A contribution to the systematics of stochastic volatility models

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Abstract

We compare systematically several classes of stochastic volatility models of stock market fluctuations. We show that the long-time return distribution is either Gaussian or develops a power-law tail, while the short-time return distribution has generically a stretched-exponential form, but can assume also an algebraic decay, in the family of models which we call "GARCH"-type. The intermediate regime is found in the exponential Ornstein-Uhlenbeck process. We calculate also the decay of the autocorrelation function of volatility.

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1. Introduction

The modelling based on stochastic volatility [1, 2] is a natural way how to implement non-Markovian property of certain stochastic processes into the machinery of stochastic differential equations, which are Markovian by definition. The idea is old, well-known and straightforward and amounts postulating an auxiliary process (or processes), which together with our process of interest form a multicomponent process, which is Markovian, while each of the components itself is not. The auxiliary processes may have a direct physical interpretation or not, depending on the situation.

When modelling the stock-market fluctuations, the process of interest is the price (which is directly measurable) and the auxiliary process is the instantaneous volatility, or some function of it, which is, however, purely hypothetical and in principle not measurable. We can estimate its properties from the volatility measured within certain time interval, but nevertheless, the instantaneous volatility remains an elusive ghost. Supposing it is subject to a process described by a stochastic differential equation, we can solve the equation pair for price and volatility and deduce for example the return distribution, autocorrelations etc.

Since this methodology is quite general, it is of interest not only for economists but can be useful in various purely physical situations. One of such examples might be the

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stochastic resonance under the influence of coloured noise. In this paper we present some generalisations of previously known results, stressing the generic and the special features of several classes of stochastic volatility models.

2. Systematics

Quite general ensemble of stochastic volatility models is described by a pair of stochastic differential equations (in Ito convention) with algebraic coefficients

$$\begin{aligned} \mathrm{d}X_t &= S_t^{\gamma}\,\mathrm{d}W_{1t} \\ \mathrm{d}S_t &= a\,(\sigma - S_t)\,S_t^{\alpha}\,\mathrm{d}t + g\,S_t^{\beta}\,\mathrm{d}W_{2t}\ . \end{aligned} \tag{1}$$

We denote W_{1t} and W_{2t} two independent Wiener processes with unit diffusion constant, $[\mathrm{d}W_{1t}]^2 = [\mathrm{d}W_{2t}]^2 = \mathrm{d}t$. In this article we disregard the possible dependence of the processes W_{1t} and W_{2t}, which was investigated e. g. in [3, 4, 5, 6, 7] in the context of the leverage effect. We also omit the possibility that either W_{1t} or W_{2t} is a Lévy noise [8], because that would lead us to a completely different area.

The process X_t describes the quantity of interest, which is observable and if we remain in the field of econophysics, it is typically the logarithm of the price of a commodity or a security. We shall call it simply the price and allow both positive and negative values for it. S_t is the auxiliary process, representing a hypothetical quantity which will be called volatility. Contrary to the true volatility, which is (whatever its definition might be) genuinely two-time quantity, like the return, the "volatility" S_t is an instantaneous, one-time stochastic variable. This does not mean that it is completely inaccessible to observations, but implies that its relation to observations is not a direct one and poses a separate problem, not to be investigated here. We consider necessary to recall these trivial observations, as they are often neglected in the econophysics literature.

Some of the combinations of the exponents bear their usual names. We make an overview in the following table.

α	β	γ	name or acronym of the model
0	0	1	Stein-Stein
0	0	1/2	"Ornstein-Uhlenbeck" or "OU"
0	1/2	1/2	Heston
0	1	1/2	"GARCH"
1	1	1/2	"geometric OU"
0	3/2	1/2	3/2-model

(The names in quotation marks " " are somewhat abuses of notation, because properly speaking they are already in use in more or less different sense.) Besides these models which contain only coefficients depending on S as a power, we shall investigate also the exponential Ornstein-Uhlenbeck process (acronym expOU) [2, 6, 7, 9, 10, 11, 12, 13, 14, 15], which in its simplest form is described by the pair of equations

$$\begin{aligned} \mathrm{d}X_t &= \mathrm{e}^{\sigma + S_t}\,\mathrm{d}W_{1t} \\ \mathrm{d}S_t &= -a\,S_t\,\mathrm{d}t + g\,\mathrm{d}W_{2t}\ . \end{aligned} \tag{2}$$

Again, X_t is the logarithm of price and S_t the hypothetical hidden variable. The instantaneous volatility in this case is $\mathrm{e}^{2(\sigma+S_t)}$.

If we were interested only in the auxiliary process S_t, we would need to solve only the second of the equations (1). Processes of this type were solved in many contexts and are considerably easier compared to solving the coupled pair (X_t, S_t). The older works relevant to our study are e. g. [16] and [17]. In the former, the special case $\alpha = 0$, $\beta = 1$, $\sigma = 0$ is solved exactly using the Lie-group technique. The latter treats approximately the difficult case $\alpha = 1$, $\beta = 1/2$ which we avoid completely, as we never found it used in the stochastic volatility context.

Let us make a very short historical remark at the end of this section. The study of stochastic volatility models in economy can be traced back to the pioneering works of Vasicek [18] and Cox, Ingersoll and Ross [19] on interest rates. In fact, the same problem was investigated, on the level of Fokker-Planck equation, much earlier by Feller [20]. Hull and White [21] formulated the problem on a more general basis. These results were adapted for the modelling of price fluctuations by Scott [9], Hull and White [22], Stein and Stein [23]. The Stein-Stein model was thoroughly studied [23, 24, 25], although it exhibits some unnatural features, namely negative volatilities. These are absent in the Heston model [26], which became very popular [24, 27, 28, 29, 30, 31, 32]. Of course, the potential of stochastic volatility models was used for better prediction of investment risks, see e. g. [22, 9, 26, 24, 33, 13]. For further variants of the stochastic volatility models used in econophysics see e. g. references [34, 35, 36, 37, 38, 39, 40, 41, 42, 43].

