q \in (0,1)$, we have

$$|\mathbb{P}[X_t \geq x] - \mathbb{P}[X_t^\epsilon \geq x]| \leq C_{x,t,q}\sigma(\epsilon)^{1-q}.$$

For the third part of the proposition, set

$$I = |\mathbb{P}[M_t \geq x] - \mathbb{P}[M_t^\epsilon \geq x]|.$$

By proposition 3.11, there exists a constant $K'_{x,t} > 0$ such that

$$I \leq K'_{x,t}\delta + \frac{\mathbb{E}(M_t - M_t^\epsilon)^p}{\delta^p}.$$

On the other hand

$$\begin{aligned}\mathbb{E}(M_t - M_t^\epsilon)^p &\leq \mathbb{E}\left(\sup_{0\leq s\leq t}|X_s - X_s^\epsilon|\right)^p \\ &\leq \mathbb{E}\left(\sup_{0\leq s\leq t}|R_s^\epsilon|\right)^p.\end{aligned}$$

By Doob's inequality, we have

$$\begin{aligned}\mathbb{E}(M_t - M_t^\epsilon)^p &\leq \left(\frac{p}{p-1}\right)^p \mathbb{E}|R_t^\epsilon|^p \\ &\leq K_{p,t}\left(\frac{p}{p-1}\right)^p \sigma_0(\epsilon)^p.\end{aligned}$$

Therefore, by choosing $\delta = \sigma(\epsilon)^{\frac{p}{p+1}}$, we have

$$I \leq \max\left(K_{x,t}^{'}, K_{p,t}\left(\frac{p}{p-1}\right)^{p}\right)\sigma_0(\epsilon)^{\frac{p}{p+1}}.$$

So for $q \in (0,1)$, we have

$$I \leq C_{x,t,q}\sigma_0(\epsilon)^{1-q}.$$

□

REMARK 3.13. *When there is no assumption on the boundedness of the probability density function of M, the upper bounds for cumulative distribution functions are bigger. If $\sigma > 0$ then*

$$|\mathbb{P}[M_t \geq x] - \mathbb{P}[M_t^{\epsilon} \geq x]| \leq C\sigma_0(\epsilon)^{\frac{2}{3}}$$

If $\sigma = 0$ and there exists $\alpha > 0$ such that

$$\liminf_{\epsilon\downarrow 0} \epsilon^{-\alpha}\int_{-\epsilon}^{\epsilon}|x|^2 d\nu(x) > 0 \tag{3.3}$$

then for any $\theta > 0$

$$|\mathbb{P}[M_t \geq x] - \mathbb{P}[M_t^{\epsilon} \geq x]| \leq C_{\theta}\sigma_0(\epsilon)^{\frac{\alpha}{(1+\theta)\alpha+1}}$$

Constants C et C_θ are independent of ϵ. The proof can be found in [6].

4. Approximation of small jumps by a Brownian motion. In this method we will replace R^{ϵ} by a Brownian motion. This method gives better results than the truncation one, subject to a convergence assumption. In fact, Asmussen and Rosinski proved ([1], theorem 2.1) that, if X is a Lévy process, then the process $\sigma(\epsilon)^{-1}R^{\epsilon}$ converges in distribution to a standard Brownian motion, when $\epsilon \to 0$, if only if for any $k > 0$

$$\lim_{\epsilon\to 0}\frac{\sigma\left(k\sigma(\epsilon)\wedge\epsilon\right)}{\sigma\left(\epsilon\right)} = 1. \tag{4.1}$$

This result is implied by the condition

$$\lim_{\epsilon\to 0}\frac{\sigma(\epsilon)}{\epsilon} = +\infty. \tag{4.2}$$

The conditions (4.1) and (4.2) are equivalent, if ν does not have atoms in some neighborhood of zero ([1], proposition 2.1).

4.1. Estimates for smooth functions. The errors resulting from Brownian approximation have not been much studied in the literature, at least theoretically. There is a result which we can find [5] (proposition 6.2).

PROPOSITION 4.1. *Let X be an infinite activity Lévy process with generating triplet (γ, σ, ν) and $t > 0$,*

1. *If $f \in C^1(\mathbb{R})$ and satisfies $\mathbb{E}\left|f'\left(X_t^{\epsilon}\right)\right| < \infty$, and if there exists $\beta > 1$ such that $\left(\sup_{\epsilon\in[0,1]}\mathbb{E}\left|f'\left(X_t^{\epsilon} + \theta\sigma(\epsilon)\hat{W}_t\right) - f'\left(X_t^{\epsilon}\right)\right|^{\beta}\right)^{\frac{1}{\beta}}$ and*

$\left(\sup_{\epsilon\in[0,1]}\mathbb{E}\left|f^{'}\left(X_t^\epsilon+\theta R_t^\epsilon\right)-f'\left(X_t^\epsilon\right)\right|^\beta\right)^{\frac{1}{\beta}}$ *are finite and integrable with respect to* θ *on* $[0,1]$*, then*

$$\mathbb{E}\left(f\left(X_t\right)-f\left(\hat{X}_t^\epsilon\right)\right)=o\left(\sigma_0(\epsilon)\right).$$

2. If $f\in C^2(\mathbb{R})$ *and satisfies* $\mathbb{E}\left|f'\left(X_t^\epsilon\right)\right|+\mathbb{E}\left|f''\left(X_t^\epsilon\right)\right|<\infty$*, and if there exists* $\beta>1$ *such that* $\left(\sup_{\epsilon\in[0,1]}\mathbb{E}\left|f''\left(X_t^\epsilon+\theta\sigma(\epsilon)\hat{W}_t\right)-f''\left(X_t^\epsilon\right)\right|^\beta\right)^{\frac{1}{\beta}}$ *and* $\left(\sup_{\epsilon\in[0,1]}\mathbb{E}\left|f''\left(X_t^\epsilon+\theta R_t^\epsilon\right)-f''\left(X_t^\epsilon\right)\right|^\beta\right)^{\frac{1}{\beta}}$ *are finite and integrable with respect to* θ *on* $[0,1]$*, then*

$$\mathbb{E}\left(f\left(X_t\right)-f\left(\hat{X}_t^\epsilon\right)\right)=o\left(\sigma_0(\epsilon)^2\right).$$

The remark we did next to the proposition 3.2 is still true here.

Proof. By proposition 3.2, we have

$$\mathbb{E}\left(f\left(X_t\right)-f\left(X_t^\epsilon\right)\right)=\frac{\sigma(\epsilon)^2t}{2}\mathbb{E}f^{''}\left(X_t^\epsilon\right)+o\left(\sigma_0(\epsilon)^2\right)$$

On the other hand, using the same reasoning as the proof of proposition 3.2 (we will replace R^ϵ by $\sigma(\epsilon)\hat{W}$) we get

$$\mathbb{E}\left(f\left(X_t^\epsilon+\sigma(\epsilon)\hat{W}_t\right)-f\left(X_t^\epsilon\right)\right)=\frac{\sigma(\epsilon)^2t}{2}\mathbb{E}f^{''}\left(X_t^\epsilon\right)+o\left(\sigma_0(\epsilon)^2\right)$$

Hence

$$\mathbb{E}\left(f\left(X_t\right)-f\left(\hat{X}_t^\epsilon\right)\right)=o\left(\sigma_0(\epsilon)^2\right)$$

□

The combination of proposition 6.2 of [5] and the Spitzer's identity for Lévy processes ([6], proposition 3.2) leads to the following result.

PROPOSITION 4.2. *Let* X *be an integrable infinite activity Lévy process with generating triplet* (γ,σ^2,ν)*, then*

$$\left|\mathbb{E}M_t-\mathbb{E}\hat{M}_t^\epsilon\right|\le 4\max\left(1+\sqrt{\frac{2}{\pi}},A\right)\sigma(\epsilon)\rho(\epsilon)\left(1+\log\left(\frac{\sqrt{t}}{2\rho(\epsilon)}\right)\right),$$

where A *is a positive constant* <16.5. Compared with the estimate of theorem 3.6, we have gained a factor of about $\rho(\epsilon)$.

