

Modelling savings behavior of agents in the kinetic exchange models of market

Anindya S. Chakrabarti

Indian Statistical Institute, 203 B. T. Road, Kolkata-700108

(Dated: June 28, 2010)

Kinetic exchange models have been successful in explaining the shape of the income/wealth distribution in the economies. However, such models usually make some ad-hoc assumptions when it comes to determining the savings factor. Here, we examine a few models in and out of the domain of standard neo-classical economics to explain the savings behavior of the agents. A number of new results are derived and the rest conform with those obtained earlier. Connections are established between the reinforcement choice and strategic choice models with the usual kinetic exchange models.

PACS numbers: 89.65 Gh Economics, Econophysics- 05.20 Kinetic theory- 05.65.+b Self-organized systems

I. INTRODUCTION

The distributions of income and wealth have long been found to possess some robust and stable features independent of the specific economic, social and political conditions of the economies. Traditionally, the economists have preferred to model the left tail and the mode of the distributions of the workers' incomes with a log-normal distribution and the heavier right tail with a Pareto distribution. For a detailed survey of the distributions used to fit the income and wealth data see Ref. [1]. However, there have been several studies recently that argue that the left tail and the mode of the distribution fit well with the gamma distribution and the right tail of the distribution follows a power law [2–4]. It has been argued that this feature might be considered to be a natural law for economics [2, 5].

The Chakraborti-Chakrabarti [6] model (CC model henceforth) can explain the gamma-like distribution very well whereas the Chatterjee-Chakrabarti-Manna [7] model (CCM model henceforth) explains the origin of the power law. Both models fall in the category of the kinetic exchange models. However, the first model assumes a constant savings propensity and the second model assumes an uniformly distributed savings propensity of the agents. It may be noted that the distribution of the savings factor is exogenous in these models. It is not derived from any optimization on the agent's part or by some other mechanism.

Ref. [8] considers an exchange economy populated with agents having a particular type of utility function and derives the CC model in the settings of a competitive market. Here, we show that the same methodology could be applied to derive the CCM model and we can explain the savings factor accordingly. A further possibility is also studied where the constancy of the savings propensity over time is relaxed. More specifically, we examine the cases where the savings propensity is dependent on the current money holding of the corresponding agent.

This model, as we shall see, shows self-organization and in some cases, it gives rise to bimodality in the money distribution.

It is well known the models of utility maximization has been severely criticized on the grounds of limitations of computational capability of human beings [9]. Hence, to model the savings behavior of the agents, we study some simple thumb-rules and derive the distributions of savings therefrom and finally the income/wealth distribution. In particular, we consider the savings propensity as a strategy variable to the agents. Two cases are explored here. In the first case, the agents take their savings decisions of their own by reinforcing their choices. In the second case, the agents adopt the winning strategy. In all cases, we study the final money distributions.

The plan of this paper is as follows. In section, II we explain the savings behavior of the agents using arguments from neoclassical economics. In the next section, we study a self-organizing market where the savings is a function of the current money holding of the corresponding agent. In section V and VI, we study the savings behavior of the agents where the savings decision follows some very simple thumb-rules. Then follows a summary and discussion.

II. KINETIC EXCHANGE MODELS IN A COMPETITIVE MARKET

By competitive market we mean a market with atomistic agents who trade with each other knowing that their individual actions can not possibly influence the market outcome (that implies there is no *strategic* interaction between the agents). We assume that markets are always cleared by equating supply and demand; that is the market is completely free of any friction. Below, we elaborate on the market structure and the behavior of the agents more explicitly.