3. Long-time properties

The properties of the process (1) can be studied directly by the method of moments. This way we can easily find indirect evidence on the asymptotic distribution of price change. Indeed, we shall find that in the most important cases the whole set of asymptotic moments can be calculated and that they fall into two groups. Either the moments imply the Gaussian distribution of returns, or some of them diverge, indicating power-law tail in the return distribution.

On the other hand, the method of moments has also serious disadvantages. Most importantly, it does not allow an easy implementation of the requirement of positive volatility. For $\gamma = 1$ this problem does not occur, because the "physical" volatility is the square S_t^2, but for $\gamma = 1/2$ we must consider it. When solving the set of Fokker-Planck equations, it is imposed as a boundary condition. On the level of moments we do not have such possibility. Therefore, we either use the fact that the process itself prohibits trajectories connecting positive and negative volatilities (this is the case of $\beta > 0$) or assume that the events with negative volatilities are rare enough to be neglected. Typically it means that $\sigma \gg g/\sqrt{a}$. This is how we shall treat the case $\beta = 0$.

The second disadvantage resides in the fact that we are unable to calculate the moments for all values of the exponents α, β, and γ. If all of them are rational, we can bring them to a common denominator, say, q, and calculate moments which are integer multiples of $1/q$. For large q it is possible but difficult. However, for irrational and incommensurate values of the exponents this way is unfeasible.

The general moment is defined as $\mu_{m,n}(t) = \langle X_t^m \, S_t^n \rangle$ and its time evolution according to the process (1) is contained in the equation

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mu_{m,n}(t) = na\Big(\sigma\,\mu_{m,n+\alpha-1}(t) - \mu_{m,n+\alpha}(t)\Big) +\\ + \frac{1}{2}m(m-1)\,\mu_{m-2,n+2\gamma}(t) + \frac{g^2}{2}n(n-1)\,\mu_{m,n-2+2\beta}(t)\ .\end{aligned} \tag{3}$$

We do not have a solution in a closed form in the general case, but we can investigate each model separately. We shall follow essentially the order in which the models are listed in the table above.

We start with the Stein-Stein model and proceed from small values of m and n to higher ones. For $m = 0$ the equation is

$$\dot\mu_{0,n} + na\,\mu_{0,n} = na\sigma\,\mu_{0,n-1} + \frac{g^2}{2}n(n-1)\,\mu_{0,n-2} \tag{4}$$

and we can see that knowing $\mu_{0,n-1}$ and $\mu_{0,n-2}$ we obtain $\mu_{0,n}$ by trivial integration. The starting point $\mu_{0,0}(t) = \langle 1\rangle = 1$ is obvious. The next two moments are

$$\begin{aligned}\mu_{0,1}(t) &= \sigma + \big(\mu_{0,1}(0) - \sigma\big)\,\mathrm{e}^{-at}\\ \mu_{0,2}(t) &= \sigma^2 + \frac{g^2}{2a} + \Big[\mu_{0,2}(0) - \Big(\sigma^2 + \frac{g^2}{2a}\Big)\Big]\,\mathrm{e}^{-2at} +\\ &\qquad + 2\sigma\,\big(\mu_{0,1}(0) - \sigma\big)\,\Big(\mathrm{e}^{-at} - \mathrm{e}^{-2at}\Big)\end{aligned} \tag{5}$$

and we can continue further as far as we please. The generic feature of the time evolution of moments, seen in (5) is an exponential relaxation to a stationary value. Setting $\dot\mu_{0,n} = 0$ in (4) we obtain a chain of equations for the stationary values of the moments $\mu_{0,n}(\infty)$. First two of them, $\mu_{0,1}(\infty) = \sigma$, $\mu_{0,2}(\infty) = \sigma^2 + \frac{g^2}{2a}$ can be seen in the formulae (5), and the general result is

$$\mu_{0,n}(\infty) = \sum_{k=0}^{\lfloor n/2\rfloor} \binom{n}{2k} \frac{(2k)!}{2^k\,k!} \Big(\frac{g^2}{2a}\Big)^k \sigma^{n-2k} \tag{6}$$

confirming that the stationary distribution of S_t only is Gaussian with average σ and variance $g^2/(2a)$, as expected. (The fact that the distribution is Gaussian follows immediately from the comparison with the formula for $2k$-th moment of a Gaussian distribution with variance s, which is $[(2k)!/(2^k\,k!)]\,s^k$.)

The moments with $m > 0$ are just a bit more complicated. We need to investigate only even m, because all moments for odd m are zero. Integrating the equation (3) we can get the moments for higher and higher m systematically. The most interesting part of the result is the behaviour of the moments for large times. For example, we find, using (5), that

$$\mu_{2,0}(t) = \mu_{2,0}(0) + \int_0^t \mu_{0,2}(\tau)\,\mathrm{d}\tau \simeq \Big(\sigma^2 + \frac{g^2}{2a}\Big)\,t\ , \quad \text{as} \quad t\to\infty\ . \tag{7}$$

Generally, it is possible to show that $\big(\frac{\mathrm{d}}{\mathrm{d}t}\big)^l \mu_{2l,n}(t)$ has a finite limit when $t\to\infty$ and after some easy algebra we arrive at a formula summarising the long-time behaviour of all moments

$$\lim_{t\to\infty} t^{-l}\,\mu_{2l,n}(t) = \frac{(2l)!}{2^l\,l!}\Big(\sigma^2+\frac{g^2}{2a}\Big)^l\,\mu_{0,n}(\infty)\;. \tag{8}$$

This result indicates that for long times the logarithmic returns have Gaussian distribution with variance $\big(\sigma^2+\frac{g^2}{2a}\big)\,t$.

