MARKET PRICE OF RISK AND RANDOM FIELD DRIVEN MODELS OF TERM STRUCTURE: A SPACE-TIME CHANGE OF MEASURE LOOK

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Abstract. No-arbitrage models of term structure have the feature that the return on zero-coupon bonds is the sum of the short rate and the product of volatility and market price of risk. Well known models restrict the behavior of the market price of risk so that it is not dependent on the type of asset being modeled. We show that the models recently proposed by Goldstein and Santa-Clara and Sornette, among others, allow the market price of risk to depend on characteristics of each asset, and we quantify this dependence. A key tool in our analysis is a very general space-time change of measure theorem, proved by the first author in earlier work, and covers continuous orthogonal local martingale measures including space-time white noise.

1. Introduction

Let $P(t,T)$ denote the market price of a discount bond that matures at time T. Since bond trading prices share many of the same characteristics of stock prices , several approaches to modeling bond prices use a stochastic noise term to express the uncertainty regarding future prices of a specific bond. The approach in Hull and White (1990), is to model each discount bond with an SDE of the form

$$dP = \mu(t,T)Pdt + P\sigma(t,T)dW \tag{1.1}$$

where $W(t)$ is a one-parameter Brownian motion which serves as a shared noise term. On the other hand, the coefficients μ and σ are adapted functions of t which also depend on the maturity time T. That is, these terms capture characteristics that are unique to the different maturity dates. By making some technical assumptions regarding μ and σ, one can derive a short–term interest rate process, $r(t)$ from the bond prices as well as forward interest rates, which form the modeling equations of Heath, Jarrow, and Morton term structure models [7]. HJM interest rate models are consistent with such SDE families given by (1.1) (see Baxter, Rennie [2]).

One attempts to choose the"parameters" μ and σ so that various correlations of bond price behavior can be attained while a consistency of bond prices is maintained. By this, we mean that arbitrage opportunities are ruled out within the model.

1.1. Condition for no–arbitrage.

The drift $\mu(t,T)$, for each bond maturity is a function of the short (interest) rate and the *market price of risk*, a single function

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$\lambda(t)$ that determines the underlying return on each bond price: There is an adapted function $\lambda(t,\omega)$, such that for $0 \le t \le T$,

$$\mu(t,T) = r(t) + \lambda(t,\omega)\sigma(t,T)$$

This peculiar feature of one-factor as well as multi-factor bond models states that the "excess" return on bonds of different maturities are proportional to their volatilities. This seems an unrealistic oversimplification that fails to take into account different characteristics of bonds with different maturities.

More elaborate bond models have been proposed by Goldstein, Santa-Clara and Sornette, and others. These models replace a Brownian motion by a Gaussian random field, $Z(t,T)$. More general correlations of bond prices may be modeled with these random field noises.

We show that a more general market price of risk is compatible with no-arbitrage models for a variety of random fields generated by the two-parameter Brownian sheet process. We also characterize the generality of market price of risk within these classes of models.

2. Random Field Driven Bond Models

Assume that the random field $Z(t,T)$ is a Brownian motion for each fixed T, and that the process is continuous in both variables. We use the bond model

$$dP = \mu(t,T)Pdt + P\sigma(t,T)d_tZ(t,T) \tag{2.1}$$

where all differentials, including $d_tZ(t,T)$, are taken in the t variable only; and so we will often refer to (2.1) as a semi-SPDE or a parametrized SDE. We assume throughout the rest of the article that σ is continuous in t. In order to find sufficient conditions (on a market price of risk) so that no arbitrage is possible, we express the drift in terms of an existing short rate process $r(t)$ and an unknown function λ as follows: For some adapted function $\lambda(t,T,\omega)$, of both time and maturity date, we have

$$\mu(t,T) = r(t) + \lambda(t,T,\omega)\sigma(t,T)$$

We find that *some dependence* on T is consistent with no arbitrage, and we obtain the space-time risk neutral measure.