[16] email-id : aschakrabarti@gmail.com

A. The Chakraborti-Chakrabarti (CC) Model

This model considers homogenous agents characterized by a single savings propensity. We briefly review the derivation of the model following Ref. [8]. The structure of the economy is the following. It is an exchange economy populated with N agents each producing a single perishable commodity. There is complete specialization in production which means none of the agents produce the commodity produced by another. Money is not produced in the economy. All agents are endowed with a certain amount of money at the very beginning of all tradings. Money can be treated as a non-perishable commodity which facilitates transactions. All commodities along with money can enter the utility function of any agent as arguments. These agents care for their future consumptions and hence they care about their savings in the current period as well. It is natural that with successive tradings their money-holding will change with time. At each time step, two agents are chosen at random to carry out transactions among themselves competitively. We also assume that the preference structure of the agents are time-dependent that is the parameters of the utility functions vary over time (Ref. [10, 11]). For a detailed discussion on the derivation of the resulting money-transfer equations, see Ref. [8]. Below, we provide the formal structure and the solution to the model only.

Let us assume that at time t , agent i and j have been chosen. Also, assume that agent i produces Q_i amount of commodity i only and agent j produces Q_j amount of commodity j only and the amounts of money in their possession at time t are $m_i(t)$ and $m_j(t)$ respectively (for simplicity, $m_k(0) = 1$ for $k=1,2$). Notice that the notion of complete specialization in production process provides the agents with a reason for trading with each other. Naturally, at each time step there would be a net transfer of money from one agent to the other due to trade. Our focus is on how the amounts money held by the agents change over time due to the repetition of such a trading process. For notational convenience, we denote $m_k(t+1)$ as m_k and $m_k(t)$ as M_k (for $k = 1, 2$).

Utility functions are defined as follows. For agent i , $U_i(x_i, x_j, m_i) = x_i^{\alpha_i} x_j^{\alpha_j} m_i^\lambda$ and for agent j , $U_j(y_i, y_j, m_j) = y_i^{\alpha_i} y_j^{\alpha_j} m_j^\lambda$ where the arguments in both of the utility functions are consumption of the first (i.e., x_i and y_i) and second good (i.e., x_j and y_j) and amount of money in their possession respectively. For simplicity, we assume that the sum of the powers is normalized to 1 i.e., $\alpha_1 + \alpha_2 + \lambda = 1$. Let the commodity prices to be determined in the market be denoted by p_i and p_j . Now, we can define the budget constraints as follows. For agent i the budget constraint is $p_i x_i + p_j x_j + m_i \leq M_i + p_i Q_i$ and similarly, for agent j the constraint is $p_i y_i + p_j y_j + m_j \leq M_j + p_j Q_j$. In this set-up, we get the market clearing price vector (\hat{p}_i, \hat{p}_j) as $\hat{p}_k = (\alpha_k/\lambda)(M_i + M_j)/Q_k$ for $k = 1, 2$.

By substituting the demand functions of x_k , y_k and p_k for

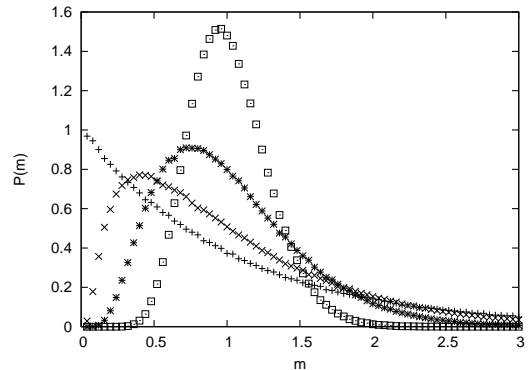


FIG. 1:

The usual CC model: money distribution among the agents. Four cases are shown above, viz., $\lambda = 0$ (+), $\lambda = 0.2$ (x), $\lambda = 0.5$ (*), $\lambda = 0.8$ (□). All simulations are done for $O(10^6)$ time steps with 100 agents and averaged over $O(10^3)$ time steps.

$k = 1, 2$ in the money demand functions, we get the most important equation of money exchange in this model. To get the final result, we substitute $\alpha_i/(\alpha_i + \alpha_j)$ by ϵ to get the money evolution equations as

$$\begin{aligned} m_i(t+1) &= \lambda m_i(t) + \epsilon(1-\lambda)(m_i(t) + m_j(t)) \\ m_j(t+1) &= \lambda m_j(t) + (1-\epsilon)(1-\lambda)(m_i(t) + m_j(t)) \end{aligned} \quad (1)$$

where $m_k(t) \equiv M_k$ and $m_k(t+1) \equiv m_k$ (for $k = i, j$). Note that for a fixed value of λ , if α_i is a random variable with uniform distribution over the domain $[0, 1-\lambda]$, then ϵ is also uniformly distributed over the domain $[0, 1]$. For the limiting value of λ in the utility function (i.e., $\lambda \rightarrow 0$), we get the money transfer equation describing the random sharing of money without savings.