Analogous steps ought to be taken when we want to find the moments in the case of the "OU" model. Note that the moments including only the process S_t, i. e. those with $m=0$, are identical to the Stein-Stein model. Therefore, the formulae (5) and (6) remain in force. The long-time asymptotics of the remaining moments is obtained in the same way as Eq. (8). This time, the result is

$$\lim_{t\to\infty} t^{-l}\,\mu_{2l,n}(t) = \frac{(2l)!}{2^l\,l!}\,\sigma^l\,\mu_{0,n}(\infty) \tag{9}$$

and again suggests Gaussian distribution of the returns.

The next in line is the Heston model. The moments evolve according to the set of equations

$$\dot{\mu}_{m,n} + na\,\mu_{m,n} = na\big(\sigma+\frac{g^2}{2a}(n-1)\big)\,\mu_{m,n-1}+\frac{1}{2}m(m-1)\,\mu_{m-2,n+1} \tag{10}$$

and solving them for the lowest non-trivial moments we get

$$\begin{aligned}\mu_{0,1}(t) &= \sigma + \big(\mu_{0,1}(0)-\sigma\big)\,\mathrm{e}^{-at}\\ \mu_{2,0}(t) &= \mu_{2,0}(0)+\sigma\,t+\frac{1}{a}\big(\mu_{0,1}(0)-\sigma\big)\big(1-\mathrm{e}^{-at}\big)\;.\end{aligned} \tag{11}$$

The long-time behaviour is again obtained assuming that the derivatives $\big(\frac{\mathrm{d}}{\mathrm{d}t}\big)^l\mu_{2l,n}(t)$ have finite limit for $t\to\infty$ and checking a posteriori that this assumption does not contain any contradiction. We can do even better than we could for Stein-Stein and "OU" models. The limits of the moments are expressed by the closed formula

$$\lim_{t\to\infty} t^{-l}\,\mu_{2l,n}(t) = \frac{(2l)!}{2^l\,l!}\,\sigma^l\Big(\frac{g^2}{2a}\Big)^n\frac{\Gamma\big(\frac{2a\sigma}{g^2}+n\big)}{\Gamma\big(\frac{2a\sigma}{g^2}\big)} \tag{12}$$

showing again without any doubt that the long-time distribution of returns is Gaussian.

The same procedure is effective also for the "GARCH" model, where the moments evolve according to

$$\dot{\mu}_{m,n} + na\Big(1-\frac{g^2}{2a}(n-1)\Big)\mu_{m,n} = na\,\sigma\,\mu_{m,n-1}+\frac{1}{2}m(m-1)\,\mu_{m-2,n+1}\;. \tag{13}$$

The evolution of the lowest moments $\mu_{0,1}(t)$ and $\mu_{2,0}(t)$ follows exactly the same law (11) as in the case of the Heston model. Moreover, we have

$$\begin{aligned}\mu_{0,2}(t) = \frac{2a\,\sigma^2}{2a-g^2} &+ \Big(\mu_{0,2}(0)-\frac{2a\,\sigma^2}{2a-g^2}\Big)\,\mathrm{e}^{-(2a-g^2)t}+\\ &+2a\,\sigma\,\frac{\mu_{0,1}-\sigma}{a-g^2}\,\mathrm{e}^{-at}\big(1-\mathrm{e}^{-(a-g^2)t}\big)\;.\end{aligned} \tag{14}$$

We must be more careful here when we compute the asymptotics of the moments. Some of them diverge exponentially, rather than algebraically, with t, becoming effectively infinite. We find that the limit

$$\lim_{t\to\infty} t^{-l}\,\mu_{2l,n}(t) = \frac{(2l)!}{2^l\,l!}\,\sigma^{l+n}\left[\prod_{j=1}^{n-1}\big(1-j\frac{g^2}{2a}\big)\right]^{-1} \tag{15}$$

is finite as long as $l+n<1+\frac{2a}{g^2}$. Otherwise the moment should be considered diverging. The interpretation of this result is, that the distribution develops power-law tails. It should hold both for the marginal distribution of returns, described by the moments with $n=0$, and the marginal distribution of volatility, corresponding to the set of moments with $l=0$. Hence, the tail exponent of the return distribution is

$$\tau = 3+\frac{4a}{g^2}\ . \tag{16}$$

Therefore, the exponent is always bounded from below by $\tau>3$, a result which is well consistent with the empirical findings [44, 45]. Note also that the empirical volatility distribution [46, 47, 48, 49, 2] shows log-normal central part, followed by a power-law tail, which is consistent with the power-law tail in the distribution of the hypothetical volatility S_t, as is evident from (15). All of that suggests that the "GARCH" member of the stochastic-volatility family of stock market models is close to reality.

The next two models in the list are the "geometric OU" and the 3/2 models. Unfortunately, the method of moments does not lead to a recursive chain which would allow computing all the moments. The only way forward is to solve directly the corresponding Fokker-Planck equations. We shall see later how we can bypass it in the short-time approximation.

