

Identification and well-posedness in a class of nonparametric problems

Victoria Zinde-Walsh*

McGill University and CIREQ

victoria.zinde-walsh@mcgill.ca

(514) 398 4834

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Abstract

This is a companion note to Zinde-Walsh (2010) to clarify and extend results on identification in a number of problems that lead to a system of convolution equations. Examples include identification of the distribution of mismeasured variables, of a nonparametric regression function under Berkson type measurement error, some nonparametric panel data models, etc. The reason that identification in different problems can be considered in one approach is that they lead to the same system of convolution equations; moreover the solution can be given under more general assumptions than those usually considered, by examining these equations in spaces of generalized functions. An important issue that did not receive sufficient attention is that of well-posedness. This note gives conditions under which well-posedness obtains, an example that demonstrates that when well-posedness does not hold functions that are far apart can give rise to observable arbitrarily close functions and discusses misspecification and estimation from the stand-point of well-posedness.

1 Introduction

The results of this paper apply to a number of econometric problems, including the examples below.

Example 1. The distribution of a mismeasured variable with another

observation.

See, e.g., reviews of Carroll, Rupert and Stefanski (1995); Chen, Hong and Nekipelov (2009); the problem is examined in Cunha, Heckman and Schennach (2010).

Suppose that g is the density of a mismeasured variable, x^* , z is observed and has density w_1 ; $z = x^* + u$, where u is measurement/contamination error independent of x^* with a density, f . Another observation, x , on x^* is available: $x = x^* + u_x$, where u_x is not necessarily independent but $E(u_x|x^*, u) = 0$.

$$x = x^* + u_x; \tag{1}$$

$$z = x^* + u. \tag{2}$$

Example 2. Errors in variables regression (EIV) model with Berkson type measurement error.

Review Chen, Hong and Nekipelov (2009); examined by Newey (2001), univariate case in Schennach (2007) and Zinde-Walsh (2009), multivariate Zinde-Walsh (2010).

Consider

$$y = g(x^*) + u_y; \tag{3}$$

$$x = x^* + u_x; \tag{4}$$

$$z = x^* + u. \tag{5}$$

Here (3)-(5) provide a regression with z representing a second measurement or possibly a given projection onto a set of instruments for the unobserved x^* . Here y, z or x, y, z are observed; u is a Berkson type measurement error independent of z ; u_y, u_x have zero conditional (on z and the other errors) expectations. Denote $w_1 = E(y|z)$, density of measurement error f .

Example 3. Panel data model with two periods.

Evdokimov (2010).

Here let x (or z) represent the observed variable in the first period, and z (x) for the second, x^* is the nonparametric function $m(X, \alpha)$, where α is the idiosyncratic component and the densities are conditional on the same value X for the two periods; the same distributional assumptions as in Example 1 are used.

The models lead to the same system of convolution equations. All vectors are in R^d .

By independence in all cases we get

$$g * f = w_1.$$

For examples 1 and 3 define density of z by f_z ; by x_k the k th component of the vector x and consider

$$E(f_z x_k | z) = E(x_k^* f_z | z) = \int (z - u)_k g(z - u) f(u) du = x_k g * f.$$

Denote the observable $E(f_z x_k | z)$ by w_{2k} , $k = 1, \dots, d$.

For example 2

$$E(x_k y | z) = E(x_k^* g(x^*) | z) = \int (z_k - u_k) g(z - u) f(u) du.$$

Denote here $E(x_k y | z)$ by w_{2k} , $k = 1, \dots, d$.

Thus for all the examples we need to solve the system of convolution equations

$$\begin{aligned} g * f &= w_1; \\ x_k g * f &= w_{2k}, k = 1, \dots, d. \end{aligned} \tag{6}$$

It is advantageous to consider the functions as generalized functions. The interest is often in distributions of the unobservables and there is no reason to restrict those to be absolutely continuous; density may not necessarily exist but can be represented as a generalized derivative of the distribution function rather than an ordinary function. Since solving the convolution equations is done via Fourier transforms restricting regression functions in Example 2 to have ordinary Fourier transforms excludes binary choice or polynomial regression and can be overcome by using generalized functions. Also, if some variables have singular distributions, or if only some variables are subject to measurement error, or there is a mass point in the error distribution (e.g. in measurement error from surveys with some portion of true responses)

convoluting with a generalized δ -function is natural when considering the problems in spaces of generalized functions.

The spaces of generalized functions most relevant for solving these problems are the space S' (tempered distributions); space D' and some related spaces also play a role in the proofs. See e.g. Zinde-Walsh, 2010 for the definitions, discussion and summary of useful properties.