Interpretation of λ : Here, it is clearly shown that λ in the CC model is nothing but the power of money in the utility function of the agents and finally this turns out to be the fraction of money holding that remains unaffected by the trading action. However, in this form it can not be directly interpreted as the propensity to save. Below, we try to derive λ from an utility maximization problem while retaining the kinetic exchange structure and we show that in this slightly alternative formulation, λ is indeed the savings propensity as has been postulated.

B. The Chatterjee-Chakrabarti-Manna (CCM) Model

As is clear from above, in the CC model the savings decision, the market clearance, the prices are all determined at the same instant. But the savings decision is usually made in separation. More specifically, we can model the savings decision and the market clearance distinctly. The CCM model takes into account the heterogeneity of the

agents. In particular, it assumes that it is the savings propensity of the agents which differs from each other. To derive the same, we assume that the agents take decisions in two steps. First, they decide how much to save and in the second step, they go to the market with the rest of the money and take the trading decisions. Formally, we can analyze a typical agent's behavior at any time step t in the following two steps.

- (i) Each agent's problem is to make the decision regarding how much to save. For simplicity, we assume that the utility function is of Cobb-Douglas type. Briefly, at time t the i -th agent's problem is to maximize $U(f_t, c_t) = f_t^{\lambda_i} c_t^{(1-\lambda_i)}$ subject to $f_t/(1+r) + c_t = m(t)$ where f is the amount of money kept for future consumption, c is the amount of money to be used for current consumption, $m(t)$ is the amount of money holding at time t and r is the interest rate prevailing in the market which can be assumed to be zero in a conservative framework. This is a standard utility maximization problem and solving it by Lagrange multiplier, we get the optimal allocation for the i -th agent as $f_t^* = \lambda_i m(t)$ and $c_t^* = (1 - \lambda_i)m(t)$. Clearly, this decision is independent of what other agents are doing. So now the agents will go to the market with $(1 - \lambda_i)m(t)$ only.

- (ii) Now that each agent has made the savings decision, they can engage in competitive trade with each other in the fashion described in subsection II A with $\lambda \rightarrow 0$ (but $\lambda \neq 0$; it is a mathematical requirement). Note that the amount of money used by the i -th agent is $c_t^* = (1 - \lambda_i)m(t)$ only.

The resultant asset exchange equations are those given by the CCM model [7].

$$\begin{aligned} m_i(t+1) &= \lambda_i m_1(t) + \epsilon[(1 - \lambda_i)m_i(t) + \\ &(1 - \lambda_j)m_j(t)] \\ m_j(t+1) &= \lambda_j m_2(t) + (1 - \epsilon)[(1 - \lambda_i)m_i(t) + \\ &(1 - \lambda_j)m_j(t)] \end{aligned} \quad (2)$$

Interpretation of λ_i : First we recall the solution of the savings decision which is $f_t^* = \lambda_i m(t)$. Note that it implies

$$\lambda_i = \frac{f_t^*}{m(t)}$$

that is λ_i is nothing but the proportion of money kept for future usage to the current money holding and this is by definition the savings propensity.