The treatment of the expOU model using the method of moments is slightly more involved, as it requires expansion of the exponential in powers and later resummation of a series back, to get a compact result. The form of the equations (2) ensures that the odd moments are zero, $\mu_{m,2k-1}=\mu_{2l-1,n}=0$, if the initial condition is $X_0=S_0=0$. In that case, we find, for the even moments

$$\lim_{t\to\infty} t^{-l}\,\mu_{2l,2k}(t) = \frac{(2l)!}{2^l\,l!}\,\exp\Big[l\big(2\sigma+\frac{g^2}{a}\big)\Big]\,\frac{(2k)!}{2^k\,k!}\,\Big(\frac{g^2}{2a}\Big)^k\ . \tag{17}$$

This type of formula is already familiar to us. The interpretation is that at long times the logarithm of price is Gaussian-distributed with variance $t\exp(2\sigma+g^2/a)$.

4. Autocorrelations

The autocorrelation function of returns

$$C_{\mathrm{ret}\,q}(\Delta t;\delta t) = \langle|(X_{t+\delta t}-X_t)(X_{t+\Delta t+\delta t}-X_{\Delta t})|^q\rangle \tag{18}$$

is a four-time quantity, not easy to calculate by the method of moments. Instead, we shall investigate a simpler quantity, which is supposed to share relevant properties with the autocorrelation function (18). Indeed, for short time intervals δt the absolute increments of price are proportional to S_t^γ and we can consider the autocorrelation function

$$C_q(\Delta t) = \langle |S_t\, S_{t+\Delta t}|^{q\gamma} \rangle \tag{19}$$

as an approximation for $C_{\mathrm{ret}\,q}$ of $\delta t \to 0$. In stationary state it is a function of one time argument only and we obtain for it exactly the same differential equations as we have had for the moments. To this end we define a slightly more general autocorrelation

$$C_{u,v}(\Delta t) = \langle S_t^u\, S_{t+\Delta t}^v \rangle \tag{20}$$

and obtain the following equation for it

$$\frac{\mathrm{d}C_{u,v}(\Delta t)}{\mathrm{d}\Delta t} = -va\big(C_{u,v+\alpha}(\Delta t) - \sigma\, C_{u,v-1+\alpha}(\Delta t)\big) + \frac{g^2}{2} v(v-1)\, C_{u,v-2+2\beta}(\Delta t)\ . \tag{21}$$

We can see that the autocorrelation function with $v = 1$ does not depend on the value of β. Especially, all the models examined in detail above have $\alpha = 0$ and the solution for all of them is

$$C_{u,1}(\Delta t) = \sigma\, \mu_{0,u}(\infty) + \big(\mu_{0,u+1}(\infty) - \sigma\, \mu_{0,u}(\infty)\big)\, \mathrm{e}^{-a\Delta t}\ . \tag{22}$$

For $v > 1$ the value of β is relevant. We could proceed iteratively finding the correlation function for $v = 2$, then for $v = 3$ etc. However, it is possible to express the solution as a linear combination of exponentials and then solve the set of equations for the coefficients. Thus we find the full solution in relatively compact form. In the case $\alpha = \beta = 0$ (i. e. Stein-Stein and "OU" models), we get

$$C_{u,v}(\Delta t) = \sum_{m=0}^{v} \binom{v}{m} \left[\sum_{n=0}^{\lfloor \frac{m}{2} \rfloor} \frac{(2n)!}{2^n\, n!} \binom{m}{2n} \sigma^{m-2n} \Big(\frac{g^2}{2a}\Big)^n \right] f_{v-m}\, \mathrm{e}^{-(v-m)a\,\Delta t} \tag{23}$$

where f_l are constants determined from the initial conditions $C_{u,v}(0) = \mu_{0,u+v}(\infty)$.

Similarly, for the Heston model ($\beta = 1/2$) we have

$$C_{u,v}(\Delta t) = \sum_{l=0}^{v} \binom{v}{l} \frac{\Gamma\Big(\frac{2a\sigma}{g^2} + v\Big)}{\Gamma\Big(\frac{2a\sigma}{g^2}\Big)} \Big(\frac{g^2}{2a}\Big)^v f_l\, \mathrm{e}^{-la\,\Delta t}\ . \tag{24}$$

The case $\beta = 1$ ("GARCH") is more complicated. We could also try to find general expressions like (23) and (24), but the decay rates of the exponentials are not ordered in arithmetic sequence $\{la\}_{l=0}^{\infty}$, but form a two-parametric set $\{l_1\, a - l_2\, g^2/2 \mid l_1, l_2 = 0, 1, 2 \ldots\}$. Some of these rates are negative, which indicates that some of the correlation functions do not exist in stationary state. This feature stems from the divergence of higher moments of the volatility in stationary regime. However, the general conclusion holds, that if the stationary autocorrelation function does exist, it decays as a combination of exponentials.

We can conclude that the common feature of all stochastic volatility models investigated here is the exponential decay of autocorrelations. This is natural, because the process S_t has Markov property. For practical use, we note that the multiple timescales found in the expOU model [10, 11] can successfully reproduce the slow decay of autocorrelations found in empirical data. Special cases when the autocorrelation does decay as a power law are examined e. g. in [50], but these cases do not seem to have much relevance for stock-market modelling.

5. Short-time properties

From the point of view of return distributions, the long-time properties of the stochastic volatility models discussed so far do not exhibit much diversity. The return distribution is mostly Gaussian, with the exception of the "GARCH" model, where power-law tails develop. At short times, however, more variable results are found. Indeed, the empirical return distribution tend all to a Gaussian for large enough times, so it is the range of medium time distances at which the interesting fat tails are observed. So, it is well sensible to discuss the special features of the stochastic volatility models appearing at short times, i. e. at times shorter than those at which Gaussian limit is reached. We have seen that the typical decay time of autocorrelations is $1/a$. We expect that this is also the time when the Gaussian distribution starts to prevail. Therefore, we shall investigate the return distributions for time distances $\Delta t \ll a^{-1}$. This kind of approach is sometimes related [51] to the Born-Oppenheimer approximation used in quantum chemistry.

