Low energy states dynamic of entanglement for spin systems *

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We have composed the ideas of quantum renormalization group and quantum information by exploring the low energy states dynamic of entanglement resources of a system close to its quantum critical point. We demonstrate the low energy states dynamical quantities of the one dimensional magnetic systems could show the quantum phase transition point and shows the scaling behavior in the vicinity of the transition point. To present our idea, we study the evolution of two spins entanglement in the one-dimensional Ising model in the transverse field. The system is initialized as the so-called thermal ground state of the pure Ising model. We investigate evolvement of the generation of entanglement with increasing the magnetic field. We have obtained that the derivative of the time at which the entanglement reaches its maximums with respect to the transverse field, diverges at the critical point and its scaling behaviors versus the size of the system are as same as the static ground state entanglement of the the system.

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A fundamental difference between quantum and classical physics is the possible existence of nonclassical correlations in quantum systems. The physical property responsible for this quantum correlation is called Entanglement [1]. Entanglement has been recognized as an important resource for quantum information and computation [2]. However the role of entanglement in quantum phase transition (QPT) [3] is of considerable interest. QPT as well as classical ones are characterized by detecting nonanalytic behaviors in some physical properties of the system. It is often accompanied by divergence in some correlation functions, but quantum systems possess additional correlations which do not exist in a classical counterpart, the entanglement. Entanglement is a direct measure of quantum correlations and shows nonanalytic behavior such as discontinuity in the vicinity of the quantum phase transition point [4]. An important motivation to study the interconnection between condensed matter and quantum information is to investigate whether it is possible to better characterize condensed matter states by looking at their entanglement properties. Recently, there has been extensive analysis of entanglement in quantum spin models [5]. Various models were considered for entanglement generation and their static [4] as well as dynamical properties [6]were investigated. A thorough understanding of the dynamical evolution of entanglement in the spin models has obviously implications for the performance of quantum information processing, as well as for understanding of fundamental quantum mechanics.

Our main purpose in this work is to compose the ideas of quantum renormalization group [7] and quantum information theory to study the evolution of the dynamical properties of the spin models in low energy states. To have a concrete discussion, the one dimensional $S = \frac{1}{2}$ Ising model in transverse field (ITF) has been considered by implementing the quantum renormalization group (QRG) approach [8–10].

The main idea of the RG method is the mode elimination or thinning of the degrees of freedom followed by an iteration which reduces the number of variables step by step until reaching a fixed point. In Kadanoff's approach, the lattice is divided into blocks. Each block is treated independently to build the projection operator onto the lower energy subspace. The projection of the inter-block interaction is mapped to an effective Hamiltonian (H^{eff}) which acts on the renormalized subspace [11].

We have considered the ITF model on a periodic chain of N sites with Hamiltonian

$$H = -J\sum_{i=1}^{N}(\sigma_i^x\sigma_{i+1}^x + g\sigma_i^z). \qquad (1)$$

where $J > 0$ is the exchange coupling and g is the transverse field. From the exact solution [12] it is known that a second order phase transition occurs for $g_c = 1$ where the behavior of the order parameter or magnetization is given by $< \sigma^x >= (1-g)^{1/2}$ for $g < 1$ and $< \sigma^x >= 0$ for $g > 1$.

To implement QRG the Hamiltonian is divided to two-site blocks, $H^B = \sum_{I=1}^{N/2} h_I^B$ with $h_I^B = -J(\sigma_{1,I}^x\sigma_{2,I}^x + g\sigma_{1,I}^z)$. The remaining part of the Hamiltonian is included in the inter-block part, $H^{BB} = -J\sum_{I=1}^{N/2}(\sigma_{2,I}^x\sigma_{1,I+1}^x + g\sigma_{2,I}^z)$. where $\sigma_{j,I}^{\alpha}$ refers to the α-component of the Pauli matrix at site j of the block labeled by I. The Hamiltonian of each block (h_I^B) is diagonalized exactly and the projection operator (P_0) is constructed from the two lowest eigenstates, $P_0 = |\psi_0\rangle\langle\psi_0| + |\psi_1\rangle\langle\psi_1|$, where $|\psi_0\rangle$ is the ground state and $|\psi_1\rangle$ is the first excited state. In this respect the effective Hamiltonian ($H^{eff} = P_0[H^B + H^{BB}]P_0$) is similar to the original one (Eq.(1)) replacing the couplings with

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the following renormalized coupling constants.