III. λ AS A FUNCTION OF MONEY: A SELF-ORGANIZING ECONOMY

A distinct possibility is that the savings propensity is a function of money-holding itself. To examine

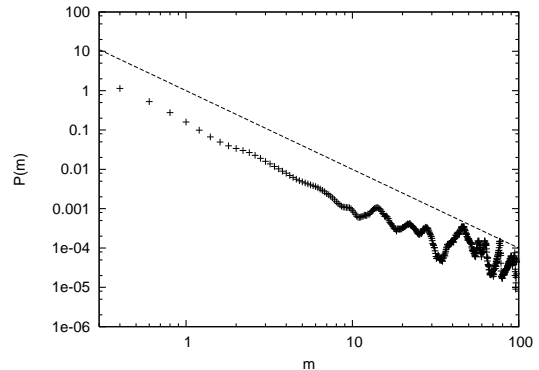


FIG. 2:

The usual CCM model: money distribution among the agents. All simulations are done for $O(10^6)$ time steps with 100 agents and averaged over $O(10^3)$ time steps. The straight line is a guide for a power law with slope -2.

that case, we need $\lambda: [0, \infty) \rightarrow [0, 1]$. However, there are two possibilities. The savings propensity can be an increasing or decreasing function of money-holding. The simplest forms that we may assume are the following.

- (i) $\lambda_t = c_1 e^{-(c_2 m(t))}$ with $c_1 < 1$: Savings propensity is a decreasing function of money holding. The restriction on c_1 does not allow any agent to have savings propensity equals to 1. The system shows self-organization and assumes a stable probability density function in the steady state (See Fig. 3 where, for purpose of illustration, c_1 has been kept constant at 0.95). It is seen numerically that as c_2 increases the distribution converges to an exponential density function. In the other extreme, it tends to the CC model with $\lambda = c_1$. For moderate values of c_2 , the distribution resembles gamma function. For other values of c_1 also, the system behaves similarly. Note that since c_1 is the maximum possible savings propensity, for a very low value of it, the system becomes indistinguishable from an exponential distribution. While it seems counter-intuitive that savings propensity falls with the money holding, this might in fact be possible since poorer people can not take any chance to gamble whereas richer people can.

- (ii) $\lambda_t = c_1(1 - e^{-(c_2 m(t))})$ with $c_1 < 1$: Savings propensity is an increasing function of money holding. The economy again organizes itself and the distribution of money becomes stable over time. However, there is something more. It is seen numerically that bimodality may appear spontaneously in the density function of money. See Fig. 4. Ref. [12] discusses such bimodal distribution of wealth (or money). There a mixture of the agents was used where two classes of agents were characterized by two different and widely separated (but fixed!) savings propen-

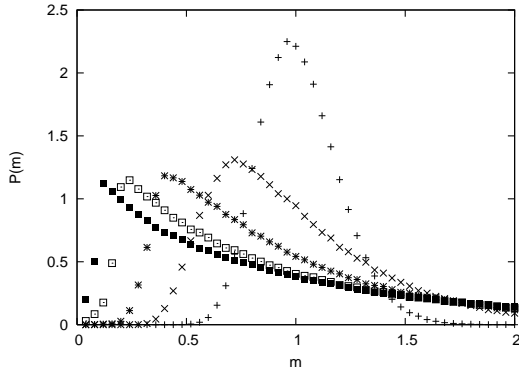


FIG. 3:

Savings propensity is negatively related to the level of money. All simulations are done for $O(10^5)$ time steps with 100 agents and averaged over $O(10^3)$ time steps. Here, c_1 has been kept constant at 0.95 and c_2 has been changed. The plots include $c_2 = 0.1$ (+), 0.5 (x), 1 (*), 2 (□), 4 (■). It is seen that as c_2 increases, the distribution becomes more and more skewed finally converging to an exponential density function.

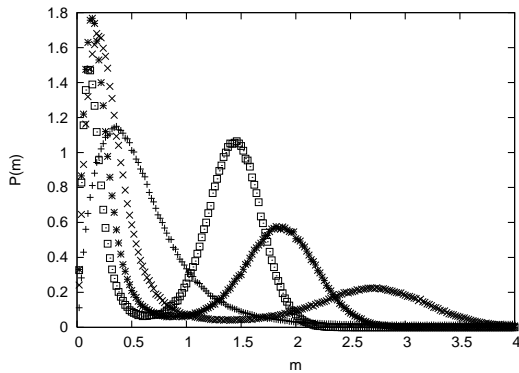


FIG. 4:

Savings propensity is positively related to the level of money. All simulations are done for $O(10^6)$ time steps with 100 agents and averaged over $O(10^3)$ time steps. Here, $c_1=0.95$ and the curves are plotted for $c_2=1$ (+), 2 (x), 3 (*) and 4 (□). Bimodality is clearly seen in the distribution of money.

sities. Such a segregated population gave rise to bimodal distributions. However, such segregations are exogenous since the λ s are given from outside the system. Here, however, we have a new model in which savings decision is completely endogenous and the economy organizes itself in such a way that it gives rise to a class of bimodal distributions. It may be noted that bimodality in the income/wealth distribution has actually been observed in many cases (see e.g., [13]).

