

Path Integral and Asian Options

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Abstract

In this paper we analytically study the pricing of the arithmetically averaged Asian option in the path integral formalism. By a trick about the Dirac delta function, the measure of the path integral is defined by an effective action whose potential term is an exponential function, i.e. the Liouville Hamiltonian, which can be explicitly solved. After working out some auxiliary integrations involving Bessel and Whittaker functions, we arrive at the spectral expansion expression of the value of an Asian option.

1 Introduction

It has been known for a long time that the path integral formalism can be applied to the pricing of financial securities. The more standard methods in quantitative finance are the stochastic calculus and partial differential equations. In almost all cases the relevant equations are diffusion type, whose solution is determined by the heat kernel. It is well-known that [1] the heat kernel can be written in terms of a path integral. This is the starting point for the financial applications of the path integral formalism. In [2, 3] it has been applied to the European options and the one-factor term-structure models. In [4, 5] it is shown how the pricing of path-dependent options can be incorporated into the path integral formulation. It is applied in [6] to models with stochastic volatility, and in [7] by the same author to the Heath-Jarrow-Morton model of forward interest rates. See e.g. [8]-[15] for more works in this direction.

Among many exotic options in the financial market, the asian option is a very popular one. Its payoff depends on the arithmetic average of the price of the underlying asset during the life of this option contract. The Asian options has the advantage that it is usually less expensive than standard options due to its smaller volatility, and its value is harder to be manipulated by a large market participant. So it is more safe to hold it. On the theoretical side, the exact pricing of the arithmetically averaged Asian option is a challenging problem, since the arithmetical average of a stochastic variable, which is logarithmic-normal, is not logarithmic-normally distributed any more. In the pioneering work [16], Geman and Yor derived a closed form expression for the Laplace transformation of the value of the Asian option. In [17] Linetsky obtained a spectral expansion expression of its value in terms of confluent hypergeometric functions. This is the main motivation of the present paper.

In this paper we will study the the arithmetically averaged Asian option in the path integral formalism. In section 2, we review the path integral formulation for a general path-dependent option. For the Asian option, the resulting effective action is the (imaginary-time) Liouville quantum mechanics. In section 3, we obtain the corresponding heat kernel by solving the differential equation it satisfies. We work out two auxiliary integrals in the following section 4 and 5. In section 6, we specify the payoff function and study the value of the put and call options. For the put option we can directly calculate its value, which is equivalent to the result of [17], while for the call option we have to reply on the put-call parity relation.

2 Path integral formulation

If S denotes the price of a stock, it is commonly assumed that $X := \log S$ follows the rule of a Brown motion. A stock option is a kind of financial derivative whose value \mathcal{O} depends on the behavior of the underlining stock price [18]. For a general path-dependent option, its payoff function $\Phi[X]$ is a functional of X . That is the final yield of this option contract depends on the whole history of the stock price before its maturity. The usual European option is just a special case, whose payoff functional is local, i.e. only depends on the stock price at the maturity day.

The value of a (newly written) stock option can be written in a path integral form as

$$\mathcal{O} = e^{-rt} \int_{-\infty}^{\infty} dx' \int_{X(0)=x'}^{X(t)=x} DX \exp \left\{ -\frac{1}{2\sigma^2} \int_0^t \left(\frac{dX}{dt} + \mu \right)^2 dt \right\} \Phi[X]. \quad (2.1)$$

In the above equation, r denoted the risk-free interest rate, σ is the volatility of the stock price, and $\mu := r - \sigma^2/2$. All of these parameters are assumed to be constant. The meaning of t is the remaining time of this option contract, with $t = 0$ denote the maturity day. By introducing the following dimensionless combinations

$$R = \frac{r}{\sigma^2}, \quad \tau = \sigma^2 t, \quad \nu = \frac{2\mu}{\sigma^2}, \quad (2.2)$$

we can simplify the above equation as

$$\mathcal{O}(\tau, x) = e^{-R\tau} \int_{-\infty}^{\infty} dx' e^{\nu(x'-x)/2 - \nu^2\tau/8} \int_{X(0)=x'}^{X(\tau)=x} DX \exp \left\{ -\frac{1}{2} \int_0^\tau \dot{X}^2 d\tau \right\} \Phi[X]. \quad (2.3)$$