5.1. Volatility distribution

The ingredient needed for these investigations will be the stationary distribution for the volatility process S_t. It is easy to find it, because the instantaneous volatility influences the price process X_t, but is not affected by it, so we can find the distribution for S_t separately. We must solve the Fokker-Planck equation

$$\begin{aligned} \frac{\partial}{\partial t} P_{S,t}(s) = & a \frac{\partial}{\partial s}\big[(s-\sigma)s^{\alpha}\, P_{S,t}(s)\big] + \\ & + \frac{g^2}{2}\frac{\partial^2}{\partial s^2}\big[s^{2\beta}\, P_{S,t}(s)\big] \; . \end{aligned} \tag{25}$$

in the limit $t \to \infty$. Contrary to the method of moments, we can obtain the solution generally for any value of the parameters α and β. It is also easy to implement the reflective boundary condition at $S_t = 0$, enforcing the positivity of the volatility simply by multiplying the solution of (25) by the Heaviside function $\theta(s)$ and recomputing the normalisation factor so that $\int_0^\infty P_S(s)\mathrm{d}s = 1$. This will work for all solutions with zero total current.

We can get easily the generic solution with zero current in the form

$$P_S(s) = \mathcal{N}\,\theta(s)\, s^{-2\beta}\, \exp\Big(- \frac{2a}{g^2}\, \frac{s^{2+\alpha-2\beta}}{2+\alpha-2\beta} + \frac{2a\,\sigma}{g^2}\, \frac{s^{1+\alpha-2\beta}}{1+\alpha-2\beta}\Big) \tag{26}$$

where $\mathcal{N}$ is the appropriate normalisation constant. The Gaussian distribution is included here, for the combination of parameters giving $\alpha - 2\beta = 0$.

There are two exceptional cases, marked by the vanishing denominators in the exponent in Eq. (26). When $1 + \alpha - 2\beta = 0$ we have

$$P_S(s) = \mathcal{N}\,\theta(s)\, s^{-2\beta + 2a\,\sigma/g^2}\, \exp\Big(- \frac{2a}{g^2}\, s\Big) \; . \tag{27}$$

We shall call this case "Heston type", as the Heston model belongs to it. The second exceptional case occurs when $2 + \alpha - 2\beta = 0$. The probability density for the volatility is then

$$P_S(s) = \mathcal{N}\,\theta(s)\, s^{-2\beta - 2a/g^2}\, \exp\Big(- \frac{2a\,\sigma}{g^2\, s}\Big) \; . \tag{28}$$

This case will be named "GARCH-type", because it includes the "GARCH" model studied above.

5.2. Return distribution

We can neglect fluctuations of the volatility during time delays $\Delta t \ll \alpha^{-1}$ and consider the volatility fixed with distribution given by (26), or, in special cases, by (26) or (27). Thus, our starting formula will be

$$P_{\Delta X}(\Delta x;\Delta t) = \int_0^\infty s^{-\gamma}\,\exp\Big(\frac{(\Delta x)^2}{2\Delta t\, s^{2\gamma}}\Big) P_S(s) \frac{\mathrm{d}s}{\sqrt{2\pi\,\Delta t}}\ . \tag{29}$$

Contrary to the method of moments, we can proceed directly with the calculation for arbitrary values of the parameters α, β, γ. We shall explore this freedom for α and β, bit for the parameter γ we use only the most studied values 1/2 and 1. The general strategy will be to use a suitable change of variables to perform the integral (29) by the saddle-point method.

5.2.1. $\gamma = 1/2$

In the general case (26) we define

$$\begin{aligned} \xi =& \Big[\frac{(\Delta x)^2}{\Delta t}\Big]^{\frac{2+\alpha-2\beta}{3+\alpha-2\beta}} \Big[\frac{4a}{g^2(2+\alpha-2\beta)}\Big]^{\frac{1}{3+\alpha-2\beta}} \\ \eta =& \sigma\,\Big[\frac{g^2}{4a}\,\frac{(\Delta x)^2}{\Delta t}\Big]^{-\frac{1}{3+\alpha-2\beta}} \frac{(2+\alpha-2\beta)^{\frac{2+\alpha-2\beta}{3+\alpha-2\beta}}}{1+\alpha-2\beta} \\ G =& \frac{2}{1-4\beta}\,\Big[\frac{g^2}{4a}\,\frac{(\Delta x)^2}{\Delta t}(2+\alpha-2\beta)\Big]^{\frac{1-4\beta}{2(3+\alpha-2\beta)}} \end{aligned} \tag{30}$$

and express the integral (29) as

$$P_{\Delta X}(\Delta x;\Delta t) = \frac{\mathcal{N}\,G}{\sqrt{2\pi\Delta t}} \int_0^\infty \exp\Big(-\frac{\xi}{2}\psi(u;\eta)\Big)\,\mathrm{d}u \tag{31}$$

where

$$\psi(u;\eta) = u^{-\frac{2}{1-4\beta}} + u^{\frac{2(2+\alpha-2\beta)}{1-4\beta}} - \eta\, u^{\frac{2(1+\alpha-2\beta)}{1-4\beta}}\ . \tag{32}$$