Proof. Using Spitzer's identity (see Proposition 3.2 of [3] for details), we have

$$\mathbb{E}\sup_{0\le s\le t}X_s=\int_0^t\frac{\mathbb{E}X_s^+}{s}ds$$

$$\mathbb{E}\sup_{0\le s\le t}\left(X_s^\epsilon+\sigma(\epsilon)\hat{W}_s\right)=\int_0^t\frac{\mathbb{E}\left(X_s^\epsilon+\sigma(\epsilon)\hat{W}_s\right)^+}{s}ds.$$

Set

$$I^\epsilon=\left|\mathbb{E}\sup_{0\le s\le t}X_s-\mathbb{E}\sup_{0\le s\le t}\left(X_s^\epsilon+\sigma(\epsilon)\hat{W}_s\right)\right|.$$

Let $\delta > 0$, then

$$\begin{aligned}
I^\epsilon &= \left| \int_0^t \frac{\mathbb{E}X_s^+ - \mathbb{E}\left(X_s^\epsilon + \sigma(\epsilon)\hat{W}_s\right)^+}{s} ds \right| \\
&= \left| \int_0^\delta \frac{\mathbb{E}X_s^+ - \mathbb{E}\left(X_s^\epsilon + \sigma(\epsilon)\hat{W}_s\right)^+}{s} ds + \int_\delta^t \frac{\mathbb{E}X_s^+ - \mathbb{E}\left(X_s^\epsilon + \sigma(\epsilon)\hat{W}_s\right)^+}{s} ds \right| \\
&\leq \left| \int_0^\delta \frac{\mathbb{E}X_s^+ - \mathbb{E}\left(X_s^\epsilon + \sigma(\epsilon)\hat{W}_s\right)^+}{s} ds \right| + \left| \int_\delta^t \frac{\mathbb{E}X_s^+ - \mathbb{E}\left(X_s^\epsilon + \sigma(\epsilon)\hat{W}_s\right)^+}{s} ds \right| \\
&\leq \int_0^\delta \left| \mathbb{E}X_s^+ - \mathbb{E}\left(X_s^\epsilon + \sigma(\epsilon)\hat{W}_s\right)^+ \right| \frac{ds}{s} + \int_\delta^t \left| \mathbb{E}X_s^+ - \mathbb{E}\left(X_s^\epsilon + \sigma(\epsilon)\hat{W}_s\right)^+ \right| \frac{ds}{s}.
\end{aligned}$$

We will call I_1^ϵ (resp. I_2^ϵ) the first (resp. the second) term of the last expression. Note that the function $x \to x^+$ is 1-Lipschitz, So by proposition 6.2 of [5], we have

$$\left| \mathbb{E}\left(X_s^\epsilon + R_s^\epsilon\right)^+ - \mathbb{E}\left(X_s^\epsilon + \sigma(\epsilon)\hat{W}_s\right)^+ \right| \leq A\sigma(\epsilon)\rho(\epsilon),$$

where $A < 16.5$. So

$$\begin{aligned}
I_2^\epsilon &\leq A\sigma(\epsilon)\rho(\epsilon) \int_\delta^t \frac{ds}{s} \\
&= A\sigma(\epsilon)\rho(\epsilon) \log\left(\frac{t}{\delta}\right).
\end{aligned}$$

On the other hand

$$\begin{aligned}
\left| \mathbb{E}X_s^+ - \mathbb{E}\left(X_s^\epsilon + \sigma(\epsilon)\hat{W}_s\right)^+ \right| &\leq \mathbb{E}\left| \left(X_s^\epsilon + R_s^\epsilon\right)^+ - \left(X_s^\epsilon + \sigma(\epsilon)\hat{W}_s\right)^+ \right| \\
&\leq \mathbb{E}\left| \left(X_s^\epsilon + R_s^\epsilon\right) - \left(X_s^\epsilon + \sigma(\epsilon)\hat{W}_s\right) \right| \\
&= \mathbb{E}\left| R_s^\epsilon - \sigma(\epsilon)\hat{W}_s \right| \\
&\leq \mathbb{E}\left| R_s^\epsilon \right| + \sigma(\epsilon)\mathbb{E}\left| \hat{W}_s \right| \\
&\leq \sqrt{\mathbb{E}\left| R_s^\epsilon \right|^2} + \sigma(\epsilon)\mathbb{E}\left| \hat{W}_s \right| \\
&= \left(1 + \sqrt{\frac{2}{\pi}}\right)\sqrt{s}\sigma(\epsilon).
\end{aligned}$$

Hence

$$\begin{aligned}
I_1^\epsilon &\leq \left(1 + \sqrt{\frac{2}{\pi}}\right)\sigma(\epsilon) \int_0^\delta \frac{ds}{\sqrt{s}} \\
&= 2\left(1 + \sqrt{\frac{2}{\pi}}\right)\sigma(\epsilon)\sqrt{\delta} \\
&= A_1\sigma(\epsilon)\sqrt{\delta},
\end{aligned}$$

where $A_1 = 2\left(1+\sqrt{\frac{2}{\pi}}\right)$. Finally

$$I^\epsilon \le \max\left(A, A_1\right)\sigma(\epsilon)\left(\sqrt{\delta}+\rho(\epsilon)\log\left(\frac{t}{\delta}\right)\right).$$

The right term of the above inequality is minimal for $\delta = 4\rho(\epsilon)^2$. So

$$I^\epsilon \le 2\max\left(A, A_1\right)\sigma(\epsilon)\rho(\epsilon)\left(1+\log\left(\frac{\sqrt{t}}{2\rho(\epsilon)}\right)\right).$$

This concludes the proposition. □

4.2. Estimates by Skorokhod embedding. We will use a powefull tool to prove the results of this section. This is the the Skorokhod embedding theorem. We will begin by defining some useful notations.

Definition 4.3. *We define*

$$\beta(\epsilon) = \frac{\int_{|x|<\epsilon} x^4 \nu(dx)}{\left(\sigma_0(\epsilon)\right)^4}$$
$$\beta_1(\epsilon) = \beta(\epsilon)^{\frac{1}{6}}\left(\sqrt{\log\left(\frac{t}{\beta(\epsilon)^{\frac{1}{3}}}+3\right)}+1\right)$$
$$\beta_2(\epsilon) = \beta(\epsilon)^{\frac{1}{4}}\left(\log\left(\frac{t}{\beta(\epsilon)^{\frac{1}{4}}}+3\right)+1\right)$$
$$\beta_{p,\theta}(\epsilon) = \beta(\epsilon)^{\frac{p\theta}{p+4\theta}}\left(\log\left(\frac{1}{\beta(\epsilon)^{\frac{p\theta}{p+4\theta}}}\right)\right)^p.$$

Remark 4.4. *Note that under the condition* (4.2), *we have*

$$\lim_{\epsilon\to 0}\beta(\epsilon) = 0. \tag{4.3}$$

The proof of proposition 4.2 cannot be extended to the Lipschitz functions, because the reformulation of the spitzer identity for Lévy processes cannot be applied in that case. We have to use an other method.

Theorem 4.5. *Let X be an integrable infinite activity Lévy process with generating triplet (γ, σ^2, ν), $\epsilon \in (0,1)$, $t > 0$ and f be a Lipschitz function, then*

$$\left|\mathbb{E}f\left(M_t\right) - \mathbb{E}f\left(\hat{M}_t^\epsilon\right)\right| \le C\sigma_0(\epsilon)\beta_1(\epsilon),$$

where C is positive constant independent of ϵ.

Compared with the estimate of theorem 3.6, we have gained the factor $\beta_1(\epsilon)$.