It is seen numerically that bimodality appears for $c_1 \geq 0.92$ and $c_2 \geq 1$. Another interesting fea-

ture of this model is that keeping c_1 constant as c_2 increases, the monomodal distribution breaks into a bimodal distribution which again becomes a monomodal distribution for even larger values of c_2 . For example, consider Fig. 4 in which the maximum value of c_2 considered is 4. But as c_2 increases further, the distribution again becomes monomodal. However, it should also be mentioned that if c_1 is too large (for example, if $c_1 \geq 0.97$), then the system produces some strange-looking bimodal distributions.

IV. ‘IRRATIONAL’ DECISION MAKING

The standard economic paradigm of market clearance via utility maximization has been criticised on the grounds of limitations of computational capability of human beings [9]. The main challenge is to derive the homogeneity in savings behavior of the agents from a very simple thumb-rule such that the final distribution of income/wealth looks realistic. A few realistic components of decision-making are noted in Ref. [14, 15]

- (i) Players develop preferences for choices associated with better outcomes even though the association may be coincident, causally spurious, or superstitious.
- (ii) Decisions are driven by the two simultaneous and distinct mechanisms of reward and punishment, which are known to operate ubiquitously in humans.
- (iii) Satisficing or persisting in a strategy that yields a positive but not optimal outcome, is common and indicates a mechanism of reinforcement rather than optimization.

Of particular interest is item (iii) which goes directly against the derivations stated above (see Section II).

V. FROM ONE TO MANY ...

To model the savings behavior of the agents, we now make the following assumptions.

- (i) The agents do not perform static optimization.
- (ii) There is reinforcement in their decision-making process.
- (iii) The agents look for better payoffs. But eventually each of them converges to a single and simple strategy or thumb rule.

To incorporate the three above-mentioned assumptions, we model the agents’ saving behavior by Polya’s urn process [14]. The model is as follows. Consider the i -th

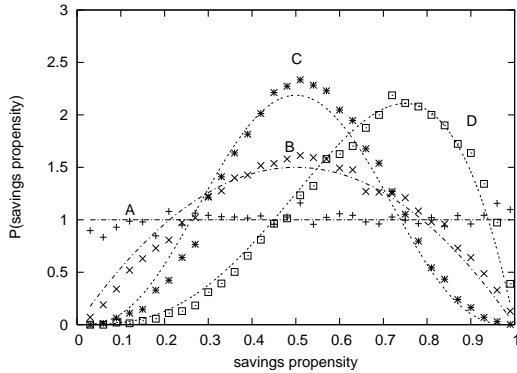


FIG. 5:

The savings distribution by reinforcement. Four cases are shown. Case A: $a=1, b=1$. Case B: $a=2, b=2$. Case C: $a=4, b=4$. Case D: $a=4, b=2$. The solid and the dotted lines show the theoretical results (the $\beta(a, b)$ distributions).