2.1. Main Result.

The following Theorem illustrates the generality of allowable market prices of risk for certain random field term structure models. It addresses the bond model in (2.1) when the noise Z has the form

$$Z \stackrel{\triangle}{=} W(t,T)/\sqrt{T} = \mathcal{W}_t([0,T])/\sqrt{T} \tag{2.2}$$

where $W(t,T)$ is the two-parameter Brownian sheet (a zero mean Gaussian process with $Cov(W(t_1,T_1),W(t_2,T_2)) = (t_1 \wedge t_2)(T_1 \wedge T_2))$ corresponding to a space-time white noise $\mathcal{W} \stackrel{\triangle}{=} \{\mathcal{W}_t(B), \mathcal{F}_t; 0 \le t \le T_0, B \in \mathcal{B}([0,T_0])\}$ on a usual probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ ($\mathcal{B}([0,T_0])$ is the Borel σ-field over $[0,T_0]$). The relation between space-time white noise as a local martingale measure and its induced Brownian sheet and the relevant definitions are detailed in [1, 10], and we refer the interested reader to these references for additional interesting details.

Theorem 2.1. *Suppose that Z has the form in* (2.2). *Then for any $T_0 > 0$ and any market price of risk λ of the form*

$$\lambda(t,T) = \int_0^T \eta(t,u,\omega)du; \qquad T \le T_0, \tag{2.3}$$

where η is an $\mathcal{F}_t$-predictable random field (*see* [1, 10]) *such that*

$$\mathbb{E}_{\mathbb{P}} \exp\left(\int_0^{T_0} \int_0^{T_0} \left[\log(T/u)\eta(t,u)\lambda(t,u)/2 + u\eta(t,u)^2 \right] \, dudt \right) < \infty, \tag{2.4}$$

the bond model (2.1) *is free from arbitrage over the time interval* $0 \le t \le T_0$.

Remark 2.1. *We make the following observations:*

(1) *This result shows that no-arbitrage is consistent with market prices of risk that are T-dependent and also that the market price of risk is absolutely continuous with respect to the maturity time.*
(2) *As shown in the next subsection, as a Corollary of the proof of Theorem* 2.1, *the integrability condition in* (2.4) *may be replaced by the simpler condition*

$$\mathbb{E}_{\mathbb{P}} \exp\left(\frac{5T_0}{4} \int_0^{T_0} \int_0^{T_0} \eta(t,u)^2 \, dudt \right) < \infty \tag{2.5}$$

(3) *The space-time setting we are adopting here and in Lemma* 2.1 *is more natural and flexible than the classical multiparameter one: it gives time its traditional role in processes while allowing the space variable* (*in this case the maturity date*) *to be free to take from any space, not necessarily symmetric to the time set. It can also accomodate a larger class of noises while avoiding the unnatural restrictions imposed by the multiparameter setting* (*see* [1]). *This makes our results extendable to more general models.*
(4) *The covariance structure of the random field $Z(t,T)$ in Theorem* 2.1 *is, of course,*

$$\mathbb{E}_{\mathbb{P}}[Z(t_1,T_1)Z(t_2,T_2)] = (t_1 \wedge t_2)\sqrt{\frac{(T_1 \wedge T_2)}{(T_1 \vee T_2)}}$$

2.2. **Proofs and Extensions of the Main Result.** We start with the aforementioned space-time change of measure result needed in our proof of Theorem 2.1. This is a special case of Corollary 2.3 along with Lemma 2.4 in [1], which we combine and state here for the reader's convenience as