The saddle point approximation is appropriate for large returns, $|\Delta x|$, which is just the regime of interest. The location of the saddle point cannot be found exactly, but it can be expressed as a series in powers of η. But as we can see, η decreases to zero when $\Delta x \to \infty$, so the tails of the return distribution are found both in the limit $\xi \to \infty$ and $\eta \to 0$. Systematic loop expansion then gives the correction of the order $O(1/\xi)$. Finally, we get the formula

$$
\begin{aligned}
P_{\Delta X}(\Delta x;\Delta t) = & R_N \Big(\frac{|\Delta x|}{\sqrt{\Delta t}}\Big)^{-\frac{1+\alpha+2\beta}{3+\alpha-2\beta}} \times \\
& \times \exp\Bigg[- R_1 \Big(\frac{|\Delta x|}{\sqrt{\Delta t}}\Big)^{\frac{2(2+\alpha-2\beta)}{3+\alpha-2\beta}} + R_2\,\sigma \Big(\frac{|\Delta x|}{\sqrt{\Delta t}}\Big)^{\frac{2(1+\alpha-2\beta)}{3+\alpha-2\beta}} + \\
& + O\Big(\sigma^2\,|\Delta x|^{\frac{2(\alpha-2\beta)}{3+\alpha-2\beta}}\Big)\Bigg] \times \\
& \times \Bigg(1 + R_3\,\sigma \Big(\frac{|\Delta x|}{\sqrt{\Delta t}}\Big)^{-\frac{2}{3+\alpha-2\beta}} + O\Big(\sigma^2\,|\Delta x|^{-\frac{4}{3+\alpha-2\beta}}\Big)\Bigg) \times \\
& \times \Bigg(1 + O\Big(|\Delta x|^{-\frac{2(2+\alpha-2\beta)}{3+\alpha-2\beta}}\Big)\Bigg)
\end{aligned}
\tag{33}
$$

where the constants $R_N,\ R_1,\ R_2,\ R_3$ contain the combination $4a/g^2$ of the remaining parameters and depend also on α and β. The most important aspect of this result is the stretched-exponential decay of the tail of the distribution. Therefore, it is fatter than the Gaussian, but still not fat enough to explain the power-law tails.

There are several special cases to be investigated separately. First, it is clear that the algebraic substitution which leads to the function (32) fails if $\beta = 1/4$. In this case, exponential substitution must be used instead, in the form

$$
\psi(u;\eta) = \mathrm{e}^{-u} + \mathrm{e}^{(\alpha+3/2)\,u} - \eta\,\mathrm{e}^{(\alpha+1/2)\,u} \tag{34}
$$

which leads to a result very similar to (33). In fact, it is equivalent to simply setting $\beta = 1/4$ in (33).

More fundamental changes occur in the cases we called previously “Heston-type” and “GARCH-type”. The saddle-point method is unnecessary here and the integral (29) can be performed explicitly. Thus, for $1+\alpha-2\beta = 0$ we obtain

$$
\begin{aligned}
P_{\Delta X}(\Delta x;\Delta t) = & \frac{2\mathcal{N}}{(1-4\beta+4a\,\sigma/g^2)\sqrt{2\pi\,\Delta t}} \left(\frac{(\Delta x)^2}{\Delta t}\,\frac{g^2}{4a}\right)^{\frac{1}{4}-\beta+\sigma a/g^2} \times \\
& \times\, K_{\frac{1}{2}-2\beta+2a\sigma/g^2}\left(\frac{|\Delta x|}{\sqrt{\Delta t}}\sqrt{\frac{4a}{g^2}}\right)
\end{aligned}
\tag{35}
$$

with $K_\nu(z)$ the modified Bessel function. From the asymptotic expansion of the Bessel function we deduce exponential decay of the tail of the return distribution. This result was already found for the Heston model itself [27, 28, 29]. Now we can see that this behaviour holds unchanged for the whole “Heston-type” class of models.

For $2+\alpha-2\beta = 0$ the calculation is even simpler. The result is the algebraic decay of the return distribution

$$
P_{\Delta X}(\Delta x;\Delta t) = \frac{2\mathcal{N}}{\sqrt{2\pi\,\Delta t}}\, \frac{\Gamma\big(-\frac{1}{2}+2\beta+2a/g^2\big)}{\Big(\frac{(\Delta x)^2}{2\Delta t}+\frac{2a\sigma}{g^2}\Big)^{-\frac{1}{2}+2\beta+2a/g^2}} \tag{36}
$$

valid for entire "GARCH-type" class. We have already seen that the moment method reveals the power-law tail at large times for the "GARCH" model, $\alpha = 0$, $\beta = 1$. Now we can see that, at least for short time, the power-law tails apply to a wide ensemble of models.

5.2.2. $\gamma = 1$

The same calculation can be performed for the case $\gamma = 1$. For some values of α and β it is sensible to accept also negative values of the stochastic volatility S_t. This is the case, e.g. of the Stein-Stein model, $\alpha = \beta = 0$. However, for generic values of α and β we still need to restrict the definition range to $S_t > 0$. Therefore, we postulate the reflecting boundary at $S_t = 0$ also here. This makes the case $\gamma = 1$ entirely analogous to the previously studied $\gamma = 1/2$.