If the payoff functional takes the form $\Phi[X] = \phi(V[X])$ with ¹

$$V = V[X] = \int_0^\tau e^X d\tau, \quad (2.4)$$

then we have

$$\begin{aligned} \Phi[X] &= \frac{1}{\tau} \int_0^\infty \delta(\xi - V) \phi(\xi) d\xi \\ &= \frac{1}{\tau} \int_0^\infty d\xi \phi(\xi) \times \frac{1}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} e^{q\xi - qV} dq. \end{aligned} \quad (2.5)$$

We can restrict the integration range to $(0, \infty)$ in the first line because the functional $V = V[X]$ is always positive. In the second line we have use the Laplace transformation

¹In this paper we will call this kind of options as arithmetically averaged Asian-type option or simply Asian option. In principle our formalism applies for any payoff function ϕ .

of the Dirac delta function with ϵ being any positive real number. Inserting (2.5) into (2.3) we have

$$\begin{aligned} \mathcal{O}(\tau, x) &= \frac{e^{-R\tau}}{2\pi i\tau} e^{-\nu x/2 - \nu^2\tau/8} \int_{-\infty}^{\infty} dx' e^{\nu x'/2} \int_0^{\infty} d\xi \phi(\xi) \int_{\epsilon - i\infty}^{\epsilon + i\infty} dq e^{q\xi} \\ &\quad \times \int_{X(0)=x'}^{X(t)=x} DX \exp \left\{ - \int_0^{\tau} \left(\frac{1}{2} \dot{X}^2 + q e^X \right) d\tau \right\}. \end{aligned} \quad (2.6)$$

This is our path integral formulation of the valuation of the Asian option. We see that the system is driven by an effective action

$$A_{\text{eff}}[X] = \int_0^{\tau} \left(\frac{1}{2} \dot{X}^2 + q e^X \right) d\tau, \quad (2.7)$$

which is called (imaginary-time) Liouville quantum mechanics [19].

3 Heat kernel

Define the heat kernel

$$\mathcal{K}(\tau, x, x'; q) := \int_{X(0)=x'}^{X(t)=x} DX \exp \left\{ - \int_0^{\tau} \left(\frac{1}{2} \dot{X}^2 + q e^X \right) d\tau \right\}. \quad (3.1)$$

By use of Feynmann-Kac theorem, it satisfies the following initial value problem

$$\begin{aligned} -\frac{\partial \mathcal{K}}{\partial \tau} &= -\frac{1}{2} \frac{\partial^2 \mathcal{K}}{\partial x^2} + q e^x \mathcal{K}, \\ \mathcal{K}|_{\tau=0} &= \delta(x - x'). \end{aligned} \quad (3.2)$$

Actually this is the (imaginary-time) Schrödinger equation of the effective action (2.7). We may use the method of spectral expansion to construct the heat kernel $\mathcal{K}(\tau, x, x'; q)$. Firstly we solve the following eigenvalue/eigenfunction problem

$$-\frac{1}{2} \frac{\partial^2 \psi_u}{\partial x^2} + q e^x \psi_u = \frac{u^2}{8} \psi_u. \quad (3.3)$$

For $q \in \mathbb{C}$ with $|\arg q| < \pi$, the normalized eigenfunction is

$$\psi_u(x) = \frac{1}{\pi} \sqrt{u \sinh(\pi u)} K_{iu}(\sqrt{8q} e^{x/2}), \quad u > 0, \quad (3.4)$$

where K_{iu} is the modified Bessel function of the second kind. Then the heat kernel can be written as

$$\begin{aligned} \mathcal{K}(\tau, x, x'; q) &= \int_0^{\infty} e^{-u^2\tau/8} \psi_u(x) \psi_u(x') du \\ &= \frac{1}{\pi^2} \int_0^{\infty} e^{-u^2\tau/8} K_{iu}(\sqrt{8q} e^{x/2}) K_{iu}(\sqrt{8q} e^{x'/2}) \sinh(\pi u) u du. \end{aligned} \quad (3.5)$$