Proof. Set

$$I_f^\epsilon = \left|\mathbb{E}\left(f\left(\sup_{0\le s\le t} X_s\right) - f\left(\sup_{0\le s\le t}\left(X_s^\epsilon + \sigma(\epsilon)\hat{W}_s\right)\right)\right)\right|$$
$$I_f^\epsilon(n) = \left|\mathbb{E}\left(f\left(\sup_{0\le k\le n} X_{\frac{kt}{n}}\right) - f\left(\sup_{0\le k\le n}\left(X^\epsilon_{\frac{kt}{n}} + \sigma(\epsilon)\hat{W}_{\frac{kt}{n}}\right)\right)\right)\right|.$$

Because f is, say, K-Lipschitz, we have

$$\begin{aligned}
I_f^\epsilon(n) &\le K\mathbb{E}\left|\sup_{0\le k\le n} X_{\frac{kt}{n}} - \sup_{0\le k\le n}\left(X^\epsilon_{\frac{kt}{n}} + \sigma(\epsilon)\hat{W}_{\frac{kt}{n}}\right)\right| \\
&= K\mathbb{E}\left|\sup_{0\le k\le n}\left(X^\epsilon_{\frac{kt}{n}} + R^\epsilon_{\frac{kt}{n}}\right) - \sup_{0\le k\le n}\left(X^\epsilon_{\frac{kt}{n}} + \sigma(\epsilon)\hat{W}_{\frac{kt}{n}}\right)\right| \\
&\le K\mathbb{E}\sup_{0\le k\le n}\left|R^\epsilon_{\frac{kt}{n}} - \sigma(\epsilon)\hat{W}_{\frac{kt}{n}}\right| \\
&\le K\mathbb{E}\left(\sup_{0\le k\le n}\left|R^\epsilon_{\frac{kt}{n}}\right| + \sigma(\epsilon)\sup_{0\le k\le n}\left|\hat{W}_{\frac{kt}{n}}\right|\right) \\
&\le K\mathbb{E}\left(\sup_{0\le s\le t}\left|R^\epsilon_s\right| + \sigma(\epsilon)\sup_{0\le s\le t}\left|\hat{W}_s\right|\right).
\end{aligned}$$

As the last expression is integrable, by dominated convergence we can deduce that

$$\lim_{n\to+\infty} I_f^\epsilon(n) = I_f^\epsilon.$$

So we will consider $I_f^\epsilon(n)$. For $k\in\{1,\dots,n\}$, we have

$$R^\epsilon_{\frac{kt}{n}} = \frac{1}{\sqrt{n}}\sum_{j=1}^k V_j^n,$$

where

$$V_j^n = \sqrt{n}\left(R^\epsilon_{\frac{jt}{n}} - R^\epsilon_{(j-1)\frac{t}{n}}\right).$$

The r.v. $\left(V_j^n\right)_{j\in\{1,\dots,n\}}$ are i.i.d. and have the same distribution as $\sqrt{n}R^\epsilon_{\frac{t}{n}}$. But $\mathbb{E}V_1 = 0$, and $var\left(V_1\right) = \sigma(\epsilon)^2 t$, by theorem 1 of [12] (see pp. 163) there exists positive i.i.d. r.v., $(\tau_j)_{j\in 1,\dots,n}$, and a standard Brownian motion, $\hat{B}$, such that $\left(\sum_{j=1}^k V_j^n, k\in\{1,\dots,n\}\right)$ and $\left(\hat{B}_{\tau_1+\dots+\tau_k}, k\in\{1,\dots,n\}\right)$ have the same join distribution. Thus $\left(R^\epsilon_{\frac{kt}{n}}, k\in\{1,\dots,n\}\right)$ and $\left(\hat{B}_{\frac{\tau_1+\dots+\tau_k}{n}}, k\in\{1,\dots,n\}\right)$ have the same join distribution. Furthermore

$$\mathbb{E}\tau_1 = var(V_1), \tag{4.4}$$

and for any $q\ge 1$, there exists a positive constant L_q, such that

$$\mathbb{E}\tau_1^q \le L_q\mathbb{E}V_1^{2q}. \tag{4.5}$$

We also have

$$\left(\sigma(\epsilon)\hat{W}_{\frac{kt}{n}}, k\in\{1,\dots,n\}\right) =^d \left(\hat{B}_{\frac{\sigma(\epsilon)^2 kt}{n}}, k\in\{1,\dots,n\}\right)$$

Set

$$\begin{aligned}
T_k &= \frac{\tau_1+\dots+\tau_k}{n} \\
T_k^\epsilon &= \frac{\sigma(\epsilon)^2 kt}{n}
\end{aligned}$$

Thus

$$I_f^\epsilon(n) \leq K\mathbb{E} \sup_{1\leq k\leq n} \left|\hat{B}_{T_k} - \hat{B}_{T_k^\epsilon}\right|$$

The following theorem concludes the proof. □

THEOREM 4.6.

1. We have

$$\limsup_{n\to+\infty} \mathbb{E} \sup_{1\leq k\leq n} \left|\hat{B}_{T_k} - \hat{B}_{T_k^\epsilon}\right| \leq C\sigma_0(\epsilon)\beta_1(\epsilon),$$

2. and

$$\limsup_{n\to+\infty} \mathbb{E} \sup_{1\leq k\leq n} \left|\hat{B}_{T_k} - \hat{B}_{T_k^\epsilon}\right|^2 \leq C\sigma_0(\epsilon)^2\beta_2(\epsilon).$$

3. For any real $p \geq 1$ and for any real $\theta \in (0,1)$, we have

$$\limsup_{n\to+\infty} \mathbb{E} \sup_{1\leq k\leq n} \left|\hat{B}_{T_k} - \hat{B}_{T_k^\epsilon}\right|^p \leq C_{p,\theta}\sigma_0(\epsilon)^p\beta_{p,\theta}(\epsilon),$$

where C and $C_{p,\theta}$ are constants independent of ϵ.

This theorem is the main result of this section. The other important which we need for the sequel is the following lemma.

LEMMA 4.7. *Let $\delta > 0$, then we have*

$$\limsup_{n\to+\infty} \mathbb{P}\left[\sup_{0\leq k\leq n} |T_k - T_k^\epsilon| > \delta\right] \leq \frac{4L_2 t\sigma_0(\epsilon)^4\beta(\epsilon)}{\delta^2}.$$

Proof of lemma 4.7. By Markov's inequality, we have

$$\mathbb{P}\left[\sup_{0\leq k\leq n} |T_k - T_k^\epsilon| > \delta\right] \leq \frac{\mathbb{E}\sup_{1\leq k\leq n} |T_k - T_k^\epsilon|^2}{\delta^2}.$$

On the other hand $(T_k - T_k^\epsilon)_{1\leq k\leq n}$ is a martingale, so using Doob's inequality, we get

$$\begin{aligned}
\mathbb{P}\left[\sup_{0\leq k\leq n} |T_k - T_k^\epsilon| > \delta\right] &\leq \frac{4\mathbb{E}\left|T_n - T_n^\epsilon\right|^2}{\delta^2} \\
&\leq \frac{4}{\delta^2}\frac{var\left(\tau_1\right)}{n} \\
&\leq \frac{4}{\delta^2}\frac{\mathbb{E}\tau_1^2}{n} \\
&\leq \frac{4L_2\mathbb{E}V_1^4}{n\delta^2}, \quad \text{by}(4.5) \\
&= \frac{4L_2 n\mathbb{E}\left(R_{\frac{t}{n}}^\epsilon\right)^4}{\delta^2}.
\end{aligned}$$

But by proposition 3.1

$$\mathbb{E}\left|R_{\frac{t}{n}}^\epsilon\right|^4 = \frac{t}{n}\int_{-\epsilon}^{\epsilon} x^4\nu(dx) + 3\left(\frac{t}{n}\sigma(\epsilon)^2\right)^2.$$

So

$$\lim_{n\to+\infty} n\mathbb{E}\left(R^{\epsilon}_{\frac{t}{n}}\right)^4 = t\int_{-\epsilon}^{+\epsilon}|x|^4\nu(dx)$$

Therefore

$$\lim_{n\to+\infty} n\mathbb{E}\left(R^{\epsilon}_{\frac{t}{n}}\right)^4 = \sigma_0(\epsilon)^4\beta(\epsilon)t. \tag{4.6}$$