The special cases are $\beta = 0$, which requires exponential substitution, and again the $1 + \alpha - 2\beta = 0$ and $2 + \alpha - 2\beta = 0$ cases as before. The generic case is characterised by substitutions

$$\begin{aligned} \xi &= \Big[\frac{(\Delta x)^2}{\Delta t}\Big]^{\frac{2+\alpha-2\beta}{4+\alpha-2\beta}} \; \Big[\frac{g^2}{4a}(2+\alpha-2\beta)\Big]^{-\frac{2}{4+\alpha-2\beta}} \\ \eta &= \sigma \Big[\frac{g^2}{4a}\,\frac{(\Delta x)^2}{\Delta t}\Big]^{-\frac{1}{4+\alpha-2\beta}} \; \frac{(2+\alpha-2\beta)^{\frac{3+\alpha-2\beta}{4+\alpha-2\beta}}}{1+\alpha-2\beta} \\ G &= \frac{1}{2\beta}\Big[\frac{g^2}{4a}\,\frac{(\Delta x)^2}{\Delta t}(2+\alpha-2\beta)\Big]^{-\frac{2\beta}{4+\alpha-2\beta}} \end{aligned} \tag{37}$$

The integral to perform is

$$P_{\Delta X}(\Delta x; \Delta t) = \frac{\mathcal{N}\, G}{\sqrt{2\pi\Delta t}} \int_0^\infty \exp\Big(- \frac{\xi}{2}\psi(u;\eta)\Big)\, \mathrm{d}u \tag{38}$$

where

$$\psi(u;\eta) = u^{\frac{1}{\beta}} + u^{-\frac{2+\alpha-2\beta}{2\beta}} - \eta\, u^{-\frac{1+\alpha-2\beta}{2\beta}} \;. \tag{39}$$

Then, the return distribution for $\Delta x \to \infty$ is

$$\begin{aligned} P_{\Delta X}(\Delta x; \Delta t) = R_N \Big(\frac{|\Delta x|}{\sqrt{\Delta t}}\Big)^{-\frac{2+\alpha+2\beta}{4+\alpha-2\beta}} \;\times \\ \times \exp\Bigg[- R_1 \Big(\frac{|\Delta x|}{\sqrt{\Delta t}}\Big)^{\frac{2(2+\alpha-2\beta)}{4+\alpha-2\beta}} + R_2\,\sigma \Big(\frac{|\Delta x|}{\sqrt{\Delta t}}\Big)^{\frac{2(1+\alpha-2\beta)}{4+\alpha-2\beta}} + \\ + O\Big(\sigma^2\, |\Delta x|^{\frac{2(\alpha-2\beta)}{4+\alpha-2\beta}}\Big)\Bigg] \times \\ \times \Bigg(1 + R_3\,\sigma \Big(\frac{|\Delta x|}{\sqrt{\Delta t}}\Big)^{-\frac{2}{4+\alpha-2\beta}} + O\Big(\sigma^2\,|\Delta x|^{-\frac{4}{4+\alpha-2\beta}}\Big)\Bigg) \times \\ \times \Bigg(1 + O\Big(|\Delta x|^{-\frac{2(2+\alpha-2\beta)}{4+\alpha-2\beta}}\Big)\Bigg) \end{aligned} \tag{40}$$

with, again, some constants R_N, R_1, R_2, R_3, in full analogy with (33).

Explicit calculation shows that the case $\beta = 0$, treated by exponential substitution and saddle point method, gives again the same result as direct substitution $\beta = 0$ in (40).

The special cases are treated similarly. For $1 + \alpha - 2\beta = 0$ we have the following substitutions

$$\begin{aligned} \xi &= \Big[\frac{(\Delta x)^2}{\Delta t}\Big]^{\frac{1}{3}} \Big(\frac{g^2}{4a}\Big)^{-\frac{2}{3}} \\ G &= \frac{1}{-2\beta + 2a\sigma/g^2} \Big[\frac{g^2}{4a}\,\frac{(\Delta x)^2}{\Delta t}\Big]^{-\frac{2\beta}{3}+\frac{2a\sigma}{3g^2}} \end{aligned} \tag{41}$$

and the function within the exponential is now

$$\psi(u) = u^{-2/(-2\beta+2a\sigma/g^2)} + u^{1/(-2\beta+2a\sigma/g^2)} \,. \tag{42}$$

The resulting return distribution can be expressed by generalised hypergeometric functions, but it is more transparent to perform the saddle-point calculations, which give the tails

$$\begin{aligned} P_{\Delta X}(\Delta x; \Delta t) = R_N \Big(\frac{|\Delta x|}{\sqrt{\Delta t}}\Big)^{-\frac{1}{3}-\frac{4}{3}\beta+\frac{4a\sigma}{3g^2}} \exp\Big[-R_1\Big(\frac{|\Delta x|}{\sqrt{\Delta t}}\Big)^{\frac{2}{3}}\Big] \times \\ \times \bigg(1 + O\Big(|\Delta x|^{-\frac{2}{3}}\Big)\bigg) \end{aligned} \tag{43}$$

The remaining special case is $2+\alpha-2\beta = 0$. The integral to perform has the following form

$$P_{\Delta X}(\Delta x; \Delta t) = \frac{\mathcal{N}}{\sqrt{2\pi\Delta t}} \int_0^\infty s^{-1-2\beta-2a/g^2} \exp\Big(-\frac{(\Delta x)^2}{2\Delta t\, s^2} - \frac{2a\sigma}{g^2\, s}\Big) \,. \tag{44}$$

Here, the saddle-point method is of no use, because the corresponding function $\psi(u)$, if used, would have no local extreme. Instead, we can use a simple trick to estimate the behaviour of the tail, at $|\Delta x| \to \infty$. The exponential function within the integrand is close to 1 for $s \gtrsim s^* \simeq |\Delta x|/\sqrt{\Delta t}$ and very small for $s \lesssim s^*$. More precisely, this behaviour holds only for $|\Delta x|/\sqrt{\Delta t} \gtrsim 2a\sigma/g^2$, but for the discussion of the tail it is quite sufficient. Thus, the integral (44) can be estimated by integrating only the algebraic factor from s^* to infinity. Finally, we obtain the power-law tail of the return distribution in the form

$$P_{\Delta X}(\Delta x; \Delta t) \sim \Big(\frac{|\Delta x|}{\sqrt{\Delta t}}\Big)^{-2\beta-2a/g^2} \,. \tag{45}$$

As a summary, we can see that the short-time return distribution has generically a tail in the form of a stretched exponential. This becomes simple exponential in the case of $\gamma = 1/2$ and Heston-type combination of parameters, $1+\alpha-2\beta = 0$. The exceptional case is the family of "GARCH"-type models, characterised by the combination $2+\alpha-2\beta = 0$. In that case, we found that both for $\gamma = 1/2$ and for $\gamma = 1$ the tail of the return distribution has a power-law form.