It can be explicitly proved that, when $q > 0$, (3.5) is indeed the solution of (3.2). That it satisfies the differential equation can be easily checked. In the appendix we will show that it also satisfies the initial condition, i.e. the completeness of $\{\psi_{iu}(x) | u > 0\}$. Due to the inverse Laplace transformation in (2.6), we need q to be complex with $\text{Re } q > 0$. Since the solution of (3.2) should be a holomorphic function in the complex q -plane cut open along the negative real axis, we can insert (3.5) into (2.6) to calculate the option value, and the result turns out to be correct. Therefore the expression of the option value becomes

$$\begin{aligned} \mathcal{O}(\tau, x) &= \frac{e^{-R\tau}}{2\pi i\tau} e^{-\nu x/2 - \nu^2 \tau/8} \int_{-\infty}^{\infty} dx' e^{\nu x'/2} \int_0^{\infty} d\xi \phi(\xi) \int_{\epsilon - i\infty}^{\epsilon + i\infty} dq e^{q\xi} \\ &\quad \times \frac{1}{\pi^2} \int_0^{\infty} e^{-u^2 \tau/8} K_{iu}(\sqrt{8q} e^{x'/2}) K_{iu}(\sqrt{8q} e^{x'/2}) \sinh(\pi u) u du. \end{aligned} \quad (3.6)$$

4 Integrating out x'

In this section we will consider the integration over the variable x' . Define

$$\mathcal{M}(\tau, x; q) := \int_{-\infty}^{\infty} e^{\nu x'/2} \mathcal{K}(\tau, x, x'; q) dx', \quad (4.1)$$

which satisfies the following initial value problem

$$\begin{aligned} -\frac{\partial \mathcal{M}}{\partial \tau} &= -\frac{1}{2} \frac{\partial^2 \mathcal{M}}{\partial x^2} + q e^x \mathcal{M}, \\ \mathcal{M}|_{\tau=0} &= e^{\nu x/2}. \end{aligned} \quad (4.2)$$

To solve this problem we use the following expansion of the initial configuration [21]

$$\begin{aligned} e^{\nu x/2} &= \frac{2^{-1-\nu/2}}{\pi^2 q^{\nu/2}} \int_0^{\infty} \left| \Gamma\left(\frac{\nu + iu}{2}\right) \right|^2 K_{iu}(\sqrt{8q} e^{x/2}) \sinh(\pi u) u du \\ &\quad + \frac{2^{1-\nu/2}}{q^{\nu/2}} \sum_{n=0}^{[-\nu/2]} \frac{(-\nu - 2n)}{n! \Gamma(-\nu - n + 1)} K_{-\nu-2n}(\sqrt{8q} e^{x/2}). \end{aligned} \quad (4.3)$$

Note that when ν is not positive, the function $e^{\nu x/2}$ is not in $L^2(\mathbb{R})$, so the above equation cannot be argued by just the orthogonality. Since eigenfunctions evolve independently, the solution of (4.2) is

$$\begin{aligned} \mathcal{M}(\tau, x; q) &= \frac{2^{-1-\nu/2}}{\pi^2 q^{\nu/2}} \int_0^{\infty} e^{-u^2 \tau/8} \left| \Gamma\left(\frac{\nu + iu}{2}\right) \right|^2 K_{iu}(\sqrt{8q} e^{x/2}) \sinh(\pi u) u du \\ &\quad + \frac{2^{1-\nu/2}}{q^{\nu/2}} \sum_{n=0}^{[-\nu/2]} \frac{(-\nu - 2n)}{n! \Gamma(-\nu - n + 1)} e^{(\nu+2n)^2 \tau/8} K_{-\nu-2n}(\sqrt{8q} e^{x/2}). \end{aligned} \quad (4.4)$$