$$J' = J\frac{2q}{1+q^2},\ q = g + \sqrt{g^2+1},\ g' = g^2. \qquad (2)$$

The entanglement is a local quantity which includes the global properties of a system. Generally, the global properties of a system enters the entanglement effectively by summing over the whole degrees of freedom except the local one. In other words, a system can be supposed of a single site and a heat bath (the rest of system). It is supposed that the effect of a heat bath can be replaced by an effective single site quantity, *the entanglement*. The effective single site represents the long range properties of the model and not the microscopic ones. Therefor we can enter the global properties of the model to entanglement (the local quantity) using the renormalization group idea. In this respect, we always think of a two site model which can be treated exactly. However, the coupling constants of the two site model are the effective ones which are given by the renormalization group procedure. This can be used as an new method to calculate the low energy states dynamic of entanglement in a large system.

The two site Hamiltonian of ITF model in the space spanned by $\{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$ ($|\uparrow\rangle$ and $|\downarrow\rangle$ denote the eigenstates of σ^z), can be expressed as

$$H = -J\begin{pmatrix} 2g & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -2g \end{pmatrix} \qquad (3)$$

However the time evolution operator $U(t) = e^{-iHt}$ of the two site Hamiltonian (Eq.(3)) has the following form

$$U(t) = J\begin{pmatrix} U_{11}(t) & 0 & 0 & U_{14}(t) \\ 0 & U_{22}(t) & U_{23}(t) & 0 \\ 0 & U_{32}(t) & U_{33}(t) & 0 \\ U_{41}(t) & 0 & 0 & U_{44}(t) \end{pmatrix} \qquad (4)$$

where

$$\begin{aligned} U_{11}(t) &= i\frac{2g}{\sqrt{1+4g^2}}\sin(Jt\sqrt{1+4g^2}) + \cos(Jt\sqrt{1+4g^2}), \\ U_{14}(t) &= U_{41}(t) = \frac{i}{\sqrt{1+4g^2}}\sin(Jt\sqrt{1+4g^2}), \\ U_{22}(t) &= U_{33}(t) = \cos Jt,\ U_{23}(t) = U_{32}(t) = i\sin Jt, \\ U_{44}(t) &= U_{11}^*. \end{aligned}$$

The density matrix for the two sites system at time t is $\rho(t) = U(t)\rho(0)U^\dagger$, where $\rho(0)$ is the the density matrix of system at $t = 0$ which is the thermal equilibrium of the system. We choose the thermal ground sate of the pure Ising model (no transverse field applied) $\rho(0) = \frac{1}{2}(|\sigma^x,+\rangle\langle\sigma^x,+| + |\sigma^x,-\rangle\langle\sigma^x,-|)$ as the initial state which the concurrence equals zero.

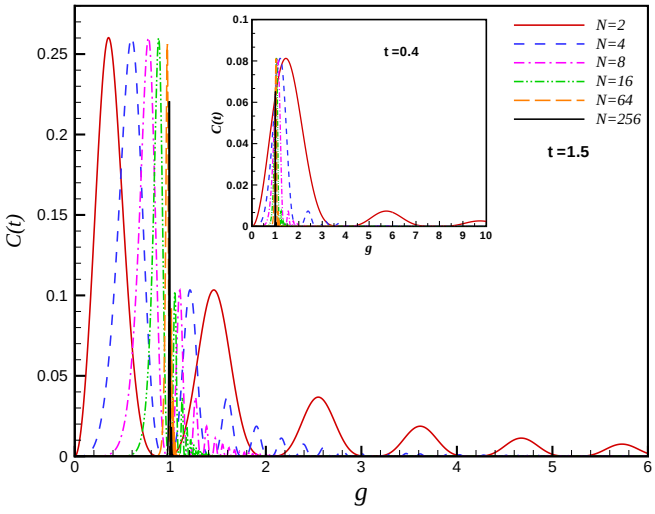

FIG. 1: (Color online) Evolution of the concurrence under RG versus g for $t = 1.5$ and in the inset for $t = 0.4$.

Here $|\sigma^x,+\rangle$ and $|\sigma^x,-\rangle$ are the two degenerate ground state of Ising model with all spins pointing respectively to the positive and negative x direction.

For bipartite entanglement, a commonly used measure for arbitrary state of two qubits is the so called concurrence [13]. The concurrence is defined as

$$C(t) = max\{0, 2\lambda_{max}(t) - tr\sqrt{\rho(t)\widetilde{\rho}(t)}\} \qquad (5)$$

where $\widetilde{\rho}(t) = (\sigma^y \otimes \sigma^y)\rho^*(t)(\sigma^y \otimes \sigma^y)$, and λ_{max} is the largest eigenvalue of the matrix $\sqrt{\rho(t)\widetilde{\rho}(t)}$.

