Two stock options at the races: Black-Scholes forecasts

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Abstract

Suppose one buys two very similar stocks and is curious about how much, after some time T, one of them will contribute to the overall asset, expecting, of course, that it should be around $1/2$ of the sum. Here we examine this question within the classical Black and Scholes (BS) model, focusing on the evolution of the probability density function $P(w)$ of a random variable $w = a_T^{(1)}/(a_T^{(1)} + a_T^{(2)})$ where $a_T^{(1)}$ and $a_T^{(2)}$ are the values of two (either European- or the Asian-style) options produced by two absolutely identical BS stochastic equations. We show that within the realm of the BS model the behavior of $P(w)$ is surprisingly different from common-sense-based expectations. For the European-style options $P(w)$ always undergoes a transition, (when T approaches a certain threshold value), from a unimodal to a bimodal form with the most probable values being close to 0 and 1, and, strikingly, $w = 1/2$ being the *least* probable value. This signifies that the symmetry between two options spontaneously breaks and just one of them completely dominates the sum. For path-dependent Asian-style options we observe the same anomalous behavior, but only for a certain range of parameters. Outside of this range, $P(w)$ is always a bell-shaped function with a maximum at $w = 1/2$.

Introduction

In finance, the style of an option is a general term denoting the class into which the option falls, usually defined by the dates on which the option may be exercised. An Asian option is an option where the payoff is not determined by the underlying price at maturity, contrary to European or American-style options, but by the average underlying price over some pre-set period of time. Such average-value options are commonly traded on some Asian (but also Western) markets being somehow advantageous over European ones since the risk of asset price manipulation near the maturity date is reduced due to their path dependence.

In classical Black-Scholes settings (Black and Scholes 1973, Merton 1973), the underlying asset on which the Asian option is based is equal to a stock with price S_t which follows the so-called geometric Brownian motion. In other words, it is defined as the strong solution of the following linear stochastic differential equation:

$$dS_t = \omega \, S_t \, dt + \sigma \, S_t \, dB_t, \tag{1}$$

where σ is the volatility of S_t, B_t is a standard one-dimensional Brownian motion and ω is a constant dependent on the nature of the asset: it could be a stock, a currency, a commodity and etc. For example, if S_t is a stock paying a dividend at the continuous rate δ, one has $\omega = r - \delta$, where r is the risk-free rate of interest.

Resorting to Itô calculus, one solves (1) to find

$$S_t = S_0 \exp\left(-\frac{\sigma^2}{2}\mu t + \sigma B_t\right), \tag{2}$$

where $S_0 = S_{t=0}$ is the initial price and $\mu = 1 - 2\omega/\sigma^2$ is a constant, which may be positive, equal to zero or negative. Further on, for continuous averaging with equal time-independent weights, the random variable of interest - the value of an Asian option - is defined as the following functional of a Brownian trajectory:

$$A_T = \frac{S_0}{T - T_0}\,\tau, \quad \tau = \int_{T_0}^{T} dt\, \exp\left(-\frac{\sigma^2}{2}\mu t + \sigma B_t\right), \tag{3}$$

where T is the maturity date and T_0 is the time moment when one starts to monitor the evolution of S_t. Without lack of generality, we set $T_0 = 0$ in what follows.

In this paper we pose the following question which is, we believe, of a considerable conceptual interest: Suppose that one has not a single equation (1) but, say, two such stochastic equations, having the same volatility, the same ω, the same initial price S_0 and random noise terms, $dB_t^{(1)}$ and $dB_t^{(2)}$, which have identic distributions. These two equations produce two identical European-style assets, $S_t^{(1)}$ and $S_t^{(2)}$, which, in turn, generate two identical Asian-style random variables, $A_T^{(1)}$ and $A_T^{(2)}$. What can be said about the distribution functions $P(W)$ and $P(\mathcal{W})$ of random variables:

$$W = \frac{S_T^{(1)}}{S_T^{(1)} + S_T^{(2)}}, \tag{4}$$

and

$$\mathcal{W} = \frac{A_T^{(1)}}{A_T^{(1)} + A_T^{(2)}} = \frac{\tau_1}{\tau_1 + \tau_2}? \tag{5}$$

These random variables describe a realization-dependent relative weight of one asset in the sum of two assets. Noticing that $S_t^{(1)}$ ($A_T^{(1)}$) and $S_t^{(2)}$ ($A_T^{(2)}$) are obviously equal to each other on average, and moreover, that all their higher moments are equal, one might be tempted to say that the distribution functions $P(W)$ and $P(\mathcal{W})$ should be bell-shaped functions with a maximum at $1/2$. They may broaden (initially both are delta-functions) with growth of the maturity T, but still $W = 1/2$ and $\mathcal{W} = 1/2$ should remain the most probable values.

Curiously, within the realm of the Black-Scholes model most often this is not the case. We set out to show here that the behavior of $P(W)$ and $P(\mathcal{W})$ is, in general, surprisingly different from these common-sense-based expectations.

1 Competition of two uncorrelated European-style variables.

Note first that W (and hence, $P(W)$) is independent of μ, since the factors $\exp(-\sigma^2 \mu T/2)$ in the nominator and the denominator cancel each other.