Therefore the option value can be written as

$$\mathcal{O}(\tau, x) = \frac{e^{-R\tau}}{2\pi i\tau} e^{-\nu x/2 - \nu^2\tau/8} \int_0^\infty d\xi \phi(\xi) \int_{\epsilon-i\infty}^{\epsilon+i\infty} dq e^{\xi q} \mathcal{M}(\tau, x; q). \quad (4.5)$$

Let us define the pricing kernel $\mathcal{P}(\tau, x, \xi)$ by

$$\begin{aligned} \mathcal{P}(\tau, x, \xi) &:= \frac{1}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} 4 e^{\xi q} e^{(1-\nu/2)x - \nu^2\tau/8} \mathcal{M}(\tau, x; q) dq \\ &= \frac{2^{1-\nu/2}}{\pi^2} e^{(1-\nu/2)x} \int_0^\infty du \sinh(\pi u) u e^{-(u^2+\nu^2)\tau/8} \left| \Gamma\left(\frac{\nu+iu}{2}\right) \right|^2 \\ &\quad \times \frac{1}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} e^{\xi q} q^{-\nu/2} K_{iu}(\sqrt{8q} e^{x/2}) dq \\ &\quad + 2^{3-\nu/2} e^{(1-\nu/2)x} \sum_{n=0}^{[-\nu/2]} \frac{(-\nu-2n)}{n! \Gamma(-\nu-n+1)} e^{n(\nu+n)\tau/2} \\ &\quad \times \frac{1}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} e^{\xi q} q^{-\nu/2} K_{-\nu-2n}(\sqrt{8q} e^{x/2}) dq. \end{aligned} \quad (4.7)$$

Then the option value becomes

$$\mathcal{O}(\tau, x) = \frac{e^{-R\tau}}{4\tau e^x} \int_0^\infty \mathcal{P}(\tau, x, \xi) \phi(\xi) d\xi. \quad (4.8)$$

From this formula we can see that $\mathcal{P}(\tau, x, \xi)$ is essentially the probability density transition function of the stochastic process $V_\tau = \int_0^\tau e^X d\tau$.

5 Integrating out q

In this section we will work out the inverse Laplace transformation in (4.7). For this we consider the following integration

$$I := \frac{1}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} e^{\xi q} q^{-\nu/2} K_\rho(\sqrt{8q} e^{x/2}) dq. \quad (5.1)$$

To calculate this inverse Laplace transformation, we use the contour as in Figure 1. It can be shown that the integration along C_Λ^\pm tend to zero as $\Lambda \rightarrow \infty$. When $\nu + |\operatorname{Re} \rho| < 2$ the integration along C_δ also vanishes as $\delta \rightarrow 0$. Therefore the original integration along L is related to the integration along L^+ and L^- . Nevertheless it can be checked that the result we obtain in this way is still true for more general ν . Note that due to the

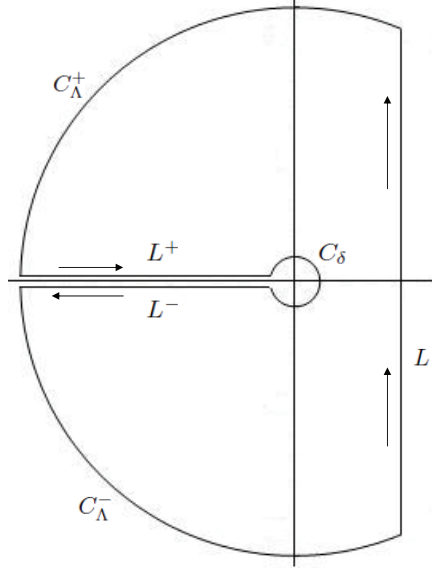


Figure 1: The integration contour for the calculation of the inverse Laplace transformation (5.1). The radius of C_Λ^\pm and C_δ are Λ and δ , respectively.