Therefor the analytic expression of the concurrence in terms of the parameters defined for the two site system is

$$C(t) = \frac{1}{2}\Big[1 - \sqrt{1 - (\frac{4g}{1+4g^2})^2\sin^4(\sqrt{1+4g^2}Jt)}\Big]. \qquad (6)$$

We have plotted the evolution of $C(t)$ under RG steps versus g for $t = 1.5$ and $t = 0.4$ in Fig.1 which shows that the concurrence changes from the equilibrium state and start to oscillating when the external magnetic filed is turned on. As g increases, the height of each peak decreases gradually and finally vanish as $g \to \infty$. Increasing the length of chain enhances the oscillation of the concurrence versus the magnetic field. However as the length of chain increases the first peak of concurrence approaches the critical point ($g_c = 1$) and at the thermodynamic limit the system becomes disentangled except at the critical point. Surviving of the concurrence at the critical point is the results of the correlation length divergence at $g_c = 1$.

The evolution of concurrence under RG has been plotted in Fig.2 versus t for $g = 0.9$ and $g_c = 1$. From

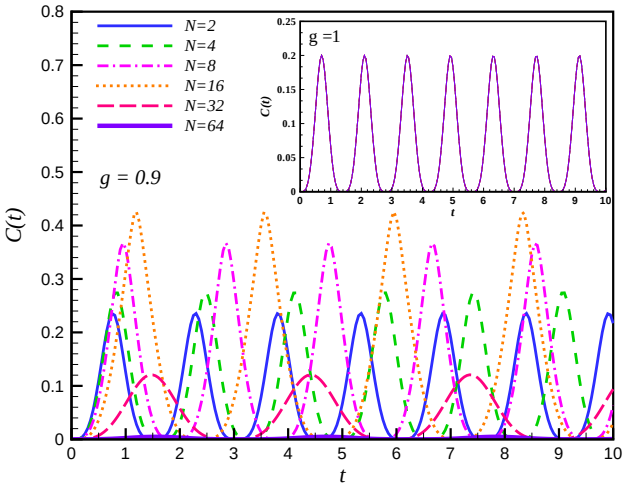

FIG. 2: (Color online) Concurrence of the ITF model as a function of t for different length chain for $g = 0.9$. The inset shows the concurrence of different lattice sizes collapse on a single curve at the critical point ($g_c = 1$).

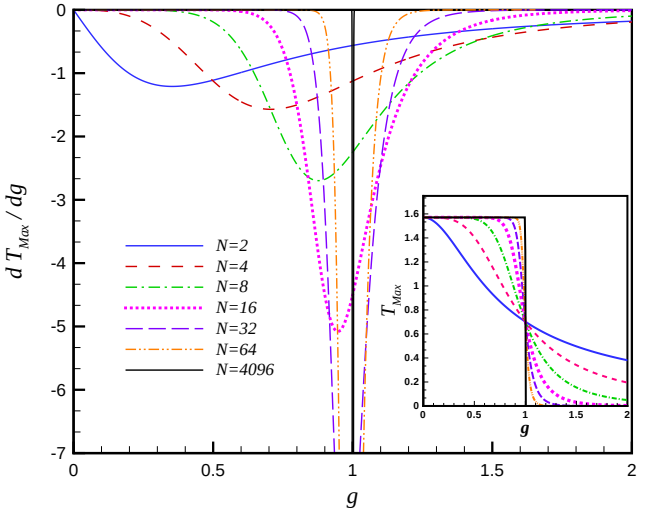

FIG. 3: (Color online) Evolution of the first derivative of T^1_{max} respect to magnetic field (g) under RG. The inset shows the T^1_{max} at different RG steps.

the Eq.(6), it is easy to see that the concurrence is periodically fluctuating with the time t with period of $T = \frac{2\pi}{J\sqrt{1+4g^2}}$. Figure.2 shows that the concurrence reduces under RG (increasing the size of system) and disappears in the large system. But for $g_c = 1$ there is no concurrence reduction under RG and the concurrence of different length chains coincide with each other (inset of Fig. 2).