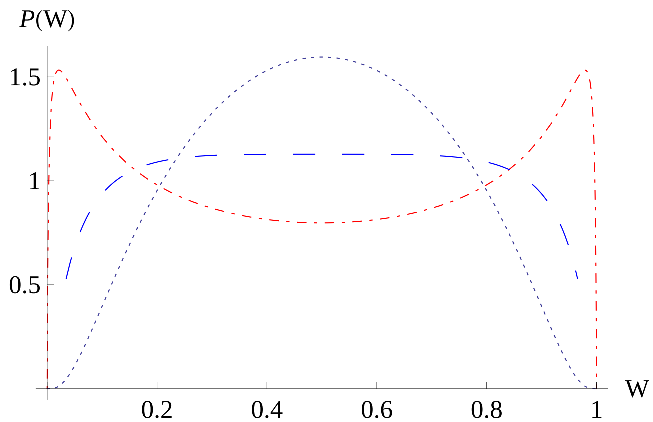

Figure 1: Probability density $P(W)$ in (6) for different values of the effective maturity α_T: Dotted curve corresponds to $\alpha_T = 0.25$, dashed - to $\alpha_T = 0.5$, while dot-dashed one - to $\alpha_T = 0.7$.

For uncorrelated increments $dB_t^{(1)}$ and $dB_t^{(2)}$, the distribution function $P(W)$ can be calculated exactly (see Appendix A):

$$P(W) = \frac{1}{\sqrt{8\pi\alpha_T}} \frac{1}{W(1-W)} \exp\left(-\frac{1}{8\alpha_T} \ln^2\left(\frac{W}{1-W}\right)\right), \tag{6}$$

where $\alpha_T = \sigma^2 T/2$ is an effective maturity.

Despite a relatively simple form, $P(W)$ in (6) contains a surprise: it shows a markedly different behavior for α_T less or greater than $\alpha_T^c = 1/2$ (see Fig. 1). For $\alpha_T < 1/2$ the distribution has a maximum at $W = 1/2$, which means that at early stages both $S_T^{(1)}$ and $S_T^{(2)}$ contribute proportionally; it is thus most likely that each variable defines just one half of the sum. However, when α_T exceeds $1/2$, $P(W)$ changes its shape from a unimodal to a bimodal, M-shaped form with $P(W=0) = P(W=1) = 0$ and maximal values progressively closer to 0 and 1 as $\alpha_T \to \infty$. Strikingly, $W = 1/2$ is now the *least* probable value. It means that at sufficiently large maturities the symmetry breaks and one of the variables completely dominates the sum, while the second one becomes a complete loser. Certainly, this is not the behavior one may expect on intuitive grounds.

2 Competition of two uncorrelated Asian-style variables.

Let $\Psi(\tau)$ denote the distribution function of τ variables in (5). Then, for two uncorrelated Asian-style variables the probability density $P(\mathcal{W})$ can be written as (see Appendix B):

$$P(\mathcal{W}) = \int_0^\infty u\, du\, \Psi\left(\mathcal{W}u\right)\, \Psi\left((1-\mathcal{W})\, u\right). \tag{7}$$

Clearly, $P(\mathcal{W})$ is symmetric around $\mathcal{W} = 1/2$. The question is whether $\mathcal{W} = 1/2$ is always the maximum of the distribution?

The form of the distribution function $\Psi(\tau)$ was discussed in the literature on mathematical finance (see, e.g., (Yor 1992, Geman and Yor 1993, Dufresne 2004)). In addition, such τ variables appear in different domains of the probability theory: they are the continuous counterparts of the so-called Kesten variables (Kesten 1973) that play an important role in multiplicative stochastic processes and in the renewal theory for products of random matrices. In physical literature, variables τ emerge in different contexts related to transport in disordered media. In particular, τ defines a resistance of a finite interval of length T offered to a passage (Redner 2001) of particles diffusing in presence of a random, time-independent Gaussian force with average value $\sigma^2\mu/2$ and variance σ^2 (Kesten et al 1975, Sinai 1982, Derrida and Pomeau 1982). Consequently, inverse moments of τ define the moments of stationary currents through the interval boundaries. Within this context, moments of τ (Burlatsky et al 1992, Oshanin et al 1993a, Oshanin et al 1993b, Monthus and Comtet 1994, Monthus et al 1998) and the distribution function $\Psi(\tau)$ (Monthus and Comtet 1994, Monthus et al 1998) were also calculated exactly.

For arbitrary μ, $\Psi(\tau)$ is determined by (Geman and Yor 1993):

$$\Psi(\tau) = \Psi_{con}(\tau) + \Psi_{dis}(\tau), \tag{8}$$

with

$$\Psi_{dis}(\tau) = \left(\frac{1}{\tau'}\right)^{-\mu/2} \exp\left(-\frac{1}{\tau'}\right) \sum_{0\leq n\leq\mu/2} e^{-\alpha_T n(n-\mu)} \frac{(-1)^n(\mu-2n)}{\Gamma\left(1+\mu-n\right)} L_n^{\mu-2n}\left(\frac{1}{\tau'}\right),$$

and

$$\Psi_{con}(\tau) = \frac{\sigma^2}{8\pi^2}\left(\frac{1}{\tau'}\right)^{(1+\mu)/2} \int_0^\infty u\,du\left|\Gamma\left(-\frac{\mu}{2}+\frac{iu}{2}\right)\right|^2 W_{\frac{1+\mu}{2},\frac{iu}{2}}\left(\frac{1}{\tau'}\right)\times$$
$$\times\exp\left(-\frac{\alpha_T}{4}\left(\mu^2+u^2\right)-\frac{1}{2\tau'}\right)\sinh(\pi u),$$

where $\tau' = \sigma^2\tau/2$, $L_n^\gamma(x)$ are Laguerre polynomials, $\Gamma(x)$ is the Gamma function, and $W_{\rho,\nu}(x)$ are Whittaker functions (Abramowitz and Stegun 1972). For $\mu > 0$ the distribution function in (8) consists of discrete ($\Psi_{dis}(\tau)$) and continuous branches ($\Psi_{con}(\tau)$), while for $\mu \leq 0$ the distribution is determined by the continuous branch only. Hence, unlike the distribution of W, $P(\mathcal{W})$ will depend on the sign (and value) of μ. Consequently, we will consider the cases of positive and negative μ separately.