multivaluedness of the integrand, its values along L^+ and L^- are not same. We choose $\arg q = \pi$ on L^+ , while $\arg q = -\pi$ on L^- . Explicitly we have

$$\begin{aligned}
I &= -\frac{1}{2\pi i} \left(\int_{L^+} + \int_{L^-} \right) e^{\xi q} q^{-\nu/2} K_\rho(\sqrt{8q} e^{x/2}) dq \\
&= -\frac{1}{2\pi i} \left\{ \int_\infty^0 (-dr) e^{-\xi r} (re^{i\pi})^{-\nu/2} K_\rho(\sqrt{8e^x} r^{1/2} e^{i\pi/2}) \right. \\
&\quad \left. + \int_0^\infty (-dr) e^{-\xi r} (re^{-i\pi})^{-\nu/2} K_\rho(\sqrt{8e^x} r^{1/2} e^{-i\pi/2}) \right\} \\
&= -\frac{1}{2\pi i} \left\{ e^{-i\pi\nu/2} \int_0^\infty dr e^{-\xi r} r^{-\nu/2} K_\rho(\sqrt{8e^x} r^{1/2} e^{i\pi/2}) \right. \\
&\quad \left. - e^{i\pi\nu/2} \int_0^\infty dr e^{-\xi r} r^{-\nu/2} K_\rho(\sqrt{8e^x} r^{1/2} e^{-i\pi/2}) \right\} \\
&= \frac{1}{2\pi} \frac{\xi^{(\nu-1)/2}}{\sqrt{8e^x}} e^{-e^x/\xi} \Gamma\left(\frac{2-\nu+\rho}{2}\right) \Gamma\left(\frac{2-\nu-\rho}{2}\right) \\
&\quad \times \left\{ e^{-i\pi\nu/2} W_{\frac{\nu-1}{2}, \frac{\rho}{2}}\left(\frac{2e^x}{\xi} e^{i\pi}\right) + e^{i\pi\nu/2} W_{\frac{\nu-1}{2}, \frac{\rho}{2}}\left(\frac{2e^x}{\xi} e^{-i\pi}\right) \right\}. \quad (5.2)
\end{aligned}$$

In the last line above we have used the formula (6.643.3) of [20] to evaluate the integration in terms of the Whittaker function $W_{\kappa,\mu}(z)$. Note that $z = 0$ is the branch point of $W_{\kappa,\mu}(z)$, so $W_{\kappa,\mu}(ze^{i\pi}) \neq W_{\kappa,\mu}(ze^{-i\pi})$. To further simplify (5.2), we use the relation between $W_{\kappa,\mu}(z)$ and the other Whittaker function $M_{\kappa,\mu}(z)$

$$W_{\kappa,\mu}(z) = \frac{\Gamma(-2\mu)}{\Gamma\left(\frac{1}{2} - \kappa - \mu\right)} M_{\kappa,\mu}(z) + \frac{\Gamma(2\mu)}{\Gamma\left(\frac{1}{2} - \kappa + \mu\right)} M_{\kappa,-\mu}(z), \quad (5.3)$$

and the Kummer's relation $M_{\kappa,\mu}(ze^{\pm i\pi}) = e^{\pm i\pi(\mu+1/2)} M_{-\kappa,\mu}(z)$ to take out the minus sign, together with $\Gamma(1/2+z)\Gamma(1/2-z) = \pi/\cos(\pi z)$, then we have

$$I = \frac{1}{\sqrt{8e^x}} \xi^{(\nu-1)/2} e^{-e^x/\xi} W_{\frac{1-\nu}{2}, \frac{\rho}{2}}\left(\frac{2e^x}{\xi}\right) \quad (5.4)$$