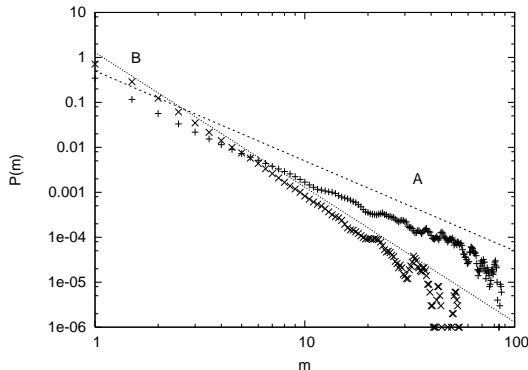


FIG. 6:

The steady state money distributions for two cases:- (A) $a=1, b=1$ and (B) $a=4, b=2$. Case (A) produces the same distribution of money as the CCM model (slope -2 in log-log plot) whereas in case (B) we get a distribution with slope -3 in the log-log plot. The lines drawn have slopes -2 and -3 respectively. All simulations are done for $O(10^6)$ time steps with 100 agents and averaged over $O(10^3)$ time steps.

agent. The choice is binary, he can take any of the two decisions, to consume (c) or to save (s). His strategies (c and s) such that $c, s = 0, 1$ and $c + s = 1$. At each instant, he chooses the values for c and s . Define C_t (S_t) as the number of times c (s) has been assigned a value of unity in t time periods. The savings propensity at time t (that is λ_t) is defined as the ratio of S_t to $(S_t + C_t)$. We can assume that initially $S_0 = a$ and $C_0 = b$. The reinforcement mechanism is incorporated by assuming that the probability of choosing $s = 1$ at any time $t + 1$, is simply λ_t . Basically, it is the Polya's urn model and the famous result that follows from it is the following (Ref. [14]). *The random variable λ_t converge almost surely to a limit λ . The distribution of λ is $\beta(a, b)$.* (See Fig. 5).

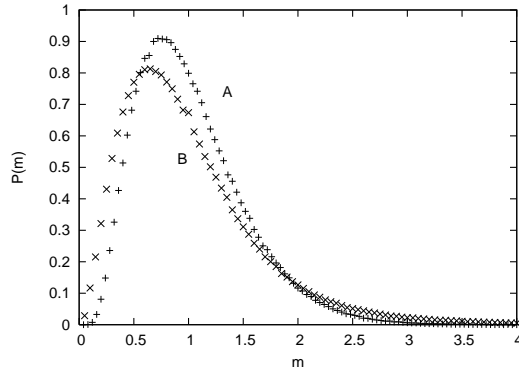


FIG. 7:

The steady state money distributions for two cases:- (A) $a = b; a, b \rightarrow \infty$ and (B) $a=4, b=4$. Note that case (A) is identical to CC model with $\lambda = 0.5$. All simulations are done for $O(10^5)$ time steps with 100 agents and averaged over $O(10^2)$ time steps.

A. Two limits

- (i) ($a=b=1$): From the above result, λ_t converges to λ where $\lambda \sim \text{uni}[0,1]$. The resultant distribution of money follows a power law. This is the basic CCM model. See Fig. 6.
- (ii) ($a=b; a, b \rightarrow \infty$): λ_t converges to λ where λ is a delta function at 0.5. This corresponds to a special case of the CC model where $\lambda = 0.5$. See Fig. 7.

B. For moderate values of a and b

Clearly, ($1 < a, b < \infty$): This model gives the gamma-like part as well as the Pareto tail of the income/money distribution for different values of a and b . For example, we show the results of two cases.

- (i) ($a=4, b=2$): The resulting distribution of savings propensity is clearly $\beta(4, 2)$. The distribution of money in the steady state follows a power law with a slope -3 in the log-log plot. See Fig. 6.
- (ii) ($a=4, b=4$): The resulting distribution of savings propensity is $\beta(4, 4)$. The distribution of money in the steady state is gamma function-like. See Fig. 7.

VI. FROM MANY TO ONE ...

In Section V, we have discussed how one can derive a set of distributed savings propensities starting from a unique value. Here, we discuss the reverse side of the same coin. We shall show that the agents with different

savings propensities, can converge to a single value over time.