Lemma 2.1. *Suppose that $\mathcal{W} = \{\mathcal{W}_t(B), \mathcal{F}_t; 0 \le t \le T_0, B \in \mathcal{B}([0,T_0])\}$ is a space-time white noise on the usual probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$. Suppose further that g is an $\mathcal{F}_t$-predictable random field satisfying*

$$\mathbb{E}_{\mathbb{P}} \exp\left(\frac{1}{2} \int_0^{T_0} \int_0^{T_0} g(s,u)^2 \, dsdu \right) < \infty, \tag{2.6}$$

then

$$\tilde{\mathcal{W}}_t(B) \triangleq \mathcal{W}_t(B) + \int_0^t \int_B g(s,u)duds; \qquad 0 \le t \le T_0, B \in \mathcal{B}([0,T_0])$$

is a white noise on $(\Omega, \mathcal{F}_{T_0}, \{\mathcal{F}_t\}, \tilde{\mathbb{P}})$*, where*

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \exp\left[-\int_0^{T_0}\int_0^{T_0} g(s,u)\mathcal{W}(ds,du) - 1/2\int_0^{T_0}\int_0^{T_0} g(s,u)^2 ds du\right].$$

I.e., the random field $\left\{\tilde{W}(t,T) \triangleq \tilde{\mathcal{W}}_t([0,T]); 0 \le t, T \le T_0\right\}$ *is a Brownian sheet with respect to* $\tilde{\mathbb{P}}$*.*

We now turn to the

Proof of Theorem 2.1. We assume that the market price of risk λ satisfies (2.3), and we consider the process

$$\tilde{Z}(t,T) \triangleq Z(t,T) + \int_0^t \lambda(s,T)ds = \frac{1}{\sqrt{T}}W(t,T) + \int_0^t\int_0^T \eta(s,u)duds \tag{2.7}$$

We wish to determine an absolutely continuous probability measure $\tilde{\mathbb{P}}$ defined on the σ-field $\mathcal{F}_{T_0}$ so that the process in (2.7) is a martingale for each fixed $T > 0$. For this purpose, it suffices to find $\tilde{\mathbb{P}}$ making the process

$$\tilde{W}(t,T) \triangleq W(t,T) + \sqrt{T}\int_0^t\int_0^T \eta(s,u)duds \tag{2.8}$$

a Brownian sheet over the parameter range $[0,T_0]\times[0,T_0]$ (each process in (2.7), for fixed T, is then a standard Brownian motion with respect to such a measure $\tilde{\mathbb{P}}$). Towards this end, we need to choose g satisfying the generalized Novikov condition (2.6) such that

$$\int_0^t\int_0^T g(s,u)duds = \sqrt{T}\int_0^t\int_0^T \eta(s,u)duds \tag{2.9}$$

Letting

$$g(s,u) \triangleq \frac{d}{du}\left(\sqrt{u}\int_0^u \eta(s,y)dy\right) = \frac{1}{2\sqrt{u}}\lambda(s,u) + \sqrt{u}\eta(s,u), \tag{2.10}$$

where $\lambda(s,u) = \int_0^u \eta(s,y)dy$, it is formally clear that (2.9) holds. We verify the validity of this formal computation by computing the $L^2[0,T_0]$ norm of each term in the function g, and we will show that each L^2 norm is finite a.s. In fact, we will compute the L^2 norm in both variables t and T over the square.

First,

$$\begin{aligned}
\left\| \frac{1}{2\sqrt{u}}\lambda(s,u)\right\|_2^2 &= \int_0^{T_0}\int_0^{T_0}\frac{1}{4u}\lambda^2(s,u)duds \\
&= \int_0^{T_0}\int_0^{T_0}\frac{1}{4u}\int_0^u\int_0^u \eta(s,r)\eta(s,\tau)drd\tau duds \\
&= \int_0^{T_0}\int_0^{T_0}\int_0^{T_0}\eta(s,r)\eta(s,\tau)\int_{r\vee\tau}^{T_0}\frac{1}{4u}dudrd\tau ds \\
&= \frac{1}{4}\int_0^{T_0}\int_0^{T_0}\int_0^{T_0}\eta(s,r)\eta(s,\tau)\log\left(\frac{T_0}{r\vee\tau}\right)drd\tau ds \\
&= \frac{1}{2}\int_0^{T_0}\int_0^{T_0}\eta(s,\tau)\int_0^{\tau}\eta(s,r)\log\left(\frac{T_0}{\tau}\right)drd\tau ds \\
&= \frac{1}{2}\int_0^{T_0}\int_0^{T_0}\eta(s,\tau)\lambda(s,\tau)\log\left(\frac{T_0}{\tau}\right)d\tau ds
\end{aligned}$$