$\mu = 0$.

In this marginal case τ defines an inverse probability current in a finite Sinai chain (Burlatsky et al 1992, Oshanin et al 1993a, Oshanin et al 1993b). Occasionally, for this case the distribution $P(\mathcal{W})$ has been already calculated in (Oshanin and Redner 2009), which studied the probability that a partially melted heteropolymer at the melting temperature will either denaturate completely or return back to a native helix state. Here $\mathcal{W}$ defines the so-called splitting probability (Redner 2001) - the probability that the boundary between the helix and coil phases (which performs Sinai-type diffusion (Sinai 1982)) will first hit one of the extremities of the chain without having ever reached the second extremity.

Adapting to our notations the result of (Oshanin and Redner 2009), we have

$$P(\mathcal{W}) = \frac{1}{\pi\alpha_T\sqrt{\mathcal{W}(1-\mathcal{W})^3}} \int_{-\infty}^{\infty} du\,\frac{\cosh(u)}{\cosh(\eta)}\cos\left(\frac{\pi u}{\alpha_T}\right)\times \tag{9}$$
$$\times\exp\left(\frac{\pi^2-4u^2-4\eta^2}{4\alpha_T}\right),\ \eta = \operatorname{arcsinh}\left(\sqrt{\frac{\mathcal{W}}{1-\mathcal{W}}}\cosh(u)\right).$$

In Fig. 2 we plot $P(\mathcal{W})$ for several values of an effective maturity α_T. For sufficiently low values of α_T the distribution is unimodal and centered around $\mathcal{W} = 1/2$. Hence, in this early-time regime both options

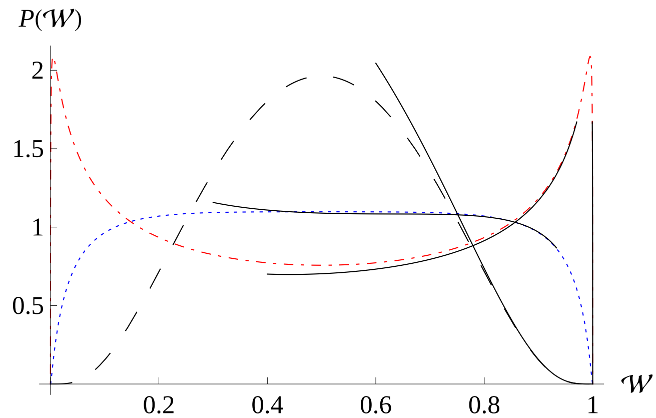

Figure 2: Exact distribution $P(\mathcal{W})$, (9), for $\alpha_T = 0.5$ (dashed curve), $\alpha_T = 1.63$ (dotted), and $\alpha_T = 3.5$ (dash-dotted). Thin solid lines are the corresponding asymptotic results in (16) (see Appendix B).

equally contribute to the total asset. However, if we allow the options to "mature" longer, we observe the same surprising *anomaly*, which we have already encountered in the previous section: when α_T reaches a critical value $\alpha_T^c \approx 1.63$, the maximum at $\mathcal{W} = 1/2$ ceases to exist and the distribution becomes close to uniform for $0.2 \leq \mathcal{W} \leq 0.8$. Thus any value of $\mathcal{W}$ in this range is nearly equally probable. For α_T exceeding α_T^c, two maxima emerge continuously and the distribution changes its shape becoming M-shaped bimodal, with most probable values close to 0 and 1. Although the *average* value $\mathcal{W}$ is still equal to $1/2$ in this regime, the probability density now has a *minimum* at $\mathcal{W} = 1/2$. Therefore, for $\mu = 0$ the path-dependence of the Asian-style variables does not suppress the transition to the disproportionate behavior but only shifts it to later times; α_T^c is more than three times larger than the corresponding value for the European-style variables.

$\mu > 0$.

In this case the distribution function in (8) converges (Oshanin et al 1993b, Monthus and Comtet 1994, Monthus et al 1998), as $\alpha_T \to \infty$, to a limiting form defined by the first term of a discrete branch,

$$\Psi(\tau) \to \Psi_\infty(\tau) = \frac{2^\mu \sigma^2}{\Gamma(\mu)} \frac{1}{(\sigma^2 \tau)^{1+\mu}} \exp\left(-\frac{2}{\sigma^2 \tau}\right). \tag{10}$$

Hence, for $\alpha_T = \infty$ the variable τ has a very broad distribution characterized by a "fat" algebraic tail. For $0 < \mu \leq 1$ this distribution is normalized but does not have moments.

We do not attempt here to present an exact solution for $P(\mathcal{W})$ for arbitrary $\mu > 0$ and arbitrary α_T. Instead, our aim will be to get a conceptual understanding whether $P(\mathcal{W})$ undergoes, at a certain unknown value of α_T, a transition from a unimodal to a bimodal form. This question can be immediately answered if we find that the corresponding limiting form of $P(\mathcal{W})$ exhibits such a transition.

Plugging the limiting form in (10) into (7) and performing integrations, we find that in the limit $\alpha_T \to \infty$, the probability density $P(\mathcal{W})$ converges to a limiting form

$$P(\mathcal{W}) \to P_\infty(\mathcal{W}) = \frac{\Gamma(2\mu)}{\Gamma^2(\mu)} \mathcal{W}^{\mu-1} (1 - \mathcal{W})^{\mu-1}, \tag{11}$$

i.e., it tends to a beta-distribution.

The distribution on the right-hand-side of (11) has a different shape for $0 < \mu < 1$, $\mu = 1$ or $\mu > 1$. For $0 < \mu < 1$ the distribution $P_\infty(\mathcal{W})$ is a U-shaped bimodal. For $\mu = 1$ it is uniform. Finally, $P_\infty(\mathcal{W})$ is unimodal, centered at $\mathcal{W} = 1/2$ for $\mu > 1$.