The non-analytic behaviour in some physical quantity is a feature of second-order quantum phase transition. It is also accompanied by a scaling behaviour since the correlation length diverges and there is no characteristic length in the system at the critical point. As we have stated in the RG approach for ITF model, a large system, i.e. $N = 2^{n+1}$, can be effectively describe by two sites with the renormalized coupling of in the n-th RG step. Thus, the concurrence between the two renormalized sites represents the entanglement between two parts of the system each containing $N/2$ sites effectively. In this respect we can speak of *block entanglement* -the entanglement between a block and the rest of system- in a large system provided the size of the block and the rest of system is equal.

For any g, there is a time $T^k_{max}(g)$ at which the $C(t)$ reaches its kth maximum (Fig.(2)) and is analyzed as a function of coupling g at different RG steps which manifest the size of system. The first derivative of T^k_{max} with respect to the coupling constant ($\frac{dT^k_{max}}{dg}$) shows a singular behavior at the critical point.

We have plotted $\frac{dT^k_{max}}{dg}$ for $k = 1$ (first maximum of the $C(t)$) versus g in Fig.3 for different RG steps which shows the singular behaviour as the size of system becomes large. Surveying the detail shows that the position of the minimum (g_m) of $\frac{dT^k_{max}}{dg}$ tends towards the critical point like $g_m = g_c - N^{1/\theta}$ in which $\theta = 1$ (inset of figure 4). Moreover, we have derived the scaling behavior of $\ln \frac{dT^1_{max}}{dg}|_{g_m}$ versus N. This has been plotted in Fig.(4), which shows a linear behavior of $\ln \frac{dT^1_{max}}{dg}|_{g_m}$ versus $ln(N)$. The scaling behavior is $\ln \frac{dT^1_{max}}{dg}|_{g_m} \propto N^{\theta}$ with exponent $\theta = 1$.

It is easy to show that the exponent θ is directly re-

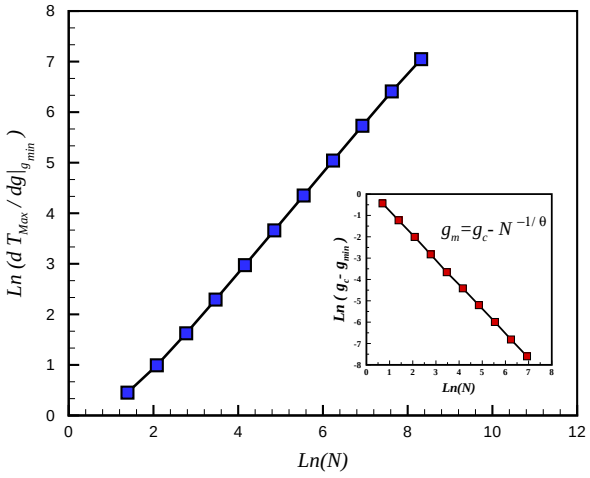

FIG. 4: (Color online) scaling the minimum of $\frac{dT^1_{max}}{dg}$ for various size of system. The inset shows the scaling of the position (g_m) of $\frac{dT^1_{max}}{dg}$ for different length chains.

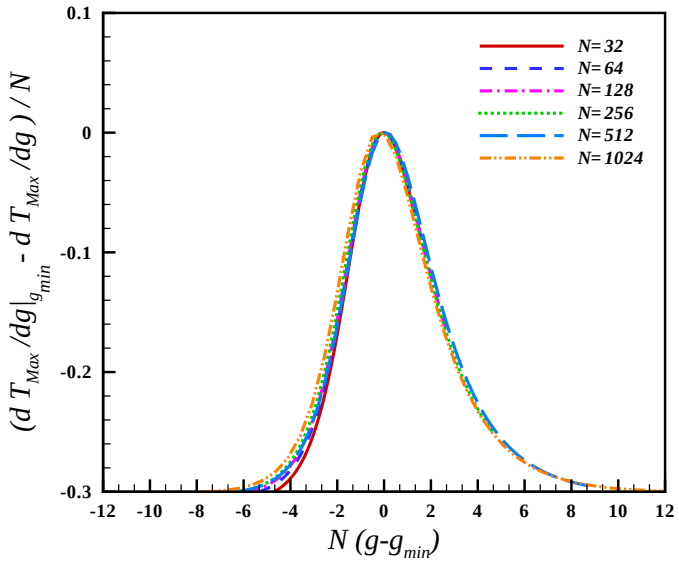

FIG. 5: (Color online) Finite-size scaling through the RG treatment for different lattice sizes. The curves which correspond to different system sizes clearly collapse on a single curve.