Thus

$$\limsup_{n\to+\infty}\mathbb{P}\left[\sup_{0\le k\le n}|T_k - T_k^{\epsilon}| > \delta\right] \le \frac{4L_2 t\sigma_0(\epsilon)^4\beta(\epsilon)}{\delta^2}.$$

∎

Proof of theorem 4.6. For $\delta > 0$, have

$$\mathbb{E}\sup_{1\le k\le n}\left|\hat{B}_{T_k} - \hat{B}_{T_k^{\epsilon}}\right| \le \mathbb{E}\sup_{1\le k\le n}\left|\hat{B}_{T_k} - \hat{B}_{T_k^{\epsilon}}\right|\mathbb{1}_{\{\sup_{1\le k\le n}|T_k - T_k^{\epsilon}|\le\delta\}}$$
$$+\mathbb{E}\sup_{1\le k\le n}\left|\hat{B}_{T_k} - \hat{B}_{T_k^{\epsilon}}\right|\mathbb{1}_{\{\sup_{1\le k\le n}|T_k - T_k^{\epsilon}|>\delta\}}.$$

We set

$$I_1 = \mathbb{E}\sup_{1\le k\le n}\left|\hat{B}_{T_k} - \hat{B}_{T_k^{\epsilon}}\right|\mathbb{1}_{\{\sup_{1\le k\le n}|T_k - T_k^{\epsilon}|\le\delta\}}$$
$$I_2 = \mathbb{E}\sup_{1\le k\le n}\left|\hat{B}_{T_k} - \hat{B}_{T_k^{\epsilon}}\right|\mathbb{1}_{\{\sup_{1\le k\le n}|T_k - T_k^{\epsilon}|>\delta\}}.$$

On $\{\sup_{1\le k\le n}|T_k - T_k^{\epsilon}|\le\delta\}$, set, for k fixed

$$s_1 = T_k^{\epsilon}\wedge T_k$$
$$s_2 = T_k^{\epsilon}\vee T_k.$$

We have $s_1 \le s_2 \le s_1 + \delta$. Let j be such that $j\delta \le s_1 < (j+1)\delta$, we have $s_1 \le s_2 \le (j+2)\delta$. If $j\delta \le s_1 \le s_2 \le (j+1)\delta$, we have

$$\left|\hat{B}_{s_1} - \hat{B}_{s_2}\right| \le \left|\hat{B}_{s_1} - \hat{B}_{j\delta}\right| + \left|\hat{B}_{j\delta} - \hat{B}_{s_2}\right|$$
$$\le 2\sup_{0\le j\le\left[\frac{\sigma(\epsilon)^2 t}{\delta}\right]+1}\ \sup_{j\delta\le u\le(j+1)\delta}\left|\hat{B}_u - \hat{B}_{j\delta}\right|.$$

If $j\delta \le s_1 \le (j+1)\delta \le s_2 \le (j+2)\delta$, we have

$$\left|\hat{B}_{s_1} - \hat{B}_{s_2}\right| \le \left|\hat{B}_{s_1} - \hat{B}_{j\delta}\right| + \left|\hat{B}_{j\delta} - \hat{B}_{(j+1)\delta}\right| + \left|\hat{B}_{(j+1)\delta} - \hat{B}_{s_2}\right|$$
$$\le 3\sup_{0\le j\le\left[\frac{\sigma(\epsilon)^2 t}{\delta}\right]+2}\ \sup_{j\delta\le u\le(j+1)\delta}\left|\hat{B}_u - \hat{B}_{j\delta}\right|.$$

Hence

$$I_1 \le 3\mathbb{E}\sup_{0\le j\le\left[\frac{\sigma(\epsilon)^2 t}{\delta}\right]+2}\ \sup_{j\delta\le u\le(j+1)\delta}\left|\hat{B}_u - \hat{B}_{j\delta}\right|$$
$$= 3\mathbb{E}\sup_{1\le j\le\left[\frac{\sigma(\epsilon)^2 t}{\delta}\right]+3}\ \sup_{(j-1)\delta\le u\le j\delta}\left|\hat{B}_u - \hat{B}_{j\delta}\right|.$$

The r.v. $\left(\sup_{(j-1)\delta\le u\le j\delta}\left|\hat{B}_u-\hat{B}_{j\delta}\right|\right)_{1\le j\le\left[\frac{\sigma(\epsilon)^2t}{\delta}\right]+3}$ are i.i.d. withe the same distribution as $\sup_{0\le u\le\delta}\left|\hat{B}_u\right|\sim\sqrt{\delta}\sup_{0\le u\le 1}\left|\hat{B}_u\right|$. Then

$$I_1\le 3\sqrt{\delta}\mathbb{E}\sup_{1\le j\le\left[\frac{\sigma(\epsilon)^2t}{\delta}\right]+3}V_j,$$

where $(V_j)_{1\le j\le\left[\frac{\sigma(\epsilon)^2t}{\delta}\right]+3}$ are i.i.d. r.v. with the same distribution as $\sup_{0\le u\le 1}\left|\hat{B}_u\right|$. On the other hand, we know that if $(V_j)_{1\le j\le m}$ are i.i.d. r.v. satisfying $\mathbb{E}e^{\alpha V_1^2}<\infty$ where α is a positive real, then

$$\mathbb{E}\sup_{1\le j\le m}V_j\le g\left(m\mathbb{E}e^{\alpha V_1^2}\right),$$

where $g:x\in[1,+\infty)\to\sqrt{\frac{1}{\alpha}\log(x)}$. Indeed g being concave, we have

$$\begin{aligned}\mathbb{E}\sup_{1\le j\le m}V_j&=\mathbb{E}\sup_{1\le j\le m}g\left(e^{\alpha V_j^2}\right)\\&=\mathbb{E}g\left(\sup_{1\le j\le m}e^{\alpha V_j^2}\right),\text{ because g is non-decreasing}\\&\le g\left(\mathbb{E}\sup_{1\le j\le m}e^{\alpha V_j^2}\right),\text{ by Jensen's inequality}\\&\le g\left(\mathbb{E}\sum_{j=1}^m e^{\alpha V_j^2}\right),\text{ because g is non-decreasing}\\&\le g\left(m\mathbb{E}e^{\alpha V_1^2}\right).\end{aligned}$$

In our case $V_1=\sup_{0\le u\le 1}\left|\hat{B}_u\right|$. So

$$V_1\le\sup_{0\le u\le 1}\hat{B}_u+\sup_{0\le u\le 1}\left(-\hat{B}_u\right).$$

Thus

$$\begin{aligned}e^{\alpha V_1^2}&\le e^{2\alpha\left(\left(\sup_{0\le u\le 1}\hat{B}_u\right)^2+\left(\sup_{0\le u\le 1}\left(-\hat{B}_u\right)\right)^2\right)}\\&\le\left(\mathbb{E}e^{4\alpha\left(\sup_{0\le u\le 1}\hat{B}_u\right)^2}\right)^{\frac{1}{2}}\left(\mathbb{E}e^{4\alpha\left(\sup_{0\le u\le 1}\left(-\hat{B}_u\right)\right)^2}\right)^{\frac{1}{2}}\\&\le\mathbb{E}e^{4\alpha\left(\sup_{0\le u\le 1}\hat{B}_u\right)^2}.\end{aligned}$$

But the probability density function of $\sup_{0\le u\le 1}\hat{B}_u$ is $2\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\mathbb{1}_{x\ge 0}$, so an easy computation give that for any real $\beta<\frac{1}{2}$

$$\mathbb{E}e^{\beta\left(\sup_{0\le u\le 1}\hat{B}_u\right)^2}=(1-2\beta)^{-\frac{1}{2}}.\tag{4.7}$$