Therefore, for $0 < \mu < 1$ the distribution $P(\mathcal{W})$ will change its shape from a unimodal to a bimodal at a certain value α_T^c. For the marginal $\mu = 1$ case, the distribution $P(\mathcal{W})$ will tend to a uniform distribution as $\alpha_T \to \infty$. Finally, for $\mu > 1$, the distribution will always remain unimodal and centered around the most probable value $\mathcal{W} = 1/2$.

$\mu < 0$.

In this case the distribution $\Psi(\tau)$ is given by the continuous branch in (8). Note that the expression in (8) can not be used directly, since it does not allow one to perform an integration over τ in (7). Thus, we will first try to obtain a plausible approximation for $\Psi(\tau)$ valid for large α_T, which will allow us to perform the integration over τ. Then, on base of this result we will see whether the transition to the disproportionate behavior indeed takes place or not. After some straightforward manipulations (see Appendix B), we find

$$\Psi(\tau) \sim C\,\frac{\exp(-1/\tau')}{\tau'^{1+\mu/2}}\,U\left(-\frac{\mu}{2}, 1, \frac{1}{\tau'}\right)\,\exp\left(-\frac{\mathrm{arcsinh}^2(\sqrt{\tau'})}{\alpha_T}\right), \tag{12}$$

where C is a constant (see Appendix B) and $U(a, b, z)$ is the confluent hypergeometric function (Abramowitz and Stegun 1972). Note that for sufficiently large α_T the approximation in (12) works fairly well for any value of τ (see Fig. 5 in Appendix B). Moreover, it exhibits exact asymptotic behaviors in the limits $\tau \to 0$ and $\tau \to \infty$.

Equation (12) predicts that $\Psi(\tau)$ has a log-normal tail as $\tau \to \infty$. Given such a slow decay, one expects that the integral in (7) is dominated by large values of u. One finds then that, for sufficiently large α_T, and $\mathcal{W}$ not too close to 0 or 1,

$$P(\mathcal{W}) \sim \frac{1}{\sqrt{\alpha_T}}\frac{1}{W(1-W)}, \tag{13}$$

while for $W \to 0$ or 1 one finds that $P(\mathcal{W})$ obeys (6) (with possible logarithmic corrections, see Appendix B, (16)). Therefore, for $\mu < 0$ the distribution $P(\mathcal{W})$ does not reach a limiting form when $\alpha_T \to \infty$. This means, in turn, that *for all* $\mu < 0$, the distribution $P(\mathcal{W})$ exhibits a generic transition from a bell-shaped to an M-shaped form, as α_T passes through some critical value α_T^c.

In fact, this is a rather counter-intuitive result. The random variables A_T are integrals of a geometric Brownian motion - an exponential of a symmetric Brownian motion plus a constant drift term $-\sigma^2\mu/2$. One may naturally expect that for sufficiently large μ a contribution due to symmetric Brownian motion will be insignificant. This is precisely the behavior we observed in the $\mu > 0$ case, for which the transition to the disproportionate behavior takes place only for $\mu \in [-1, 0[$ and is absent for $\mu < -1$. Surprisingly, this is not the case for $\mu > 0$ and "disorder" embodied in Brownian terms B_t turns out to be always relevant, despite the fact that the constant drift term clearly provides a dominant contribution (for sufficiently large μ) in the exponential.

We have performed numerical simulations of the race between two uncorrelated Asian-style random variables which confirm our conclusions on the transition to a disproportionate behavior for any $\mu < 0$. In Fig. 3 we plot $P(\mathcal{W})$ for $\mu = -1.7$ for three different values of the maturity α_T, which clearly shows a transition

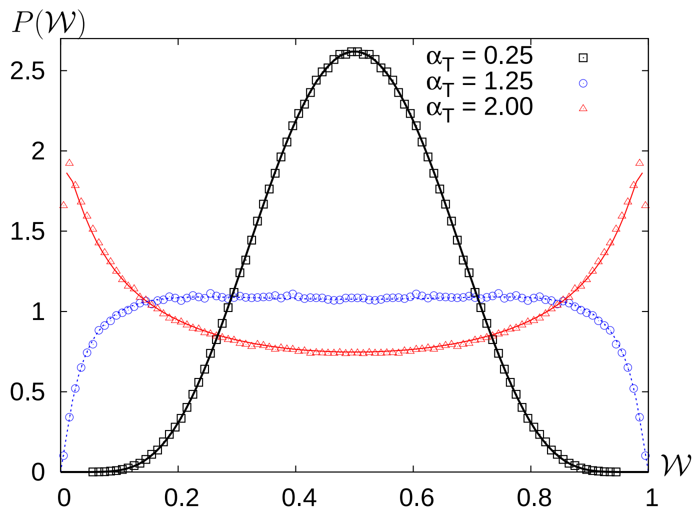

Figure 3: Probability density $P(\mathcal{W})$ obtained from numerical simulations (symbols) of the race between two uncorrelated Asian-style variables for for $\mu = -1.7$ and for two different maturities $\alpha_T = 0.25$ (a bell-shaped curve) and $\alpha_T = 1.5$ (an M-shaped curve). The critical value $\alpha_T^c \simeq 1.12$. The dashed lines correspond to a fit according to Eq. (6) with α_T replaced with a fitting parameter $\tilde{\alpha}_T$.

from a unimodal to an M-shaped form, as the maturity exceeds a critical value $\alpha_T^c \simeq 1.12$. Guided by the above analytic argument (13), we have fitted these distributions using the expression for the European-style variable, $P(W)$ in (6), in which we replaced α_T by some effective maturity $\tilde{\alpha}_T$ used as a fitting parameter. As one may notice, the quality of the fit in Fig. 3 is very good. We also found numerically that $\alpha_T^c \equiv \alpha_T^c(\mu)$ is a slowly decreasing function of μ for $\mu < 0$, which implies that the larger, (by absolute value) μ is, the earlier the transition takes place.