The next section 2 presents the full solution to system of equations (6) extending all the results in the current literature.

Section 3 discusses well-posedness. This is to clarify two issues: under what conditions consistent estimation of the identified general model is possible and in what sense does a possibly mis-specified parametric model deliver valid analysis. The answer hinges on well-posedness of the identification of the function g . Well-posedness refers to g depending on the distributions of the observed variables in a continuous fashion. Well-posedness does not hold if both the function g and the density f are supersmooth (that is their Fourier transforms decline exponentially); on the other hand if any one of the two is such that the Fourier transform is continuously differentiable and its inverse is a regular generalized function (grows no faster than some power), then well-posedness in the weak topology of generalized functions obtains; for well-posedness in stronger topologies additional conditions need to be provided. An example shows that a Gaussian density for both functions would lead to violation of well-posedness. Classes of nonparametric models that include the Gaussian and that lead to a well-posed problem are defined.

Further, the issue of regularization is discussed.

2 The identification result

Assumption 1. The generalized functions g, f, w_1 and $w_{2k}, k = 1, \dots, d$, are in the generalized function space S' and are related by (6).

Any generalized density functions are generalized derivatives of the distribution function and belong to S' , convolution equations are defined. For a ordinary function, b , e.g. a regression function of example 2 to belong to S' it is sufficient that it belong to some class of functions on $R^d, \Phi(m, V)$ (with m a vector of integers, V a positive constant) where $b \in \Phi(m, V)$ if

$$\int \Pi ((1 + t_i^2)^{-1})^{m_i} |b(t)| dt < V < \infty. \quad (7)$$

Thus if e.g. b grows no faster than a polynomial, it is in S' , so that the analysis here applies to binary choice and polynomial regression. Convolutions with generalized functions from some classes are defined for such functions (as discussed in Zinde-Walsh, 2010). For conditional density of Example 3 some extra assumptions on the joint density of the regressors are required.

Consider now Fourier transforms (Ft) : $\gamma = Ft(g); \phi = Ft(f); \varepsilon. = Ft(w)$.

Assumption 2. Either ϕ or γ is a continuous function such that it satisfies (7).

The continuity assumption on the characteristic function is typically made; any characteristic function satisfies(7).

By Theorem 1 of Zinde-Walsh 2010 then the following system of equations holds in S' .

$$\begin{aligned}\gamma \cdot \phi &= \varepsilon_1; \\ \gamma'_k \cdot \phi &= \varepsilon_{2k}, \quad k = 1, \dots, d.\end{aligned}\tag{8}$$

Assumption 3. $\text{supp}(\phi) \supseteq \text{supp}(\gamma) = W$, where W is a convex set in R^d that includes an interior point 0.

The support assumption is necessary to solve for γ . The interior point in the case of characteristic functions is zero with the value of the continuous characteristic function equal to 1 at that point. If the system of equations involves functions with W having an interior point $a \neq 0$ consider shifted functions.

Theorem 1 *Under Assumptions 1-3 if*

(a) γ is continuously differentiable in W , $\gamma(0) = c$

$$\gamma(\zeta) = c \exp \int_0^\zeta \sum_{k=1}^d \alpha_k(\xi) d\xi_k,\tag{9}$$

with the uniquely defined continuous functions $\varkappa_k(\xi)$ that solve

$$\varkappa_k(\xi)\varepsilon_1 - i\varepsilon_{2k} = 0, k = 1, \dots, d;$$

or

(b) ϕ is continuously differentiable in W , $\phi(0) = c$

$$\gamma(\zeta) = \tilde{\phi}(\zeta)^{-1}\varepsilon_i(\zeta), \tag{10}$$

where

$$\tilde{\phi}(\zeta) = \exp \int_0^\zeta \sum_{k=1}^d \tilde{\varkappa}_k(\xi) d\xi_k,$$

with the uniquely defined continuous functions $\varkappa_k(\xi)$ that solve

$$\varepsilon_1 \tilde{\varkappa}_k - ((\varepsilon_1)'_k - i\varepsilon_{2k}) = 0, k = 1, \dots, d.$$

The proof is in proof of Theorem 3 and the Corollary of Zinde-Walsh, 2010. If support of ϕ coincides with support of γ , ϕ (and thus the function f) is identified.

The proof is set in the space S' of generalized functions and does not rely on existence of densities. The proof of (b) in the univariate case was first provided in Zinde-Walsh 2009, correcting the result of Schennach 2007. The formula for the density in Cunha et al. 2010 is valid in case (a) and can be interpreted in terms of generalized functions ("distributions"). In case

(b), though, a different solution is given here. Thus identification requires differentiability of either γ or ϕ ; when γ is not differentiable and the result in Cunha et al does not hold identification is still possible in case (b). This also extends the identification result of Evdokimov 2010.