Thus choosing $\alpha<\frac{1}{8}$, we get

$$\mathbb{E}e^{\alpha V_1^2}\le(1-8\alpha)^{-\frac{1}{2}}.$$

Hence for $\alpha \in (0, \frac{1}{8})$

$$\begin{aligned}
I_1 &\leq 3\sqrt{\delta}\sqrt{\frac{1}{\alpha}\log\left(\left(\left[\frac{\sigma(\epsilon)^2 t}{\delta}\right]+3\right)\mathbb{E}e^{\alpha V_1^2}\right)} \\
&\leq 3\sqrt{\frac{1}{\alpha}\left(1+\frac{\log\left(\mathbb{E}e^{\alpha V_1^2}\right)}{\log(3)}\right)}\sqrt{\delta\log\left(\left[\frac{\sigma(\epsilon)^2 t}{\delta}\right]+3\right)} \\
&\leq 3\sqrt{\frac{1}{\alpha}\left(1-\frac{\log(1-8\alpha)}{2\log(3)}\right)}\sqrt{\delta\log\left(\left[\frac{\sigma(\epsilon)^2 t}{\delta}\right]+3\right)} \\
&= C_\alpha\sqrt{\delta\log\left(\left[\frac{\sigma(\epsilon)^2 t}{\delta}\right]+3\right)} \\
&\leq C_\alpha\sqrt{\delta\log\left(\frac{\sigma(\epsilon)^2 t}{\delta}+3\right)}
\end{aligned}$$

Notice that $C_\alpha = 3\sqrt{\frac{1}{\alpha}\left(1-\frac{\log(1-8\alpha)}{2\log(3)}\right)}$ and the second inequality come from the fact that for any $x, y \in \mathbb{R}^+$ we have $\log(x+3)+y \leq \left(1+\frac{y}{\log(3)}\right)\log(x+3)$. Consider now I_2. We have

$$\begin{aligned}
I_2 &\leq \left(\mathbb{E}\left(\sup_{1\leq k\leq n}\left|\hat{B}_{T_k}-\hat{B}_{T_k^\epsilon}\right|\right)^2\right)^{\frac{1}{2}}\left(\mathbb{P}\left[\sup_{1\leq k\leq n}|T_k-T_k^\epsilon|>\delta\right]\right)^{\frac{1}{2}} \\
&\leq \left(\mathbb{E}\left(\sup_{1\leq k\leq n}\left|\hat{B}_{T_k}\right|+\sup_{1\leq k\leq n}\left|\hat{B}_{T_k^\epsilon}\right|\right)^2\right)^{\frac{1}{2}}\left(\mathbb{P}\left[\sup_{1\leq k\leq n}|T_k-T_k^\epsilon|>\delta\right]\right)^{\frac{1}{2}}.
\end{aligned}$$

So

$$\begin{aligned}
I_2 &\leq \left(\left(\mathbb{E}\sup_{1\leq k\leq n}\left|\hat{B}_{T_k}\right|^2\right)^{\frac{1}{2}}+\left(\mathbb{E}\sup_{1\leq k\leq n}\left|\hat{B}_{T_k^\epsilon}\right|^2\right)^{\frac{1}{2}}\right)\left(\mathbb{P}\left[\sup_{0\leq k\leq n}|T_k-T_k^\epsilon|>\delta\right]\right)^{\frac{1}{2}} \\
&\leq \left(\left(\mathbb{E}\sup_{0\leq s\leq t}|R_s^\epsilon|^2\right)^{\frac{1}{2}}+\left(\mathbb{E}\sup_{0\leq s\leq\sigma(\epsilon)^2 t}\left|\hat{B}_s\right|^2\right)^{\frac{1}{2}}\right)\left(\mathbb{P}\left[\sup_{0\leq k\leq n}|T_k-T_k^\epsilon|>\delta\right]\right)^{\frac{1}{2}}
\end{aligned}$$

Using Doob's inequality, we get

$$I_2 \leq 2\left(\left(\mathbb{E}\,|R_t^\epsilon|^2\right)^{\frac{1}{2}}+\left(\mathbb{E}\left|\hat{B}_{\sigma(\epsilon)^2 t}\right|^2\right)^{\frac{1}{2}}\right)\left(\mathbb{P}\left[\sup_{0\leq k\leq n}|T_k-T_k^\epsilon|>\delta\right]\right)^{\frac{1}{2}}.$$

Hence

$$I_2 \leq 4\sqrt{t}\sigma(\epsilon)\left(\mathbb{P}\left[\sup_{1\leq k\leq n}|T_k-T_k^\epsilon|>\delta\right]\right)^{\frac{1}{2}}$$

So, by lemma 4.7, we have

$$\limsup_{n\to+\infty} I_2 \leq 4\sqrt{t}\sigma(\epsilon)\left(\frac{4L_2 t\sigma_0(\epsilon)^4\beta(\epsilon)}{\delta^2}\right)^{\frac{1}{2}}.$$

Hence

$$\limsup_{n\to+\infty}\mathbb{E}\sup_{1\le k\le n}\left|\hat{B}_{T_k}-\hat{B}_{T_k^\epsilon}\right| \le C_\alpha\sqrt{\delta\log\left(\frac{\sigma(\epsilon)^2t}{\delta}+3\right)}+\frac{8t}{\delta}\sigma(\epsilon)\sigma_0(\epsilon)^2\sqrt{\beta(\epsilon)L_2}$$
$$\le C_\alpha\sqrt{\delta\log\left(\frac{\sigma_0(\epsilon)^2t}{\delta}+3\right)}+\frac{8\sqrt{L_2}t}{\delta}\sigma_0(\epsilon)^3\sqrt{\beta(\epsilon)}$$
$$= C_{\alpha,t}\left(\sqrt{\delta\log\left(\frac{\sigma_0(\epsilon)^2t}{\delta}+3\right)}+\frac{1}{\delta}\sigma_0(\epsilon)^3\sqrt{\beta(\epsilon)}\right),$$

where $C_{\alpha,t}=\max\left(C_\alpha,8\sqrt{L_2}t\right)$. Choosing $\delta=\sigma_0(\epsilon)^2\beta(\epsilon)^{\frac{1}{3}}$, we get

$$\limsup_{n\to+\infty}\mathbb{E}\sup_{1\le k\le n}\left|\hat{B}_{T_k}-\hat{B}_{T_k^\epsilon}\right|\le C_{\alpha,t}\sigma_0(\epsilon)\beta(\epsilon)^{\frac{1}{6}}\left(\sqrt{\log\left(\frac{t}{\beta(\epsilon)^{\frac{1}{3}}}+3\right)}+1\right).$$

For the second part of the theorem we will use the following definition for the function g

$$g(x)=\frac{1}{\alpha}\log(x),\ x\in[1,+\infty).$$

And for the third part of the theorem, the function g will be defined as follows

$$g(x)=\left(\frac{1}{\alpha}\log(x)\right)^p.$$

The proofs of the second and the third part of the theorems can be found in [6]. □

With the method used above, we can derived results for functions depending on X at a given time.

PROPOSITION 4.8. *Let X be an infinite activity Lévy process with generating triplet (γ,σ^2,ν), $t>0$, $\epsilon\in(0,1]$ and f be a Lipschitz function. Then*

$$\left|\mathbb{E}f\left(X_t\right)-\mathbb{E}f\left(\hat{X}_t^\epsilon\right)\right|\le C\beta_1(\epsilon)\sigma_0(\epsilon),$$

where C is a positive constante. If we replace condition f' bounded by f' Lipschitz, then using a Richardson-like extrapolation method, we get the following result.

PROPOSITION 4.9. *Let X be a Lévy process with generating triplet (γ,σ^2,ν), f be a continuously differentiable function satisfying f' is Lipschitz. Then*

$$\left|\mathbb{E}f\left(X_t\right)-\mathbb{E}\left(2f\left(X_t^\epsilon\right)-f\left(\hat{X}_t^\epsilon\right)\right)\right|\le C\beta_2(\epsilon)\sigma_0(\epsilon)^2,$$

where C is a positive constant.

For exotic payoffs, we have the following results.