3 Correlated increments.

So far we have concentrated on the case of independent variables, to which one may object claiming that such a transition is spurious and correlations (Laloux et al 1999) between the increments $dB_t^{(1)}$ and $dB_t^{(2)}$ will "stabilize" the behavior of the European- and/or the Asian-style variables. In the remainder, we proceed to show that the transition to the bimodal shape is robust and the only effect of correlations is to shift the transition to later times.

We focus on the case of very strongly correlated increments, when the penalty for having different $dB_t^{(1)}$ and $dB_t^{(2)}$ grows with $u_t = dB_t^{(1)} - dB_t^{(2)}$ as u_t^2. In this case the distribution function of the increments reads:

$$F\left(dB_t^{(1)}, dB_t^{(2)}\right) = \frac{\sqrt{2+\chi^2}}{2\pi\chi dt} \exp\left(-\frac{\left(dB_t^{(1)}\right)^2}{2dt} - \frac{\left(dB_t^{(2)}\right)^2}{2dt}\right) \times \tag{14}$$

$$\times \exp\left(-\frac{\left(dB_t^{(1)} - dB_t^{(2)}\right)^2}{2\chi^2 dt}\right)$$

Physically, it corresponds to the situation of two Brownian particles coupled by a Hookean spring with rigidity $1/\chi$. The parameter χ sets the scale of correlations. When $\chi \to 0$ (an infinite rigidity of the spring), the function F in (14) converges to a delta-function which signifies that the increments are forced to be equal to each other, $dB_t^{(1)} = dB_t^{(2)}$. Conversely, when χ is large, the function F in (14) will tolerate large deviations of $dB_t^{(2)}$ from $dB_t^{(1)}$. For $\chi = \infty$ we recover the limit of independent variables.

For F in (14), the distribution function $P(W)$ of the random variable W can be calculated exactly (see Appendix C):

$$P(W) = \left(\frac{2+\chi^2}{8\pi\chi^2\alpha_T}\right)^{1/2} \frac{1}{W(1-W)} \exp\left(-\frac{2+\chi^2}{8\chi^2\alpha_T} \ln^2\left(\frac{W}{1-W}\right)\right), \tag{15}$$

and appears to have essentially the same form as the result for two *uncorrelated* European-style variables, (6). The only difference is that now α_T is renormalized by χ, so that the transition to the bimodal form occurs at $\alpha_T^c = (2+\chi^2)/2\chi^2 \geq 1/2$. Hence, in presence of correlations the transition to the bimodal form is postponed for later times.

For correlated Asian options, analytic calculations are much more involved and here we again resort to numerical analysis. We have performed numerical simulations of the race between two Asian-style options with correlated increments dB_t, defined in (14). The results of our simulations are summarized in the ”phase diagram” in the $(\mu, 1/\chi)$ plane, Fig. 4. For finite χ, we find that $P(\mathcal{W})$ always undergoes a transition from

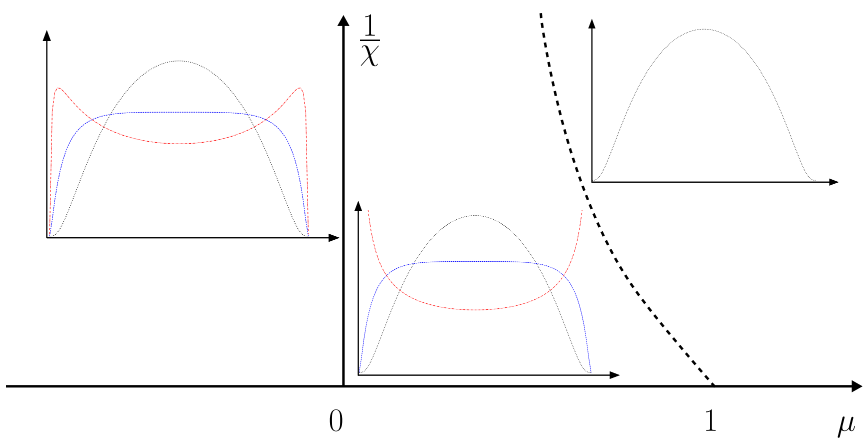

Figure 4: A sketch of the phase diagram in the $(\mu, 1/\chi)$ plane. The dotted line starting at $\mu = 1$ for $1/\chi = 0$ is the critical line $\mu_c(\chi)$. For $\mu > \mu_c$, the distribution $P(\mathcal{W})$ remains unimodal for all maturities. For $\mu < \mu_c$ we depict the shape of $P(\mathcal{W})$ for $\alpha_T < \alpha_T^c$, $\alpha_T \simeq \alpha_T^c$ and $\alpha_T > \alpha_T^c$. For $0 < \mu < \mu_c$, the distribution $P(\mathcal{W})$ evolves from a bell-shaped curve to a U-shaped one, while for $\mu < 0$, $P(\mathcal{W})$ evolves from a bell-shaped curve to an M-shaped one as α_T crosses α_T^c.

a unimodal to a bimodal form as the maturity passes through a critical value α_T^c for all $\mu < \mu_c \equiv \mu_c(\chi)$. As we expected, μ_c is a decreasing function of the strength of correlations: this is shown by the dotted line in Fig. 4. Similarly to the uncorrelated case, one also finds a different behavior for $0 < \mu < \mu_c$ and $\mu < 0$: in the first case, $P(\mathcal{W})$ changes from a bell-shaped to a U-shaped form, while for $\mu < 0$ it changes from a bell-shaped to an M-shaped form. This is depicted in Fig. 4. Finally, for $\mu < \mu_c$, we observe that α_T^c is an increasing function of the strength of correlations, similarly to the case of the European-style variables.