To model this situation, we assume that the agents treat savings propensity as a strategy which evolves over time. A very simple rule of evolution is the following. The winner in any trade retains his strategy whereas the loser adopts the winner's strategy. Note that by winning in a trading action, we simply imply that the agent who gets the lion's share in that particular trading is the winner. Since it is a relative term, (by referring to Eqn.2) winning is determined by the value of the stochastic term ϵ . If $\epsilon \geq 0.5$, then the i -th agent wins (in Eqn. 2) else the j -th agent wins. Note that the most important support of this type of strategy evolution comes from the third observation by Flache and Macy [15] noted in Section IV. Let us assume that the possible saving propensities are finite and denoted by $\lambda_1, \lambda_2, \dots, \lambda_k$ etc. Also, let us denote the number of agents with λ_i savings propensity at time t by $n_i(t)$ (for $i= 1, \dots, k$). At each time-period two of the agents are randomly selected and they trade according to Eqn. 2 and then the loser adopts the winner's savings propensity. This process is repeated until the system reaches a steady state *in terms of savings propensities*. After the whole system becomes steady with the agents with a fixed saving propensity, the system is allowed to evolve further to reach a steady state in terms of money.

A. Convergence in savings propensity

Let

$$\sum_{i=1}^k n_i(0) = N, \quad (3)$$

as the total number of agents remains fixed over time (recall that $n_i(t)$ has been defined above as the number of agents with a particular savings propensity λ_i). The agents only shift from one savings propensity to another over time. Note that at each (trading) time point, the number of agents with a particular savings propensity rises or falls by unity with equal probability (i.e. depending on whether $\epsilon \geq 0.5$ or not) or it remains unchanged if its agents are not selected to trade. To put it formally, let us assume that the two agents selected to trade have savings propensities λ_i and λ_j respectively. Then

$$n_i(t+1) = n_i(t) \pm 1 \quad \text{with equal probability} \quad (4)$$

$$n_i(t+1) + n_j(t+1) = n_i(t) + n_j(t) \quad (5)$$

$$n_k(t+1) = n_k(t) \quad \text{for all } k \neq i, j. \quad (6)$$

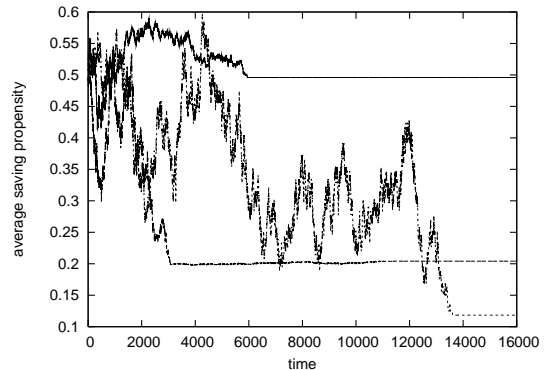


FIG. 8:

The convergence in savings propensity. As examples, three cases are shown. To study the phenomena of convergence in the savings propensities we focus on the fluctuation of the average savings propensities. Initially all agents are assigned with uniformly distributed savings propensities. Hence the average savings propensity is initially very close to half. But over time it evolves to reach a steady state where it maintains a fixed value indicating that all the agents have the same savings propensity. Thereafter the system behaves like the usual CC model.

Hence, the number of agents with a particular savings propensity performs a random walk of unit step and also note that the walk is bounded below since $n_i \geq 0$ for all i and also above by Eqn. 3. In fact, the formulation is akin to the *n-players ruin problem* [16] where the random walk occurs on a simplex given by Eqn. 3. The general *n-players ruin problem* is very difficult to solve analytically for $k \geq 4$. Ref. [16] presents a matrix-theoretic approach to the problem which reduces the complexity of the computation. However, we are not interested in finding the exact solutions to the problem. We simply note that the given enough time the system will ultimately evolve to a state where there is only one savings propensity and this is not unique. It can be any of the initial λ_i s ($i = 1, 2, \dots, k$). See Fig. 8.

B. Steady state money distribution

Once all the agents have a single savings propensity, the system then behaves like the CC model (see Fig. 1).

VII. SUMMARY AND DISCUSSION

The kinetic exchange models have been very successful in explaining the origin of the gamma function-like distribution and the power law in the income/wealth distribution. However, these models use the notion of savings extensively on an ad-hoc basis without offering much theoretical understanding of it. The aim of the present paper is to provide support to the kinetic exchange models by

deriving and explaining them from standard neo-classical economics paradigm and the not-so-standard models of reinforcement learning and strategic selection.