3 Well-posedness

We now consider whether when the distributions of the observables are close the unknown functions are also necessarily close.

A sufficient condition is provided in Zinde-Walsh 2010 (Theorem 4). When identification is based on (b) of Theorem 1 here the model class needs to be restricted to include only measurement error distributions with ϕ^{-1} in $\Phi(m, V)$ for some m, V . Equivalently, when identification is based on (a), the sufficient condition is for the class of models to be restricted to those where the latent factor distribution is such that $\gamma^{-1} \in \Phi(m, V)$.

These conditions exclude models where both g and f are supersmooth with $\text{supp}(\gamma)$ unbounded leading to a supersmooth distribution for w_1 . Although these conditions are only shown to be sufficient, an example below (from Zinde-Walsh 2009 and 2010) demonstrates that a Gaussian distribution (that violates these conditions) fails well-posedness in the weak topology of generalized functions in S' and therefore in any stronger topology or metric (uniform, L_1 , etc.).

Example 4. Consider the function $\phi(x) = e^{-x^2}$, $x \in R$. Consider in S

the function $b_n(x) =$

$$\begin{cases} e^{-n} & \text{if } n - \frac{1}{n} < x < n + \frac{1}{n}; \\ 0 < b_n(x) \leq e^{-n} & \text{if } n - \frac{2}{n} < x < n + \frac{2}{n}; \\ 0 & \text{otherwise.} \end{cases} \quad (11)$$

This $b_n(x)$ converges to $b(x) \equiv 0$ in S' . Indeed for any $\psi \in S$

$$\int_{-\infty}^{\infty} b_n(x)\psi(x)dx = \int_{n-2/n}^{n+2/n} b_n(x)\psi(x)dx \rightarrow 0.$$

Now consider $\varepsilon_n = \varepsilon + b_n \rightarrow \varepsilon$. We show that $\varepsilon_n\phi^{-1}$ does not converge in S' to $\varepsilon\phi^{-1}$. Such convergence would imply that $(\varepsilon_n - \varepsilon)\phi^{-1} = b_n\phi^{-1} \rightarrow 0$ in S' .

But the sequence $b_n(x)\phi(x)^{-1}$ does not converge. Indeed if it did then $\int b_n(x)\phi^{-1}(x)\psi(x)dx$ would converge for any $\psi \in S$. But for $\psi \in S$ such that $\psi(x) = \exp(-|x|)$

$$\begin{aligned} \int b_n(x)e^{x^2}\psi(x)dx &\geq \int_{n-2/n}^{n+2/n} b_n(x)e^{x^2}\psi(x)dx \geq e^{-n} \int_{n-1/n}^{n+1/n} e^{x^2-x}dx \\ &\geq \frac{2}{n}e^{-2n+(n-1/n)^2}. \end{aligned}$$

This diverges. ■

Thus, e.g. for the Gaussian distribution there are models with unknown functions that are far from each other in S' , but that lead to observable functions that are arbitrarily close.

When the nonparametric identification result is interpreted to support possible wider applicability, when estimation is in fact based on a parametric model the question arises as to which nonparametric models are close to a model misspecified as parametric. This question may be posed e.g. for the analysis of Cunha et al 2010 who use Gaussian and mixed Gaussian distributions in estimation. Is there some meaningful nonparametric class that includes the Gaussian where observationally close models imply closeness of latent factors?

Define a class of generalized functions $\Phi(B, \Lambda, m, V) \subset S'$ for some positive constant B and matrix Λ ; a generalized function $b \in \Phi(B, \Lambda, m, V)$ if there exists a function $\bar{b}(\zeta) \in \Phi(m, V)$ with support in $\|\zeta\| > B$ such that also $\bar{b}(\zeta)^{-1} \in \Phi(m, V)$ and $b \cdot I(\|\zeta\| > B) = \bar{b}(\zeta) \exp(-\zeta' \Lambda \zeta)$. Note that a linear combination of functions in $\Phi(B, \Lambda, m, V)$ belongs to the same class. For a sequence of $b_n \in \Phi(B, \Lambda, m, V)$ to converge to zero as generalized functions it is necessary that the corresponding \bar{b}_n converge to zero (a.e.).

Assumption 4. $\gamma \in \Phi(B, \Lambda_\gamma, m, V)$; $\phi \in \Phi(B, \Lambda_\phi, m, V)$.