We conclude with several words concerning a common feature of the transition to the disproportionate behavior observed in our paper. As one may notice, such a transition always takes place in situations when the first moment of the distribution function of the underlying S or A diverges as $T \to \infty$, and does not take place if the first moment remains finite at $T = \infty$. It is a bit intriguing to see how the behavior at $T = \infty$ defines the transition which takes place at a finite T.

A Independent European-style variables

Let $E\left\{\exp\left(-\lambda W\right)\right\}$, $\lambda \geq 0$, denote the moment generating function of the random variable W. The curly brackets here and henceforth denote averaging with respect to the distributions of Brownian motions $B_t^{(1)}$ and $B_t^{(2)}$. Explicitly, this function can be written down as

$$E\left\{e^{-\lambda W}\right\} = \frac{1}{2\pi T} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dB_1 \, dB_2 \, \exp\left(-\frac{B_1^2 + B_2^2}{2T} - \lambda \frac{e^{\sigma B_1}}{e^{\sigma B_1} + e^{\sigma B_2}}\right).$$

Integrating over dB_1, we formally change the integration variable $B_1 \to W$, which yields

$$E\left\{e^{-\lambda W}\right\} = \frac{1}{\sqrt{8\pi\alpha_T}} \int_0^1 \frac{dW}{W(1-W)} \exp\left(-\lambda W - \frac{1}{8\alpha_T} \ln^2\left(\frac{W}{1-W}\right)\right),$$

from which one immediately deduces (6).

B Independent Asian-style variables

The moment generating function of $\mathcal{W}$ can be written as

$$E\left\{e^{-\lambda\mathcal{W}}\right\} = \int_0^{\infty} d\tau_1 \int_0^{\infty} d\tau_2 \, \Psi(\tau_1) \, \Psi(\tau_2) \, \exp\left(-\lambda \frac{\tau_1}{\tau_1 + \tau_2}\right).$$

Integrating over $d\tau_2$, we formally change the integration variable from τ_2 to $\mathcal{W}$ to give

$$E\left\{e^{-\lambda\mathcal{W}}\right\} = \int_0^1 d\mathcal{W} \, e^{-\lambda\mathcal{W}} \int_0^{\infty} \tau_1 \, d\tau_1 \, \Psi(\mathcal{W}\tau_1) \, \Psi\left((1-\mathcal{W})\tau_1\right),$$

from which one may read off the result in (7).

$\mu > 0$

We note that in the limit $\mathcal{W} \to 1$, $P(\mathcal{W})$ in (6) has the following asymptotic representation (Oshanin and Redner 2009):

$$P(\mathcal{W}) \sim \frac{\left(1 + \sqrt{\mathcal{W}} \ln z\right)^{-1/2}}{\sqrt{\pi\alpha_T\mathcal{W}} \, (1-\mathcal{W})} \exp\left(-\frac{\ln^2 z}{\alpha_T} + \frac{\pi^2\sqrt{\mathcal{W}} \ln z}{4\alpha_T \left(1 + \sqrt{\mathcal{W}} \ln z\right)}\right), \tag{16}$$

where $z = (1+\sqrt{\mathcal{W}})/\sqrt{(1-\mathcal{W})}$. This asymptotic form agrees quite well with the exact result in (9), not only when $\mathcal{W} \to 1$, but also for moderate values of $\mathcal{W}$ (Fig. 2). The asymptotics of $\mathcal{P}(\mathcal{W})$ for $\mathcal{W} \to 0$ can be obtained by merely changing $\mathcal{W}$ to $1-\mathcal{W}$.

It may be also worthy to remark that in the limit $\mathcal{W} \to 1$ the asymptotic form in (16) follows, apart of a logarithmic factor $\ln^{1/2}(1/(1-\mathcal{W}))$, the asymptotic form of the parental distribution $\Psi(\tau)$ in (8) (Oshanin et al 1993b, Monthus and Comtet 1994, Monthus et al 1998):

$$\Psi(\tau) \sim \frac{1}{2\sqrt{\pi\alpha_T}} \frac{1}{\tau} \exp\left(-\frac{1}{4\alpha_T} \ln^2(\tau)\right), \quad \tau \to \infty$$

For fixed $\mathcal{W}$, the probability density $P(\mathcal{W}) \sim 1/\sqrt{\alpha_T}$ when $\alpha_T \to \infty$, which signifies that the large-T behavior of $P(\mathcal{W})$ in (16) is supported by *negative* moments of τ, which decay as $E(1/\tau^n) \sim 1/\sqrt{\alpha_T}$ regardless of the order n (Oshanin et al 1993b, Monthus and Comtet 1994, Monthus et al 1998).