Therefore the pricing kernel $\mathcal{P}(\tau; x, \xi)$ in (4.7) is

$$\begin{aligned} \mathcal{P}(\tau, x, \xi) &= \frac{1}{2\pi^2} \int_0^\infty e^{-(u^2+\nu^2)\tau/8} e^{-e^x/\xi} \left(\frac{2e^x}{\xi}\right)^{(1-\nu)/2} \\ &\quad \times W_{\frac{1-\nu}{2}, \frac{i u}{2}}\left(\frac{2e^x}{\xi}\right) \left| \Gamma\left(\frac{\nu+iu}{2}\right) \right|^2 \sinh(\pi u) u du \\ &\quad + \sum_{n=0}^{[-\nu/2]} \frac{2(-\nu-2n)}{n! \Gamma(-\nu-n+1)} e^{n(\nu+n)\tau/2} e^{-e^x/\xi} \left(\frac{2e^x}{\xi}\right)^{(1-\nu)/2} W_{\frac{1-\nu}{2}, -\frac{\nu}{2}-n}\left(\frac{2e^x}{\xi}\right). \end{aligned} \quad (5.5)$$

6 Integrating out ξ

In this section we will specify payoff functions for put and call options and then study their values.

6.1 Put options

The payoff function for the asian put option is

$$\phi_P(\xi) = (K\tau - \xi) \theta(K\tau - \xi), \quad (6.1)$$

where $\theta(\cdot)$ is the Heaviside step function. According to (4.8) its value is

$$\mathcal{O}_P = \frac{e^{-R\tau}}{4\tau e^x} \int_0^{K\tau} (K\tau - \xi) \mathcal{P}(\tau, x, \xi) d\xi. \quad (6.2)$$

By investigate the expression (5.5) of the pricing kernel $\mathcal{P}(\tau, x, \xi)$, we see that we need to consider the following type integration

$$\begin{aligned}
I_P &= \int_0^{K\tau} (K\tau - \xi) \left(\frac{2e^x}{\xi}\right)^{(1-\nu)/2} \exp\left(-\frac{e^x}{\xi}\right) W_{\frac{1-\nu}{2}, \frac{\rho}{2}}\left(\frac{2e^x}{\xi}\right) d\xi \\
&= 4e^{2x}(2k)^{(3+\nu)/2} \int_1^\infty (1-y) y^{-3+(1-\nu)/2} \exp\left(-\frac{y}{4k}\right) W_{\frac{1-\nu}{2}, \frac{\rho}{2}}\left(\frac{y}{2k}\right) dy \\
&= 4e^{2x}(2k)^{(3+\nu)/2} \exp\left(-\frac{1}{4k}\right) W_{-\frac{3+\nu}{2}, \frac{\rho}{2}}\left(\frac{1}{2k}\right), \tag{6.3}
\end{aligned}$$

where $k = K\tau/(4e^x)$, and we have use the formula (7.623.7) of [20] in the last line. Therefore the value of an Asian call option is

$$\begin{aligned}
\mathcal{O}_P &= \frac{e^{R\tau+x}}{2\pi^2\tau} \int_0^\infty e^{-(u^2+\nu^2)\tau/8} (2k)^{(3+\nu)/2} e^{-1/(4k)} W_{-\frac{3+\nu}{2}, \frac{i u}{2}}\left(\frac{1}{2k}\right) \\
&\quad \times \left| \Gamma\left(\frac{\nu + i u}{2}\right) \right|^2 \sinh(\pi u) u du \tag{6.4} \\
&+ \frac{e^{R\tau+x}}{\tau} \sum_{n=0}^{[-\nu/2]} \frac{2(-\nu - 2n)}{n! \Gamma(-\nu - n + 1)} e^{n(\nu+n)\tau/2} (2k)^{(3+\nu)/2} e^{-1/(4k)} W_{-\frac{3+\nu}{2}, -\frac{\nu}{2}-n}\left(\frac{1}{2k}\right).
\end{aligned}$$

By using the relation (see (9.237.3) of [20] ²)

$$W_{-\frac{3+\nu}{2}, -\frac{\nu}{2}-n}(z) = (-1)^n n! z^{-n-(3+\nu)/2} e^{-z/2} L_n^{-\nu-2n}(z) \tag{6.5}$$

with $L_n^{-\nu-2k}(z)$ being the generalized Laguerre polynomial, and

$$W_{\mu-1/2, \mu}(z) = z^{1/2-\mu} e^{z/2} \Gamma(2\mu, z) \tag{6.6}$$

with $\Gamma(2\mu, z)$ the incomplete Gamma function, it can be shown that (6.4) is exactly equal ³ to the result obtained in [17] through a different approach. [17] is based on an equivalence between two stochastic process, while our method seems more elementary, just doing integrations. Actually we can derive that equivalence by using the formulation in this paper.