PROPOSITION 4.10. *Let X be an infinite activity Lévy process with generating triplet (γ,σ^2,ν), $\epsilon\in(0,1]$, $t>0$ and f be a function from $\mathbb{R}^+\times\mathbb{R}$ in $\mathbb{R}$ K-Lipschitzwith respect to its second variable. We assume that $\frac{\epsilon}{\sigma(\epsilon)}$ is bounded. Then*

$$\left|\sup_{\tau\in\mathcal{T}_{[0,t]}}\mathbb{E}f\left(\tau,X_\tau\right)-\sup_{\tau\in\mathcal{T}_{[0,t]}}\mathbb{E}f\left(\tau,\hat{X}_\tau^\epsilon\right)\right|\le C\sigma_0(\epsilon)\beta_1(\epsilon),$$

where $\mathcal{T}_{[0,t]}$ is the set of stopping times with values in $[0,t]$, and C is a positive constant independent of ϵ.

PROPOSITION 4.11. *Let X be a Lévy process with generating triplet (γ, σ^2, ν), $p > 1$, $\epsilon \in (0,1]$ and $t > 0$. We assume that $\mathbb{E}e^{pM_t} < \infty$, then for any $x \in \mathbb{R}$ and for any $\theta \in (0,1)$*

$$\left| \mathbb{E}\left(e^{M_t} - x\right)^+ - \mathbb{E}\left(e^{\hat{M}_t^\epsilon} - x\right)^+ \right| \leq C_{p,\theta}\sigma_0(\epsilon)\left(\beta_{\frac{p}{p-1},\theta}(\epsilon)\right)^{1-\frac{1}{p}},$$

where $C_{p,\theta}$ is a positive constant independent of ϵ.

To prove the above propositions, we will use the same notations as in the proof of theorem 4.5.

Proof of proposition 4.8. We have

$$R_t^\epsilon =^d \hat{B}_{T_n}$$
$$\sigma(\epsilon)\hat{W}_t =^d \hat{B}_{T_n^\epsilon}.$$

So, if f is K-Lipschitz, we have

$$\begin{aligned} \left|\mathbb{E}f\left(X_t\right) - \mathbb{E}f\left(X_t^\epsilon + \sigma(\epsilon)\hat{W}_t\right)\right| &= \left|\mathbb{E}f\left(X_t^\epsilon + \hat{B}_{T_n}\right) - f\left(X_t^\epsilon + \hat{B}_{T_n^\epsilon}\right)\right| \\ &\leq K\mathbb{E}\left|\hat{B}_{T_n} - \hat{B}_{T_n^\epsilon}\right|. \end{aligned}$$

We conclude with theorem 4.6. ∎

Proof of proposition 4.9. We have

$$R_t^\epsilon =^d \hat{B}_{T_n}$$
$$\sigma(\epsilon)\hat{W}_t =^d \hat{B}_{T_n^\epsilon}.$$

We have showed in the proof of proposition 3.2 that

$$\mathbb{E}f\left(X_t\right) - \mathbb{E}f\left(X_t^\epsilon\right) = \int_0^1 \mathbb{E}f'\left(X_t^\epsilon + \theta R_t^\epsilon\right)R_t^\epsilon d\theta.$$

Because R^ϵ is independent of X^ϵ and $\hat{W}$, we have

$$\mathbb{E}f\left(X_t\right) - \mathbb{E}f\left(X_t^\epsilon\right) = \int_0^1 \mathbb{E}\left(f'\left(X_t^\epsilon + \theta R_t^\epsilon\right) - f'\left(X_t^\epsilon + \theta\sigma(\epsilon)\hat{W}_t\right)\right)R_t^\epsilon d\theta.$$

By the same way, we will get

$$\begin{aligned} \mathbb{E}f\left(\hat{X}_t^\epsilon\right) - \mathbb{E}f\left(X_t^\epsilon\right) &= \int_0^1 \mathbb{E}f'\left(X_t^\epsilon + \theta\sigma(\epsilon)\hat{W}_t\right)\sigma(\epsilon)\hat{W}_t d\theta \\ &= \int_0^1 \mathbb{E}\left(f'\left(X_t^\epsilon + \theta\sigma(\epsilon)\hat{W}_t\right) - f'\left(X_t^\epsilon + \theta R_t^\epsilon\right)\right)\sigma(\epsilon)\hat{W}_t d\theta. \end{aligned}$$

Set

$$I^\epsilon = \mathbb{E}f\left(X_t\right) - \mathbb{E}\left(2f\left(X_t^\epsilon\right) - f\left(\hat{X}_t^\epsilon\right)\right).$$

Hence

$$\begin{aligned} I^\epsilon &= \mathbb{E}\left(f\left(X_t\right)-f\left(X_t^\epsilon\right)\right)+\mathbb{E}\left(f\left(\hat{X}_t\right)-f\left(X_t^\epsilon\right)\right) \\ &= \int_0^1 \mathbb{E}\left(f^{'}\left(X_t^\epsilon+\theta R_t^\epsilon\right)-f^{'}\left(X_t^\epsilon+\theta\sigma(\epsilon)\hat{W}_t\right)\right)\left(R_t^\epsilon-\sigma(\epsilon)\hat{W}_t\right)d\theta \\ &= \int_0^1 \mathbb{E}\left(f^{'}\left(X_t^\epsilon+\theta\hat{B}_{T_n}\right)-f^{'}\left(X_t^\epsilon+\theta\hat{B}_{T_n^\epsilon}\right)\right)\left(\hat{B}_{T_n}-\hat{B}_{T_n^\epsilon}\right)d\theta. \end{aligned}$$

Therefore, if f is K-Lipschitz, we get

$$\begin{aligned} \left|\mathbb{E}f\left(X_t\right)-\mathbb{E}\left(2f\left(X_t^\epsilon\right)-f\left(\hat{X}_t^\epsilon\right)\right)\right| &\leq \int_0^1 K\theta\mathbb{E}\left(\hat{B}_{T_n}-\hat{B}_{T_n^\epsilon}\right)^2 d\theta \\ &= \frac{K}{2}\mathbb{E}\left(\hat{B}_{T_n}-\hat{B}_{T_n^\epsilon}\right)^2. \end{aligned}$$

We conclude with theorem 4.6. □

Proof of proposition 4.10. Set $t_k = \frac{kt}{n}$, and

$$\begin{aligned} p(t) &= \sup_{\tau\in\mathcal{T}_{[0,t]}} \mathbb{E}f\left(\tau, X_\tau\right) \\ p^\epsilon(t) &= \sup_{\tau\in\mathcal{T}_{[0,t]}} \mathbb{E}f\left(\tau, \hat{X}_\tau^\epsilon\right) \\ p_n(t) &= \sup_{\tau\in\mathcal{T}_{[0,t]}^n} \mathbb{E}f\left(\tau, X_\tau\right) \\ p_n^\epsilon(t) &= \sup_{\tau\in\mathcal{T}_{[0,t]}^n} \mathbb{E}f\left(\tau, \hat{X}_\tau^\epsilon\right), \end{aligned}$$

where $\mathcal{T}_{[0,t]}^n$ is the set of stopping times with values in $\{t_k,\ k=1,\ldots,n\}$. We know that

$$\begin{aligned} \lim_{n\to+\infty} p_n(t) &= p(t) \\ \lim_{n\to+\infty} p_n^\epsilon(t) &= p^\epsilon(t). \end{aligned}$$

Let $\theta\in\mathcal{T}_{[0,t]}^n$, we have

$$\begin{aligned} \mathbb{E}f\left(\theta, X_\theta\right) &= \mathbb{E}f\left(\theta, X_\theta^\epsilon\right)+\mathbb{E}\left(f\left(\theta, X_\theta\right)-f\left(\theta, \hat{X}_\theta^\epsilon\right)\right) \\ &\leq p_n^\epsilon(t)+\mathbb{E}\sup_{1\leq k\leq n}\left(f\left(t_k, X_{t_k}\right)-f\left(t_k, \hat{X}_{t_k}^\epsilon\right)\right) \\ &= p_n^\epsilon(t)+\mathbb{E}\sup_{1\leq k\leq n}\left(f\left(t_k, X_{t_k}^\epsilon+R_{t_k}^\epsilon\right)-f\left(t_k, X_{t_k}^\epsilon+\sigma(\epsilon)\hat{W}_{t_k}\right)\right) \\ &= p_n^\epsilon(t)+\mathbb{E}\sup_{1\leq k\leq n}\left(f\left(t_k, X_{t_k}^\epsilon+B_{T_k}\right)-f\left(t_k, X_{t_k}^\epsilon+B_{T_k^\epsilon}\right)\right) \\ &\leq p_n^\epsilon(t)+K\mathbb{E}\sup_{1\leq k\leq n}\left|B_{T_k}-B_{T_k^\epsilon}\right|. \end{aligned}$$