$\mu < 0$

Whittaker function has the following integral representation:

$$\left|\Gamma\left(-\frac{\mu}{2}+\frac{iu}{2}\right)\right|^2 W_{\frac{1+\mu}{2},\frac{iu}{2}}(y) =$$
$$= 2^{\mu+2}\, y^{(\mu+1)/2} \exp\left(-\frac{y}{2}\right) \int_0^\infty \frac{dx}{x^{\mu+1}} \exp\left(-\frac{x^2}{4y}\right) \mathrm{K}_{iu}(x),$$

where $\mathrm{K}_{iu}(x)$ is the modified Bessel function (Abramowitz and Stegun 1972). Using the latter equation and a standard integral representation of $\mathrm{K}_{iu}(x)$,

$$\mathrm{K}_{iu}(x) = \int_0^\infty dt\, \cos(ut) \exp\left(-x\cosh(t)\right),$$

we find

$$\Psi(\tau) = \frac{2^{\mu-1}\sigma^2 \exp\left(-\alpha_T\mu^2/4\right)}{\pi^2} \frac{\exp(-1/\tau')}{\tau'^{\mu+1}} \psi(\alpha_T, \tau'), \tag{17}$$

where $\tau' = \sigma^2\tau/2$ and $\psi(\alpha_T, \tau)$ is given by

$$\psi(\alpha_T, \tau) = \int_0^\infty \frac{dx}{x^{1+\mu}} \exp\left(-\frac{\sigma^2\tau x^2}{8}\right) \theta\left(x, \frac{\alpha_T}{2}\right), \tag{18}$$

with

$$\theta\left(x, \frac{\alpha_T}{2}\right) = \frac{2e^{\frac{\pi^2}{\alpha_T}}}{(\pi\alpha_T)^{\frac{3}{2}}} \int_0^\infty d\xi \exp\left(-x\cosh(\xi) - \frac{\xi^2}{\alpha_T}\right) \times$$
$$\times \left(\pi\cos\left(\frac{2\pi\xi}{\alpha_T}\right) - \xi\sin\left(\frac{2\pi\xi}{\alpha_T}\right)\right). \tag{19}$$

We now focus on $\psi(\alpha_T, \tau)$ seeking for a plausible approximation in the limit $\alpha_T \gg 1$. Note first that $\psi(\alpha_T, \tau)$ is a monotonically decreasing function of τ and

$$\psi(\alpha_T, \tau = 0) = \frac{\Gamma^2(-\mu/2)}{\sqrt{\pi}2^{1+\mu}\alpha_T^{3/2}} + o\left(\alpha_T^{-3/2}\right). \tag{20}$$

In the large τ limit, the integral in (18) is dominated by the small x behavior of the integrand. To obtain the small x behavior of $\theta\left(x, \frac{\alpha_T}{2}\right)$ in the large α_T limit, we take $x = 1/(2\sinh(y\sqrt{\alpha_T}))$ and also change the integration variable $\xi = u\sqrt{\alpha_T}$ in (19). Setting then $\alpha_T \to \infty$ while keeping y fixed, we have

$$\theta\left(x, \frac{\alpha_T}{2}\right) \simeq \frac{2}{\sqrt{\pi}\alpha_T} y e^{-y^2} \equiv \tag{21}$$

$$\equiv \frac{2}{\sqrt{\pi}\alpha_T^{3/2}} \operatorname{arcsinh}\left(\frac{1}{2x}\right) \exp\left(-\frac{1}{\alpha_T}\operatorname{arcsinh}^2\left(\frac{1}{2x}\right)\right),$$

which defines the small x asymptotic behavior of $\theta(x, \alpha_T/2)$.

On the other hand, for large x

$$\theta\left(x, \frac{\alpha_T}{2}\right) \sim 2K_0(x)/\sqrt{\pi}\alpha^{3/2}.$$

Hence, we may approximate $\theta\left(x, \frac{\alpha_T}{2}\right)$, for sufficiently large α_T, as

$$\theta\left(x, \frac{\alpha_T}{2}\right) \simeq \frac{2}{\sqrt{\pi}\alpha_T^{3/2}} K_0(x) \exp\left(-\frac{1}{\alpha_T}\operatorname{arcsinh}^2\left(\frac{1}{2x}\right)\right). \tag{22}$$

Note that this approximate form reproduces correctly the exact behavior of $\theta\left(x, \frac{\alpha_T}{2}\right)$ both for $x \to 0$ (21) and $x \to \infty$.

We use next the small x asymptotic behavior of $\theta\left(x, \frac{\alpha_T}{2}\right)$, (21), to obtain from (18) the following large-τ asymptotic of $\psi(\alpha_T, \tau)$:

$$\psi(\alpha_T, \tau) \simeq \frac{\Gamma(-\mu/2)}{\sqrt{\pi}\alpha_T^{3/2} 2^{1+\mu}} \tau'^{\mu/2} \log(\tau') \exp\left(-\frac{\log^2 \tau'}{4\alpha_T}\right). \tag{23}$$

The small-τ asymptotic behavior of $\psi(\alpha_T, \tau)$ can be deduced from the large-x asymptotic of $\theta\left(x, \frac{\alpha_T}{2}\right)$, which yields

$$\psi(\alpha_T, \tau) \simeq \frac{\Gamma^2(-\mu/2)}{2^{\mu+1}\sqrt{\pi}\alpha_T^{3/2}} \tau'^{\mu/2} U\left(-\frac{\mu}{2}, 1, \frac{2}{\sigma^2\tau}\right), \tag{24}$$

where $U(a, b, z)$ is the confluent hypergeometric function (Abramowitz and Stegun 1972).

Combining the estimates in (23) and (24), we obtain the approximate form for $\Psi(\tau)$ in (12) with

$$C = \frac{\sigma^2\Gamma^2(-\mu/2)}{4\pi^{5/2}\alpha_T^{3/2}} \exp\left(-\frac{\alpha_T\mu^2}{4}\right).$$

In Fig. 5, we compare our approximate expression for $\Psi(\tau)$ in (12) against the numerical evaluations of the integrals in (17) and (18) for $\mu = -1$ and different values of $\alpha_T = 20, 30, 50$. One notices that our approximation is quite accurate for all $\tau > 0$. As expected, the approximation works better as α_T increases.