If this assumption holds $\varepsilon_1 = \gamma \cdot \phi \in \Phi(B, \Lambda_\gamma + \Lambda_\phi, 2m, V^2)$ and $\varepsilon_2 = \gamma'_k \cdot \phi$ also is in $\Phi(B, \Lambda_\gamma + \Lambda_\phi, 2m, V^2)$.

Theorem 2 *Under conditions of Theorem 1 and the Assumption 4 applying to generalized functions $\varepsilon_{i,n}$, $i = 1, 2$ if $\varepsilon_{i,n} \rightarrow \varepsilon_i$ in S' , then the corresponding solutions γ_n given by (9) or (10) converge to γ in S' .*

Proof. From the conditions of the theorem $\eta_{i,n} = \varepsilon_{i,n} - \varepsilon_i$ converges in S'

to zero. From the nature of the identified solution (9) or (10) it follows that if it can be shown that $\varepsilon_1^{-1}\eta_{.,n}$ converges to zero then the problem for the distribution of the latent factor is well-posed. But this is indeed the case since the exponents cancel, $\bar{\eta}_{.,n}$ converges to zero in S' and convergence to zero follows by hypocontinuity of the product $\bar{\varepsilon}_1^{-1}\bar{\eta}_{.,n}$. ■

The values for the tail exponent have to be fixed implying that even a slight deviation in the exponent violates the well-posedness condition: there are no two different Gaussian distributions in the class. Since an estimated Gaussian will differ from the true Gaussian they cannot belong to the same non-parametric class thus there is separation between estimation in the parametric Gaussian problem and in the fully general nonparametric specification in S' .

Consider a solution regularized with a weighting function. It is high frequency components that cannot be identified in convolution with a super smooth function and regularization smooths those out. Fix a function $\psi \in D$. Using this weight on the Fourier transform is equivalent to solving convolution equations

$$\begin{aligned} g * f * (Ft^{-1}(\psi)) &= w_1 * (Ft^{-1}(\psi)); & (12) \\ x_k g * f * (Ft^{-1}(\psi)) &= w_{2k} * (Ft^{-1}(\psi)), k = 1, \dots, d. \end{aligned}$$

As in the proof of Theorem 3 of Zinde-Walsh 2010 for any such ψ the solution exists because multiplication by a continuous function γ^{-1} or ϕ^{-1}

with arbitrary growth at infinity is permitted since support of ψ is bounded.

Schwartz (1964, pp.271-273) gives a characterization of functions in S' with Fourier transform that has bounded support (in a cube $|x_k| < C, k = 1, \dots, d$) based on Wiener-Paley theorem. Such a function is a continuous function g that can be extended to a entire analytic function G of a complex argument and is of exponential type $\leq 2\pi C$, meaning

$$\limsup_{|z| \rightarrow \infty} \frac{\log |G(z)|}{|z_1| + \dots + |z_d|} \leq 2\pi C.$$

Thus as long as g is such a function it can be expressed via the regularized solution. As in Schwartz the subspace of all functions of exponential type (for any finite C) can also be considered. However, the regularized solutions may not come close to a true g that does not belong to this subspace.

References

- [1] Carroll, R. J., Ruppert, D., and Stefanski, L. A. (1995). Measurement Error in Nonlinear Models. Chapman & Hall.
- [2] Chen, X., H. Hong and D.Nekipelov (2009), Nonlinear models of measurement errors, Journal of Economic Literature, under review.
- [3] Cunha, F., J.J. Heckman and S.M.Schennach (2010), Estimating the Technology of Cognitive and Noncognitive Skill Formation, Econometrica, v.78, 883-933.

- [4] Evdokimov, K. (2010), Identification and Estimation of a Nonparametric Panel Data Model with Unobserved Heterogeneity, working paper.
- [5] Hadamard, J. (1923): Lectures on Cauchy's Problem in Linear Partial Differential Equations. Yale University Press, New Heaven.
- [6] Newey, W. (2001), Flexible Simulated Moment Estimation of Nonlinear Errors-in-Variables Models, Review of Economics and Statistics, v. 83, 616-627.
- [7] Schennach, S. (2007) Instrumental variable estimation in nonlinear errors-in-variables models, Econometrica, v.75, pp. 201-239.
- [8] Schwartz, L. (1966) "Théorie des distributions", Hermann, Paris.
- [9] Zinde-Walsh, V. (2009), Errors-in-variables models: a generalized functions approach, working paper, arXiv:0909.5390v1 [stat.ME], McGill University working paper.
- [10] Zinde-Walsh, V. (2010), Measurement error and deconvolution in generalized functions spaces, working paper, arXiv:1009.4217v1[MATH.ST].