So

$$p_n(t) \leq p_n^\epsilon(t)+K\mathbb{E}\sup_{1\leq k\leq n}\left|B_{T_k}-B_{T_k^\epsilon}\right|.$$

By the same reasoning, we get

$$p_n^\epsilon(t) \leq p_n(t) + K\mathbb{E} \sup_{1\leq k\leq n} \left| B_{T_k} - B_{T_k^\epsilon} \right| .$$

Therefore

$$|p_n(t) - p_n^\epsilon(t)| \leq K\mathbb{E} \sup_{1\leq k\leq n} \left| B_{T_k} - B_{T_k^\epsilon} \right| .$$

We conclude by theorem 4.6. ∎

Proof of proposition 4.11. Define

$$\begin{aligned} M_t^n &= \sup_{0\leq k\leq n} \left(X^\epsilon_{k\frac{t}{n}} + R^\epsilon_{k\frac{t}{n}} \right) \\ \hat{M}_t^{\epsilon,n} &= \sup_{0\leq k\leq n} \left(X^\epsilon_{k\frac{t}{n}} + \sigma(\epsilon)\hat{W}^\epsilon_{k\frac{t}{n}} \right) . \end{aligned}$$

We know that

$$\begin{aligned} \lim_{n\to+\infty} M_t^n &= M_t \quad a.s. \\ \lim_{n\to+\infty} \hat{M}_t^{\epsilon,n} &= \hat{M}_t^\epsilon \quad a.s. \end{aligned}$$

Set

$$\begin{aligned} U_t^n &= \sup_{0\leq k\leq n} \left(X^\epsilon_{k\frac{t}{n}} + \hat{B}_{T_k} \right) \\ \hat{U}_t^{\epsilon,n} &= \sup_{0\leq k\leq n} \left(X^\epsilon_{k\frac{t}{n}} + \hat{B}_{T_k^\epsilon} \right) . \end{aligned}$$

So

$$\begin{aligned} M_t^n &=^d U_t^n \\ \hat{M}_t^{\epsilon,n} &=^d \hat{U}_t^{\epsilon,n} \end{aligned}$$

By mean value theorem, we have

$$e^{U_t^n} - e^{\hat{U}_t^{\epsilon,n}} = \left(U_t^n - \hat{U}_t^{\epsilon,n} \right) e^{\bar{U}_t^{\epsilon,n}},$$

where $\bar{U}_t^{\epsilon,n}$ is between U_t^n and $\hat{U}_t^{\epsilon,n}$. Set

$$I_n^\epsilon = \left| \mathbb{E}\left(e^{U_t^n} - x \right)^+ - \mathbb{E}\left(e^{\hat{U}_t^{\epsilon,n}} - x \right)^+ \right| .$$

Thus

$$\begin{aligned}
I_n^\epsilon &\leq \mathbb{E}\left|e^{U_t^n} - e^{\hat{U}_t^{\epsilon,n}}\right| \\
&\leq \mathbb{E}\left|U_t^n - \hat{U}_t^{\epsilon,n}\right| e^{\bar{U}_t^{\epsilon,n}} \\
&\leq \mathbb{E}\sup_{0\leq k\leq n}\left|\hat{B}_{T_k} - \hat{B}_{T_k^\epsilon}\right| e^{\bar{U}_t^{\epsilon,n}} \\
&\leq \left(\mathbb{E}\sup_{0\leq k\leq n}\left|\hat{B}_{T_k} - \hat{B}_{T_k^\epsilon}\right|^{\frac{p}{p-1}}\right)^{1-\frac{1}{p}} \left(\mathbb{E}e^{p\bar{U}_t^{\epsilon,n}}\right)^{\frac{1}{p}} \\
&\leq \left(\mathbb{E}\sup_{0\leq k\leq n}\left|\hat{B}_{T_k} - \hat{B}_{T_k^\epsilon}\right|^{\frac{p}{p-1}}\right)^{1-\frac{1}{p}} \left(\mathbb{E}\left(e^{pM_t^n} + e^{p\hat{M}_t^{\epsilon,n}}\right)\right)^{\frac{1}{p}} \\
&\leq \left(\mathbb{E}\sup_{0\leq k\leq n}\left|\hat{B}_{T_k} - \hat{B}_{T_k^\epsilon}\right|^{\frac{p}{p-1}}\right)^{1-\frac{1}{p}} \left(\mathbb{E}\left(e^{pM_t} + e^{p\hat{M}_t^{\epsilon}}\right)\right)^{\frac{1}{p}}.
\end{aligned}$$

But

$$\begin{aligned}
\mathbb{E}\left(e^{pM_t} + e^{p\hat{M}_t^\epsilon}\right) &\leq \mathbb{E}\left(e^{pM_t} + e^{p\sigma(\epsilon)\sup_{0\leq s\leq t}\hat{W}_s} e^{pM_t^\epsilon}\right) \\
&\leq \mathbb{E}e^{pM_t} + 2e^{\frac{p^2}{2}\sigma(\epsilon)^2 t}\mathbb{E}e^{pM_t^\epsilon} \\
&\leq 2e^{\frac{p^2}{2}\sigma(\epsilon)^2 t}\mathbb{E}\left(e^{pM_t} + e^{pM_t^\epsilon}\right).
\end{aligned}$$

So using dominated convergence, theorem 4.6 and lemma 3.9, we get

$$\begin{aligned}
\left|\mathbb{E}\left(e^{M_t} - x\right)^+ - \mathbb{E}\left(e^{\hat{M}_t^\epsilon} - x\right)^+\right| &= \lim_{n\to+\infty}\left|\mathbb{E}\left(e^{M_t^n} - x\right)^+ - \mathbb{E}\left(e^{\hat{M}_t^{\epsilon,n}} - x\right)^+\right| \\
&= \limsup_{n\to+\infty}\left|\mathbb{E}\left(e^{U_t^n} - x\right)^+ - \mathbb{E}\left(e^{\hat{U}_t^{\epsilon,n}} - x\right)^+\right| \\
&\leq C_{p,\theta}\sigma_0(\epsilon)\left(\beta_{\frac{p}{p-1},\theta}(\epsilon)\right)^{1-\frac{1}{p}}.
\end{aligned}$$

∎

4.3. Estimates for cumulative distribution functions. The bounds obtained in this section are better than those obtained by truncation, provided that condition (4.2) is satisfied.

PROPOSITION 4.12. *Let X be an infinite activity Lévy process with generating triplet (γ, σ^2, ν), $x > 0$ and $\epsilon \in (0, 1]$. Set*

$$\tilde{\beta}_{\rho,\theta}(\epsilon) = \beta_{\frac{1}{\rho}-1,\theta}(\epsilon)$$

1. *If $\sigma > 0$, then*

$$\sup_{x\in\mathbb{R}}\left|\mathbb{P}\left[X_t \geq x\right] - \mathbb{P}\left[\hat{X}_t^\epsilon \geq x\right]\right| \leq C\sigma_0(\epsilon)\beta_1(\epsilon).$$

2. *If X_t has a locally bounded probability density function, then for any pair of reals $\theta, \rho \in (0, 1)$*

$$\left|\mathbb{P}\left[X_t \geq x\right] - \mathbb{P}\left[\hat{X}_t^\epsilon \geq x\right]\right| \leq C_{\rho,\theta}\sigma_0(\epsilon)^{1-\rho}\left(\tilde{\beta}_{\rho,\theta}(\epsilon)\right)^\rho.$$

3. If M_t has a locally bounded probability density function on $(0,+\infty)$, then for any pair of reals $\theta, \rho \in (0,1)$

$$\left|\mathbb{P}\left[M_t \geq x\right] - \mathbb{P}\left[\hat{M}_t^\epsilon \geq x\right]\right| \leq C_{\rho,\theta}\sigma_0(\epsilon)^{1-\rho}\left(\tilde{\beta}_{\rho,\theta}(\epsilon)\right)^\rho .$$

The positive constants C and $C_{\rho,\theta}$ independent of ϵ.