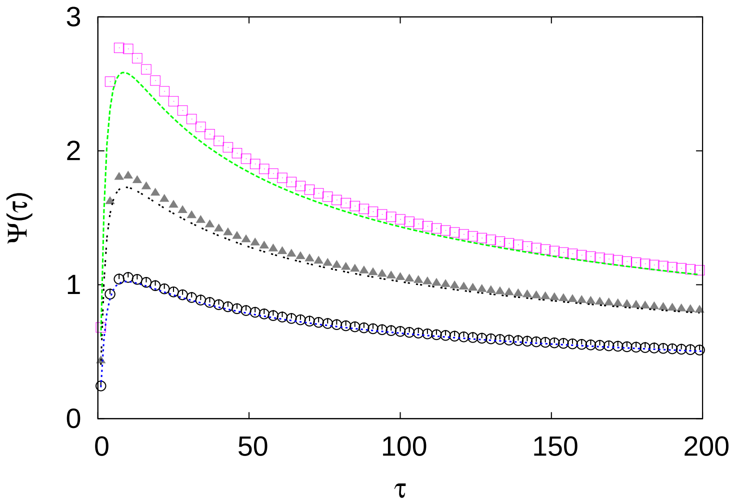

Figure 5: Probability density $\Psi(\tau)$ for $\mu = -1$ and $\alpha_T = 20, 30, 50$ (from top to bottom). Each pair of curves have been rescaled by an appropriate factor so that they can be shown on a same graph. The symbols correspond to the exact value of $\Psi(\tau)$, obtained by a numerical evaluation of the integrals in (17) and (18), while the solid lines correspond to the approximation given in (12).

C Correlated increments

We first divide the interval $[0, T]$ into N subintervals dt approximating the Brownian trajectories $B_T^{(1)}$ and $B_T^{(2)}$ by the trajectories of random walks in discrete time (with time-step dt) k:

$$B_T^{(n)} = \sum_{k=1}^{N} dB_k^{(n)},\ n = 1, 2,$$

where $dB_k^{(n)}$ are the values of the increments at time moment k. In what follows, we will be interested by the behavior in the limit $N \to \infty, dt \to 0$ with $Ndt = T$ kept fixed.

Then, a random variable W, which defines a contribution of a given European-style option into the sum of two such options, can be written formally as

$$W = \left(1 + \prod_{k=1}^{N} \exp\left(\sigma(dB_k^{(1)} - dB_k^{(2)})\right)\right)^{-1}.$$

Then, for the generating function of this random variable we have

$$E\left\{e^{-\lambda W}\right\} = \left(\frac{\sqrt{2+\chi^2}}{2\pi\chi dt}\right)^N \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{k=1}^{N} dB_k^{(1)} dB_k^{(2)} \times$$

$$\times \exp\left(-\frac{\left(dB_k^{(1)}\right)^2}{2dt} - \frac{\left(dB_k^{(2)}\right)^2}{2dt} - \frac{\left(dB_k^{(2)} - dB_k^{(2)}\right)^2}{2\chi^2 dt}\right) \times$$

$$\times \exp\left(-\frac{\lambda}{1+\prod_{k=1}^{N}\exp\left(\sigma\left(dB_k^{(2)}-dB_k^{(2)}\right)\right)}\right)=$$

$$=\left(\frac{\sqrt{2+\chi^2}}{2\pi\chi dt}\right)^N\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod_{k=1}^{N}dB_k^{(1)}du_k\times$$

$$\times \exp\left(-\frac{\left(dB_k^{(1)}\right)^2}{dt}-\frac{dB_k^{(1)}}{dt}u_k-\frac{(1+\chi^2)u_k^2}{2\chi^2 dt}\right)\times$$

$$\times \exp\left(-\frac{\lambda}{1+\prod_{k=1}^{N}\exp\left(\sigma u_k\right)}\right),$$

which reduces, upon the integration over $dB_k^{(1)}$, to

$$E\left\{e^{-\lambda W}\right\}=\frac{1}{\left(2\pi g^2 dt\right)^{N/2}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod_{k=1}^{N}du_k\times$$

$$\times \exp\left(-\frac{u_k^2}{2g^2 dt}\right)\exp\left(-\frac{\lambda}{1+\prod_{k=1}^{N}\exp\left(\sigma u_k\right)}\right),$$

where $g^2=2\chi^2/(2+\chi^2)$.

Further on, performing the integration over du_N we formally change the integration variable $u_N \to W$, which yields

$$E\left\{e^{-\lambda W}\right\}=\frac{1}{\sigma\left(2\pi g^2 dt\right)^{N/2}}\int_0^1\frac{dW}{W(1-W)}e^{-\lambda W}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod_{k=1}^{N-1}du_k\times$$

$$\times \exp\left(-\frac{u_k^2}{2g^2 dt}\right)\exp\left(-\frac{1}{2g^2 dt}\left(\frac{1}{\sigma}\ln\left(\frac{1-W}{W}\right)-\sum_{k=1}^{N-1}u_k\right)^2\right).$$

The latter equation implies that the distribution function $P(W)$ in case of two European-style variables with correlated increments is given by

$$P(W)=\frac{1}{\sigma\left(2\pi g^2 dt\right)^{N/2}}\frac{1}{W(1-W)}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod_{k=1}^{N-1}du_k\,\exp\left(-\frac{u_k^2}{2g^2 dt}\right)\times$$

$$\times \exp\left(-\frac{1}{2g^2 dt}\left(\frac{1}{\sigma}\ln\left(\frac{1-W}{W}\right)-\sum_{k=1}^{N-1}u_k\right)^2\right)$$

Finally, by taking the advantage of the integral identity

$$\frac{\exp\left(-u^2/2A\right)}{\sqrt{2\pi A}}=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\,\exp\left(i\,\omega\,u-A\omega^2/2\right)$$

we readily perform the $N-1$-fold integral in the latter equation to get the result in (15).

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