This quantity is finite a.s., since (2.4) implies that its exponential has finite mean. Secondly,

$$\left\|\sqrt{u}\eta(s,u)\right\|_2^2 = \int_0^{T_0}\int_0^{T_0} u\eta(s,u)^2 dsdu$$

is also finite a.s. for the same reason.

Now the condition (2.6) of Lemma 2.1 may be stated as $\mathbb{E}_{\mathbb{P}}\exp(\|g\|_2^2/2) < \infty$. We use the inequality

$$\|g\|_2^2 = \left\|\frac{1}{2\sqrt{u}}\lambda(s,u)+\sqrt{u}\eta(s,u)\right\|_2^2 \le 2\left\|\frac{1}{2\sqrt{u}}\lambda(s,u)\right\|_2^2 + 2\left\|\sqrt{u}\eta(s,u)\right\|_2^2$$

and the estimates previously derived to obtain

$$\frac{\|g\|_2^2}{2} \le \frac{1}{2}\int_0^{T_0}\int_0^{T_0}\eta(s,u)\lambda(s,u)\log\left(\frac{T_0}{u}\right)duds + \int_0^{T_0}\int_0^{T_0}u\eta(s,u)^2dsdu$$

But, the assumption in (2.4) implies that the exponential moment of this quantity is finite; hence, the hypothesis of Lemma 2.1 is satisfied.

Consider a bond model where the dynamics of discount bonds are given by (2.1) and where μ has the form $\mu = r + \lambda\sigma$. For each $T > 0$ we have

$$dP = (r(t)+\lambda(t,T)\sigma(t,T))Pdt + \sigma(t,T)Pd_tZ(t,T) \tag{2.11}$$

where we assume that (P, Z) is a solution of this semi-SPDE (parametrized SDE) on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$. We let $\tilde{Z}(t,T)$ denote the process defined in (2.7), and we note that

$$Z(t,T) = \tilde{Z}(t,T) - \int_0^t \lambda(s,T)ds$$

Under the measure $\tilde{\mathbb{P}}$ of Lemma 2.1, $\sqrt{T}\tilde{Z}(t,T)$ is a Brownian sheet, so that $\tilde{Z}(t,T)$ is a standard Brownian motion for fixed T. Then the integral form of (2.11), which holds pathwise, may be written in differential form as

$$\begin{aligned}
dP &= (r(t)+\lambda(t,T)\sigma(t,T))Pdt + \sigma(t,T)Pd_t\tilde{Z}(t,T) - \lambda(t,T)\sigma(t,T)Pdt \\
&= r(t)Pdt + \sigma(t,T)Pd_t\tilde{Z}(t,T)
\end{aligned} \tag{2.12}$$

We now proceed to verify Harrison and Kreps criteria for no arbitrage [6]. Since the semi-SPDE defined by (2.12) is in terms of a standard Brownian motion $\tilde{Z}$, one may fix T and use the Itô formula to investigate each process

$$D_t \triangleq \exp\left[-\int_0^t r(s)ds\right] P(t,T)$$

It follows that

$$dD_t = \exp\left[-\int_0^t r(s)ds\right] \sigma(t,T)P(t,T)d_t\tilde{Z}(t,T)$$

and hence D_t is a martingale in t, under $\tilde{\mathbb{P}}$ (since σ is assumed continuous in t, and thus bounded on $[0,T_0]$). The model then satisfies the desired Harrison and Kreps criteria developed in [6] to form a model with no arbitrage and Theorem 2.1 is proved. □