As a separate calculation, we show how the same procedure works for the expOU model. We have to perform the integral

$$P_{\Delta X}(\Delta x; \Delta t) = \sqrt{\frac{a}{\pi^2 \Delta t\, g^2}} \int_0^\infty \mathrm{e}^{-a\,\psi(u)/(2g^2)}\, \mathrm{d}u \tag{46}$$

where

$$\psi(u) = 2\xi\, u + \big(\ln u\big)^2 \tag{47}$$

and

$$\xi = \frac{g^2\,(\Delta x)^2}{2a\,\Delta t}\ . \tag{48}$$

The saddle point u^* can be expressed using the Lambert function $W_{\rm L}(y)$ defined by the equation $W_{\rm L}(y)\,{\rm e}^{W_{\rm L}(y)} = y$. Finally, we find that

$$\psi(u^*) = \big(1 + W_{\rm L}(\xi)\big)^2 - 1 \tag{49}$$

and within the saddle-point approximation

$$P_{\Delta X}(\Delta x; \Delta t) \propto \exp\Big\{ -\frac{a}{2g^2}\Big[\Big(1 + W_{\rm L}\Big(\frac{g^2\,(\Delta x)^2}{2a\,\Delta t}\Big)\Big)^2 - 1\Big]\Big\}\ . \tag{50}$$

The tails of the return distributions, $\Delta x \to \infty$, correspond again to the limit $\xi \to \infty$. However, the situation is less transparent here, because the Lambert function does not have a simple asymptotic behaviour. Naively, one could use the asymptotics

$$W_{\rm L}(y) \simeq \ln y, \qquad y \to \infty \tag{51}$$

but it holds well only for extremely large y. There is a much better approximate formula

$$W_{\rm L}(y) \simeq \ln y + \Big(\frac{1}{1+\ln y} - 1\Big)\ \ln\ln y, \qquad y \to \infty \tag{52}$$

which is reasonably accurate already for $y \gtrsim 3$. The farthest tail of the return distribution, using (50) and (51), has the form

$$P_{\Delta X}(\Delta x) \sim (\Delta x)^{-(2a/g^2)\ln \Delta x} \tag{53}$$

which is steeper than any power but fatter than any stretched exponential. Therefore, the expOU model lies somewhere between the "GARCH" model and the other models exhibiting stretched exponential tails. In this sense it is very promising, as the true power-law tails are hardly detectable in real data, but the behaviour very close to power law is well documented. The exponential Ornstein-Uhlenbeck process may serve as a convenient tool in this situation.

6. Conclusions

We showed that many properties of wide class of stochastic volatility models can be deduced quite generally without full explicit solution of the Fokker-Planck equation for the set of coupled stochastic differential equations. The class characterised by algebraic coefficients was solved systematically in the long-time and short-time regimes. Full solution for the stationary autocorrelation functions was found in two special cases, while generic features of the autocorrelation function are found generally.

Specifically, the method of moments was shown to reveal the asymptotic distribution of return and volatility. The return distribution is asymptotically Gaussian, except for the "GARCH" model, where power-law tail develops. This tail in returns originates from analogous power-law tail in the distribution of volatility. The empirical findings [47, 48] suggest that the central part of the volatility distribution is close to log-normal, while the tail is fatter, perhaps indeed a power-law. This makes the "GARCH" stochastic volatility model a good candidate for phenomenological modelling and risk assessment. Saying "phenomenological" we admit and stress that the model does not say anything of the real mechanism how the price fluctuations are produced, neither it explains why just the chosen combination of the parameters α, β and γ is superior to others.

While the price process is not Markovian any more, the compound price-volatility process does have the Markov property by definition. It results in exponential decay of volatility autocorrelation. Multiple time scales found in the expOU model may be used as explanation for the slow decay of empirical autocorrelations [11, 10]. However, to balance the opinions, we should also note that the approach of [10] was severely criticised in [52].

The fat tails in return distribution, which is the most notorious stylised fact of empirical econophysics, are in fact a short-time property. At long time distances, the return distribution approaches to a Gaussian, as if the price process was a Brownian motion. This lead us to the study of short-time properties of the stochastic-volatility models, which at long times exhibit undifferentiated (with exception of the "GARCH" class) Gaussian behaviour. We examined the return distributions for general values of the parameters α, β and γ and we found that stretched exponential is the generic form of the distribution. As a special case, we find exponential distribution in the Heston-type class, $1 + \alpha - 2\beta = 0$. One exception is the "GARCH"-type class, $(2 + \alpha - 2\beta = 0)$ where the distribution has power-law tail. Of course, this is not surprising, as we already know that the power-law tail persists for all times. The intermediate class is represented by the exponential Ornstein-Uhlenbeck process, where the tail is thinner than a power-law, but fatter than stretched exponential.

To sum up, we examined systematically whole class of stochastic volatility models and found explicitly their asymptotic and short time properties, as well as the volatility autocorrelation. Power-law tails are reproduces in the family of "GARCH"-type models, while generic form of the short-time return distribution is stretched exponential.

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