6.2 Call options

Now we consider the Asian call option, whose payoff function is

$$\phi_C(\xi) = (\xi - K\tau) \theta(\xi - K\tau). \tag{6.7}$$

²The factor $n!$ is missed in [20].

³Need the replacement $\tau_{\text{here}} = 4\tau_{\text{there}}$ and $e^x_{\text{here}} = S_{0\text{there}}$.

The integration we need to do is

$$\begin{aligned}
I_C &= \int_{K\tau}^{\infty} (\xi - K\tau) \left(\frac{2e^x}{\xi} \right)^{(1-\nu)/2} \exp\left(-\frac{e^x}{\xi}\right) W_{\frac{1-\nu}{2}, \frac{\rho}{2}}\left(\frac{2e^x}{\xi}\right) d\xi \\
&= 4e^{2x}(2k)^{(3+\nu)/2} \int_0^1 (1-y) y^{-3+(1-\nu)/2} \exp\left(-\frac{y}{4k}\right) W_{\frac{1-\nu}{2}, \frac{\rho}{2}}\left(\frac{y}{2k}\right) dy \\
&= 4e^{2x}(2k)^{(3+\nu)/2} \exp\left(-\frac{1}{4k}\right) W_{-\frac{3+\nu}{2}, \frac{\rho}{2}}\left(\frac{1}{2k}\right). \tag{6.8}
\end{aligned}$$

We have used the formula (6.623.8) of [20] in the last line. Note that, unlike (6.3) is always true, (6.8) is convergent only under the condition $\nu + |\operatorname{Re} \rho| < -2$. For the integration part of the pricing kernel $\mathcal{P}(\tau, x, \xi)$ to be convergent, we should require $\nu < -2$. Then there are at least $n = 0, 1$ two terms in the finite summation part of (5.5). But these two terms are both divergent since $\nu + (-\nu - 2n) \geq -2$ for $n = 0, 1$. Therefore for call options we cannot naively interchange the order of the integrations over u and ξ . In the theory of finance we have so-called put-call parity relation [16]

$$\mathcal{O}_C = \mathcal{O}_P + \frac{1 - e^{-R\tau}}{R\tau} e^x - e^{-R\tau} K. \tag{6.9}$$

We can use this relation to obtain the value of Asian call options from that of put options. However it is interesting to find a direct calculation method of this result.

A The completeness proof

In this appendix we will show that, for $q > 0$,

$$I(x, x') := \frac{1}{\pi^2} \int_0^{\infty} K_{iu}(\sqrt{8q} e^{x/2}) K_{iu}(\sqrt{8q} e^{x'/2}) \sinh(\pi u) u du = \delta(x - x'). \tag{A.1}$$

Since the integrand is an even function due to $K_{iu} = K_{-iu}$, we can extend the integration range to $(-\infty, \infty)$. Therefore

$$I(x, x') = \frac{i}{2\pi^2} \lim_{\Lambda \rightarrow \infty} \int_{-i\Lambda}^{i\Lambda} K_{\nu}(\sqrt{8q} e^{x/2}) K_{\nu}(\sqrt{8q} e^{x'/2}) \sin(\pi\nu) \nu d\nu. \tag{A.2}$$