We now prove the second observation in Remark 2.1

Corollary 2.1. *In Theorem* 2.1*, if we replace the integrability condition* (2.4) *by*

$$\mathbb{E}_{\mathbb{P}} \exp\left(\frac{5T_0}{4}\int_0^{T_0}\int_0^{T_0} \eta(t,u)^2 \ dudt\right) < \infty \tag{2.13}$$

then Theorem 2.1 *holds.*

Proof. We see from (2.9) and (2.10) and the following discussion that if

$$g(t,T) \triangleq \frac{1}{2\sqrt{T}}\int_0^T \eta(t,u)du + \sqrt{T}\eta(t,T)$$

it suffices to show

$$\mathbb{E}_{\mathbb{P}} \exp\left[\frac{1}{2}\|g\|_2^2\right] < \infty.$$

Applying the Cauchy-Schwarz inequality, we obtain

$$\left|\frac{1}{2\sqrt{T}}\int_0^T \eta(t,u)du\right|^2 \le \frac{1}{4T}\int_0^T \eta^2(t,u)du \cdot T \le \frac{1}{4}\int_0^{T_0} \eta^2(t,u)du$$

Then

$$\left\|\frac{1}{2\sqrt{T}}\int_0^T \eta(t,u)du\right\|_2^2 \le \frac{T_0}{4}\|\eta\|_2^2$$

so that

$$\|g\|_2^2 \le 2\frac{T_0}{4}\|\eta\|_2^2 + 2T_0\|\eta\|_2^2 = \frac{5T_0}{2}\|\eta\|_2^2$$

Now,

$$\mathbb{E}_{\mathbb{P}} \exp\left[\frac{1}{2}\|g\|_2^2\right] \le \mathbb{E}_{\mathbb{P}} \exp\left[\frac{5T_0}{4}\|\eta\|_2^2\right],$$

and that last term is finite by assumption (2.13). □

In Santa-Clara and Sornette [9], more general random field term structure models are considered. These use a noise term of the form

$$Z(t,T) = \frac{1}{h(T)} W(t, h^2(T)) = \frac{1}{h(T)} \mathcal{W}_t([0, h^2(T)]) \tag{2.14}$$

where h is an increasing, positive function on the interval $[0, T_0]$. Clearly, such random fields have the feature that for each fixed T, the process is a standard Brownian motion. Also, the random field correlation is more general than for the normalized Brownian sheet in (2.2). Theorem 2.2 below shows that such models allow essentially the same type of market price of risk behavior as in Theorem 2.1.

Theorem 2.2. *Suppose that Z is the random field appearing in* (2.14) *where the function $T \to h(T)$ is absolutely continuous and the function $h'(T)$ is L^2 on each interval $[a, T_0]$, $a > 0$. Then for market price of risk λ of the form*

$$\lambda(t,T) = \int_0^T \eta(t,u,\omega)du; \qquad T \le T_0,$$

where η is $\mathcal{F}_t$-predictable, $\mathcal{F}_t$ is a usual filtration with respect to which the white noise $\mathcal{W}$ in (2.14) *is measurable, and*

$$\mathbb{E}_{\mathbb{P}} \exp\left(\int_0^{T_0} \int_0^{T_0} \left[\eta(t,u)\lambda(t,u) \int_u^{T_0} h'(\tau)^2 d\tau/2 + h(u)^2 \eta(t,u)^2 \right] dudt \right) < \infty,$$

the bond model (2.1) *is free from arbitrage over the time interval* $0 \le t \le T_0$.

Lemma 2.1 applies just as easily in this case (in fact see Theorem 2.2 in [1] which applies to a much larger class of noises) and the proof is quite similar to that for Theorem 2.1 and will be omitted.

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