Proof. We will use the same notations as in the proofs of theorem 4.5 and proposition 4.11. Recall that

$$R_t^\epsilon =^d \hat{B}_{T_n}$$
$$\sigma(\epsilon)\hat{W}_t =^d \hat{B}_{T_n^\epsilon}.$$

Set

$$Y_t = X_t^\epsilon + \hat{B}_{T_n}$$
$$\hat{Y}_t^\epsilon = X_t^\epsilon + \hat{B}_{T_n^\epsilon}.$$

Thus

$$\begin{aligned}\left|\mathbb{P}\left[X_t \geq x\right] - \mathbb{P}\left[\hat{X}_t^\epsilon \geq x\right]\right| &= \left|\mathbb{P}\left[Y_t \geq x\right] - \mathbb{P}\left[\hat{Y}_t^\epsilon \geq x\right]\right| \\ &= \left|\mathbb{P}\left[Y_t \geq x, \hat{Y}_t^\epsilon < x\right] - \mathbb{P}\left[Y_t < x, \hat{Y}_t^\epsilon \geq x\right]\right|.\end{aligned}$$

But, in the case $\sigma > 0$

$$\begin{aligned}\mathbb{P}\left[Y_t \geq x, \hat{Y}_t^\epsilon < x\right] &= \mathbb{P}\left[x - \left(Y_t - \hat{Y}_t^\epsilon\right) \leq \hat{Y}_t^\epsilon < x\right] \\ &= \mathbb{P}\left[x - \left(\hat{B}_{T_n} - \hat{B}_{T_n^\epsilon}\right) \leq \sigma B_t + \left(\hat{Y}_t^\epsilon - \sigma B_t\right) < x\right].\end{aligned}$$

By construction, σB_t is independent of $\left(\hat{Y}_t^\epsilon - \sigma B_t\right)$ and of $\left(\hat{B}_{T_n} - \hat{B}_{T_n^\epsilon}\right)$. We know that $\frac{1}{\sigma\sqrt{2\pi t}}$ is an upper bound of the probability density function of σB_t, so

$$\mathbb{P}\left[Y_t \geq x, \hat{Y}_t^\epsilon < x\right] \leq \frac{1}{\sigma\sqrt{2\pi t}}\mathbb{E}\left|\hat{B}_{T_n} - \hat{B}_{T_n^\epsilon}\right|.$$

We get the first part of the proposition by using theorem 4.6. Consider now the second part of the proposition. By proposition 3.11, we know that there exists $K_x > 0$ such that

$$\begin{aligned}\left|\mathbb{P}\left[Y_t \geq Y\right] - \mathbb{P}\left[\hat{Y}_t^\epsilon \geq x\right]\right| &\leq K_x\delta + \frac{\mathbb{E}\left|Y_t - \hat{Y}_t^\epsilon\right|^p}{\delta^p} \\ &= K_x\delta + \frac{\mathbb{E}\left|\hat{B}_{T_n} - \hat{B}_{T_n^\epsilon}\right|^p}{\delta^p}.\end{aligned}$$

By theorem 4.6, we know that there exists $C_{p,\theta} > 0$ such that

$$\begin{aligned}\left|\mathbb{P}\left[Y_t \geq Y\right] - \mathbb{P}\left[\hat{Y}_t^\epsilon \geq x\right]\right| &\leq K_x\delta + C_{p,\theta}\frac{\sigma_0(\epsilon)^p\beta_{p,\theta}(\epsilon)}{\delta^p} \\ &\leq \max\left(K_x, C_{p,\theta}\right)\left(\delta + \frac{\sigma_0(\epsilon)^p\beta_{p,\theta}(\epsilon)}{\delta^p}\right) \\ &\leq 2\max\left(K_x, C_{p,\theta}\right)\sigma_0(\epsilon)^{1-\frac{1}{p+1}}\beta(\epsilon)^{\frac{1}{p+1}}.\end{aligned}$$

The last inequality is obtained by choosing $\delta = \sigma_0(\epsilon)^{1-\frac{1}{p+1}}\beta_{p,\theta}(\epsilon)^{\frac{1}{p+1}}$. We set

$$\begin{aligned} I(\epsilon) &= \left|\mathbb{P}\left[M_t \geq x\right] - \mathbb{P}\left[\hat{M}_t^\epsilon \geq x\right]\right| \\ I^n(\epsilon) &= \left|\mathbb{P}\left[M_t^n \geq x\right] - \mathbb{P}\left[\hat{M}_t^{\epsilon,n} \geq x\right]\right|, \end{aligned}$$

Note that $\lim_{\epsilon\to 0} I^n(\epsilon) = I(\epsilon)$. So (see the proof of proposition 4.11)

$$I^n(\epsilon) = \left|\mathbb{P}\left[U_t^n \geq x\right] - \mathbb{P}\left[\hat{U}_t^{\epsilon,n} \geq x\right]\right|.$$

Using the proof of proposition 3.11, we get

$$\begin{aligned} I^n(\epsilon) &\leq \mathbb{P}\left[x \leq U_t^n < x+\delta\right] + \frac{\mathbb{E}\left|U_t^n - \hat{U}_t^{\epsilon,n}\right|^p}{\delta^p} \\ &\leq \mathbb{P}\left[x \leq U_t^n < x+\delta\right] + \frac{\mathbb{E}\sup_{1\leq k\leq n}\left|\hat{B}_{T_k} - \hat{B}_{T_k^\epsilon}\right|^p}{\delta^p}. \end{aligned}$$

But

$$\begin{aligned} \lim_{n\to+\infty} \mathbb{P}\left[x \leq U_t^n < x+\delta\right] &= \lim_{n\to+\infty} \mathbb{P}\left[x \leq M_t^n < x+\delta\right] \\ &= \mathbb{P}\left[x \leq M_t < x+\delta\right] \\ &\leq C_x\delta, \end{aligned}$$

where C_x is a positive constant given by proposition 3.11. Thus by theorem 4.6

$$\begin{aligned} I(\epsilon) &\leq C_x\delta + \frac{C_{p,\theta}}{\delta^p}\sigma_0(\epsilon)^p\beta_{p,\theta}(\epsilon) \\ &\leq 2\max(C_x, C_{p,\theta})\sigma_0(\epsilon)^{1-\frac{1}{p+1}}\beta_{p,\theta}(\epsilon)^{\frac{1}{p+1}}. \end{aligned}$$

The last inequality is obtained by choosing $\delta = \sigma_0(\epsilon)^{1-\frac{1}{p+1}}\beta_2(\epsilon)^{\frac{1}{p+1}}$. □

REMARK 4.13. *Suppose that $\sigma > 0$ or $\sigma = 0$ and there exists $\alpha > 0$ such that (3.3) is true, then for any $\theta, \rho \in (0,1)$*

$$\left|\mathbb{P}\left[M_t \geq x\right] - \mathbb{P}\left[\hat{M}_t^\epsilon \geq x\right]\right| \leq C_{\rho,\theta}\sigma_0(\epsilon)^{\frac{\alpha}{(1+\rho)\alpha+1}}\left(\beta_{\frac{1}{\rho},\rho}(\epsilon)\right)^{\frac{\alpha\rho}{\alpha(1+\rho)+1}}$$

The constant $C_{\rho,\theta}$ is independent of ϵ. The proof can be found in [6].

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