By using the relation $K_{\nu}(z) = 2^{-1}\pi(I_{-\nu}(z) - I_{\nu}(z))/\sin(\pi\nu)$, we decompose $I(x, x')$ into three terms

$$\begin{aligned}
I(x, x') &= \frac{i}{8} \lim_{\Lambda \rightarrow \infty} \left\{ \int_{-i\Lambda}^{i\Lambda} \frac{\nu d\nu}{\sin(\pi\nu)} I_{\nu}(z) I_{\nu}(z') + \int_{-i\Lambda}^{i\Lambda} \frac{\nu d\nu}{\sin(\pi\nu)} I_{-\nu}(z) I_{-\nu}(z') \right. \\
&\quad \left. - \int_{-i\Lambda}^{i\Lambda} \frac{\nu d\nu}{\sin(\pi\nu)} [I_{\nu}(z) I_{-\nu}(z') + I_{-\nu}(z) I_{\nu}(z')] \right\}, \tag{A.3}
\end{aligned}$$

where $z = \sqrt{8q} e^{x'/2}$ and $z' = \sqrt{8q} e^{x/2}$. The first two terms are actually equal by interchanging ν and $-\nu$. Since the integrands are holomorphic in the complex ν -plane, we can deform the integration path to a semicircle $C_\Lambda := \{\Lambda e^{i\phi} | -\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2}\}$. Therefore

$$I(x, x') = \frac{i}{8} \lim_{\Lambda \rightarrow \infty} \left\{ 2 \int_{C_\Lambda} \frac{\nu d\nu}{\sin(\pi\nu)} I_\nu(z) I_\nu(z') - \int_{C_\Lambda} \frac{\nu d\nu}{\sin(\pi\nu)} [I_\nu(z) I_{-\nu}(z') + I_{-\nu}(z) I_\nu(z')] \right\}. \quad (\text{A.4})$$

When the order ν is large and z is fixed, we have

$$I_\nu(z) \sim \frac{1}{\Gamma(1+\nu)} \left(\frac{z}{2}\right)^\nu, \quad |\nu| \rightarrow \infty, \quad |\arg z| < \pi. \quad (\text{A.5})$$

By carefully analyzing the asymptotic behavior along C_Λ when $\Lambda \rightarrow \infty$, it can be shown that [22] the first term of (A.4) actually tends to zero. So we have

$$\begin{aligned} I(x, x') &= -\frac{i}{8} \lim_{\Lambda \rightarrow \infty} \int_{C_\Lambda} \frac{\nu d\nu}{\sin(\pi\nu)} \left\{ \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(1+\nu)} \frac{\left(\frac{z'}{2}\right)^{-\nu}}{\Gamma(1-\nu)} + \frac{\left(\frac{z}{2}\right)^{-\nu}}{\Gamma(1-\nu)} \frac{\left(\frac{z'}{2}\right)^\nu}{\Gamma(1+\nu)} \right\} \\ &= \lim_{\Lambda \rightarrow \infty} \frac{\Lambda}{8\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left\{ \left(\frac{z}{z'}\right)^{\Lambda e^{i\phi}} + \left(\frac{z}{z'}\right)^{-\Lambda e^{i\phi}} \right\} e^{i\phi} d\phi \\ &= \lim_{\Lambda \rightarrow \infty} \frac{\Lambda}{8\pi} \sum_{n=0}^{\infty} \left\{ \frac{\left(\frac{x-x'}{2}\right)^n \Lambda^n}{n!} + (-1)^n \frac{\left(\frac{x-x'}{2}\right)^n \Lambda^n}{n!} \right\} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i(n+1)\phi} d\phi \\ &= \frac{1}{2} \lim_{\Lambda \rightarrow \infty} \frac{\sin\left(\Lambda \frac{(x-x')}{2}\right)}{\pi \frac{(x-x')}{2}} = \frac{1}{2} \times \delta\left(\frac{x-x'}{2}\right) = \delta(x-x'). \end{aligned} \quad (\text{A.6})$$

In the second line we have used $\Gamma(1+\nu)\Gamma(1-\nu) = \pi\nu/\sin(\pi\nu)$, and in the last line the limit representation of the Dirac delta function

$$\lim_{\Lambda \rightarrow \infty} \frac{\sin(\Lambda u)}{\pi u} = \delta(u). \quad (\text{A.7})$$

Therefore we have proved the completeness relation (A.1).

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