lated to the correlation length exponent (ν) close to the critical point. The correlation length exponent, gives the behavior of correlation length in the vicinity of g_c, i.e., $\xi \sim (g - g_c)^{-\nu}$. Under the RG transformation, Eq. (2), the correlation length scales in the nth RG step as $\xi^{(n)} \sim (g_n - g_c)^{-\nu} = \xi / n_B^n$, which immediately leads to an expression for $|\frac{dg_n}{dg}|_{g_c}$ in terms of ν and n_B (number of sites in each block). Dividing the last equation to $\xi \sim (g - g_c)^{-\nu}$ gives $|\frac{dg_n}{dg}|_{g_c} \sim N^{1/\nu}$, which implies $\theta = 1/\nu$, since $\frac{dT^K_{max}}{dg}|_{g_m} \sim |\frac{dg_n}{dg}|_{g_c}$ at the critical point. It should also be noted that the scaling of the position of minimum, g_m (inset of figure 3), also comes from the behavior of the correlation length near the critical point. As the critical point is approached and in the limit of large system size, the correlation length almost covers the size of the system, i.e., $\xi \sim N$, and a simple comparison with $\xi \sim (g - g_c)^{-\nu}$ results in the following scaling form $g_m = g_c - N^{1/\nu}$.

To obtain the finite-size scaling behavior of $\frac{dT^k_{max}}{dg}|_{g_m}$, we look for a scaling function in such away that all graphs tend to collapse on each other under RG evolution which results in a large system. This is also a manifestation of the existences of the finite size scaling for the case of block entanglement. We have plotted $\frac{dT^k_{max}}{dg}|_{g_m} - \frac{dT^k_{max}}{dg}$ versus $N(g - g_m)$ for $k = 1$ in Fig.5. The upper curves which are for large system sizes clearly show that all plots fall on each other.

The similar scaling behaviours and their relation to correlation length exponent have been reported in our previous work [9] in which we have studied the static properties of the ground state entanglement of ITF model by RG method.

We would like to mention that in Ref.[14] the authors investigate the dynamics of concurrence of two nearest-neighbor sites at ITF model using the exact solution. They shows that $\frac{dT^1_{max}}{dg}$ dose not diverge at the critical point but has a minimum at $g = 1$. The divergence of $\frac{dT^1_{max}}{dg}$ at the g_c in our work and the similarity of the dynamic scaling behaviors to the static scaling behaviors originates from the low energy state properties.

To summarize, we have implement the idea of renormalization group (RG) to study the low energy states dynamic of entanglement for spin chains. In this respect we show that the RG procedure could be implemented to obtain low energy states dynamic of systems in terms of effective Hamiltonian which is described by renormalized coupling constants. This manifest the fact that some dynamical quantities of the system could show the fingerprint of quantum phase transition for an infinite size system. These notions have been observed and approved in our study of the ITF model. Moreover, the RG approach shows that as the size of the system becomes large, the derivative of the time at which the entanglement reaches its maximums with respect to the transverse field, diverges at the critical point and its scaling behaviors versus the size of the system are as same as the static ground state entanglement of the system.

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References

[1] J. S. Bell, Physics **1**, 195 (1964).
[2] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Communication* (Cambridge University Press, Cambridge, 2000).
[3] S. Sachdev, *Quantum Phase Transitions* (Cambridge University Press, Cambridge, 2000).
[4] A. Osterloh, Luigi Amico, G. Falci and Rosario Fazio, Nature **416**, 608 (2002).
[5] L. Amico, R. Fazio, A. Osterloh, and V. Vedral, Rev. Mod. Phys. **80**, 517 (2008).
[6] L. Amico, A. Osterloh, F. Plastina, R. Fazio, and G. M. Palma, Phys. Rev. A **69**, 022304 (2004).
[7] P. Pfeuty, R. Jullian, K. L. Penson, in *Real-Space Renormalization*, edited by T. W. Burkhardt and J. M. J. van Leeuwen (Springer, Berlin, 1982), Chap. 5.
[8] M. A. Martin-Delgado and G. Sierra, Int. J. Mod. Phys.

A **11**, 3145 (1996).
[9] M. Kargarian, R. Jafari and A. Langari, Phys. Rev. A **76**, 060304(R)(2007).
[10] R. Jafari, M. Kargarian, A. Langari, and M. Siahatgar, Phys. Rev. B **78**, 214414 (2008).
[11] M. A. Martin-Delgado and G. Sierra, Phys. Rev. Lett. **76**, 1146 (1996).
[12] P. Pfeuty, ANNALS of Physics, **57**, 79 (1970).
[13] W. K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).
[14] Zhe Chang and Ning Wu, Phys. Rev. A **81**, 022312 (2010).