Ref. [8] provides a microeconomic basis for the kinetic exchange models with homogenous agents. Here, we extend that model to explain the heterogenous exchange models where the agents have different savings propensities (Sec. II). A further possibility is investigated in Sec. III where the savings propensity of an agent is dependent on the money holding of that particular agent and hence it changes over time (as the money holding changes). It is shown that even in that case, the economy organizes itself in such a way that the distribution of money becomes stable over time. In some cases, the distribution produces bimodality. Bimodal income/wealth distributions have indeed been seen in many countries (see e.g., Ref. [13]).

However, it is also noted in Sec. IV that the market clearing, competitive models used extensively in the economics literature has been criticised on the grounds of

limitations of computational capability of human beings (see e.g., [9]). So we try to explain the kinetic exchange models assuming that the agents follow some simple rules of thumb. It is shown that the mechanism of reinforcing one's own choice leads to the CCM model [7] (Sec. V). The basic result regarding the distribution of the fixed points follows from the famous *Polya's Urn* problem (Ref. [14]). Next, we show in Sec. VI that the agents following a simple rule of thumb of selecting the best strategy leads to the CC model (Ref. [6]). The game of strategy selection reduces to the generalized *Gambler's Ruin* problem or the *N-player Ruin* problem (Ref. [16]).

Acknowledgments

The author is grateful to Bikas K. Chakrabarti and Arnab Chatterjee for some useful suggestions.

-
- [1] C. Kleiber, S. Kotz, *Statistical Size Distributions in Economics and Actuarial Sciences*, Wiley, New Jersey, 2003.
 - [2] A. S. Chakrabarti, B. K. Chakrabarti, *Statistical Theories of Income and Wealth Distribution*, Economics e journal **4**(2010).
 - [3] A. Chatterjee, B. K. Chakrabarti, *Kinetic exchange models for income and wealth distribution*, Eur. Phys. J. B **60** 135-149 (2007).
 - [4] A. Chatterjee, S. Yarlagadda, B. K. Chakrabarti (Eds.), *Econophysics of wealth distributions*, Springer Verlag, Milan (2005).
 - [5] V. M. Yakovenko, J. B. Rosser, *Colloquium: Statistical mechanics of money, income and wealth*, Reviews of Modern Physics **81** 1703-1725 (2009).
 - [6] A. Chakraborti, B. K. Chakrabarti, *Statistical Mechanics of Money: How savings propensity affects its distribution*, Eur. Phys. J. B **17** 167-170 (2000).
 - [7] A. Chatterjee, B.K. Chakrabarti, S. S. Manna, *Money in gas-like markets: Gibbs and Pareto laws*, Physica Scripta T **106** 36-38 (2003).
 - [8] A. S. Chakrabarti, B.K. Chakrabarti, *Microeconomics of the Ideal Gas Like Market models*, Physica A **388** 4151-4158 (2009).
 - [9] P. Bak, *How Nature Works*, Oxford University Press, Oxford, Great Britain, 1997.
 - [10] T. Lux, *Emergent statistical wealth distributions in simple monetary exchange models: A critical review* in Ref. [4].
 - [11] J. Silver, E. Slud, K. Takamoto, *Statistical equilibrium wealth distribution in an exchange economy with stochastic preference*, J. Econ. Theo. **106** 417-435 (2002).
 - [12] A. Kargupta, *Money exchange model and a general outlook*, Physica A **359** 634 (2006).
 - [13] X. Sala-i-Martin, *The world distribution of income: Falling poverty and convergence*, Quart. J. Econ. **121** 351-397 (May 2006).
 - [14] R. Pemantle, *A survey of random processes with reinforcement*, Prob. Surveys **4** 1-79 (2007).
 - [15] A. Flache, M. Macy, *Stochastic collusion and the power law of learning*, J. Conflict Res. **46** 629-653 (2002).
 - [16] Y. C. Swan, F. T. Bruss, *A matrix-analytic approach to the N-player ruin problem*, J. Appl. Prob. **43** 3 755-766 (2006)