SHORTFALL RISK APPROXIMATIONS FOR AMERICAN OPTIONS IN THE MULTIDIMENSIONAL BLACK–SCHOLES MODEL

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Abstract. We show that shortfall risks of American options in a sequence of multinomial approximations of the multidimensional Black–Scholes (BS) market converge to the corresponding quantities for similar American options in the multidimensional BS market with path dependent payoffs. In comparison to previous papers we consider the multi assets case for which we use the weak convergence approach.

1. Introduction

This paper deals with multinomial approximations of the shortfall risk for American options in the multidimensional BS (complete) model. It is well known that in a complete market an American contingent claim can be hedged perfectly with an initial capital which is equal to the optimal stopping value of the discounted payoff under the unique martingale measure. In real market conditions an investor (seller) may not be willing for various reasons to tie in a hedging portfolio the full initial capital required for a perfect hedge. In this case the seller is ready to accept a risk that his portfolio value at an exercise time may be less than his obligation to pay and he will need additional funds to fullfil the contract. Thus a portfolio shortfall comes into the picture.

We deal with a certain type of risk called the shortfall risk which is defined as the maximal expectation (with respect to the buyer exercise times) of the discounted shortfall (see [12]). An investor whose initial capital is less than the option price still want to compute the minimal possible shortfall risk and to find a portfolio strategy which minimizes or "almost" minimizes the shortfall risk. In this paper we allow only *admissible* self financing portfolios, i.e. a portfolios with nonnegative wealth process. This corresponds to the situation when the portfolio is handled without borrowing of the capital.

For discrete time markets such as the multinomial models the above problems can be solved by dynamical programming algorithm. For continuous time models such as the BS model these problems are much more complicated.

We prove that for American options, the shortfall risk in the multidimensional BS model can be approximated by a sequence of shortfall risks in an appropriate

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multinomial models. This type of results has a practical value since the shortfall risks in the multinomial models can be calculated via dynamical programming algorithm. Our main tools are the extended weak convergence theory that was developed in [1] and the tightness theorems that were obtained in [13]. Since we use the weak convergence approach we could not provide error estimates of the above approximations. Thus, to open problems remains open. The first one is to obtain error estimates of the above approximations. The second one is to find explicit formulas for optimal or "almost" optimal hedges in the BS model. It seems that both of the above problems require new tools.

So far, shortfall risk approximations were studied only in the one dimensional BS model (see [6], [7]). For this case it was proved that the shortfall risk in a BS market is a limit of the shortfall risks in an appropriate sequence of CRR markets. Furthermore, the authors obtained error estimates and dynamical programming algorithm for "almost" optimal hedges. The main tool that was used in the above papers is Skorohod embedding tool of i.i.d. random variables into the one dimensional Brownian motion. This tool can not be applied for the multidimensional Brownian motion.

Main results of this paper are formulated in the next section. In Section 3 we derive auxiliary lemmas that will be essential in the proof of the main results. In Section 4 we complete the proof of main results of the paper. In Section 5 we analyze the multinomial models and provide a dynamical programming algorithm for the shortfall risk and the corresponding optimal portfolios.

2. Preliminaries and main results

First we introduce the multidimensional BS market. Consider a complete probability space (Ω_W, P^W) together with a standard d–dimensional continuous in time Brownian motion $\{W(t) = (W_1(t), ..., W_d(t))\}_{t=0}^{\infty}$, and the filtration $\mathcal{F}_t^W = \sigma\{W(s)|s \leq t\}$. We assume that the σ–algebras contain the null sets. A BS financial market consists of a savings account $B(t)$ with an interest rate r, assuming without loss of generality that $r = 0$, i.e.

$$B(t) = B(0) > 0 \tag{2.1}$$

and of d risky stocks $S^W = (S_1^W, ..., S_d^W)$ given by the following equation

$$S_i^W(t) = S_i(0)\exp(\sum_{j=1}^{d} \sigma_{ij} W_j(t) + (b_i - \frac{1}{2}\sum_{j=1}^{d} \sigma_{ij}^2)t), \ \ S_i(0) > 0 \tag{2.2}$$

where $b \in \mathbb{R}^d$ is a constant vector and $\sigma \in M_d(\mathbb{R})$ is a constant nonsingular matrix.

Let $T < \infty$ be the maturity date of our American option and let $\mathcal{T}_{[0,T]}^W$ be the set of all stopping times with respect to $\mathcal{F}^W$ which take values in $[0, T]$. Denote by $(\mathbb{D}([0, T]; \mathbb{R}^d), \mathcal{S})$ the space of all right continuous functions with left hand limits, equipped with the Skorohod topology (see [2]). Let $F : [0, T] \times (\mathbb{D}([0, T]; \mathbb{R}^d), \mathcal{S}) \to \mathbb{R}_+$ be a measurable functions such that there exists a constant $C > 0$ which satisfies

$$\sup_{0\leq t\leq T} F(t, x) \leq C \sup_{0\leq t\leq T} |x(t)|, \quad \forall x \in \mathbb{D}([0, T]; \mathbb{R}^d). \tag{2.3}$$

Furthermore, we assume that for any $t \in [0, T]$ and $x, y \in \mathbb{D}([0, T]; \mathbb{R}^d)$:
i. $F(\cdot, x)$ is a right continuous function with left hand limits.
ii. $F(t, x) = F(t, y)$ if $x(s) = y(s)$ for any $s \leq t$.

iii. If x is continuous at t then F is continuous at (x,t) (with respect to the product topology).

Next, consider an American option with the payoff process given by

$$Y^W(t) = F(t, S^W), \ 0 \le t \le T. \tag{2.4}$$

From the assumptions above it follows that $\{Y^W(t)\}_{t=0}^T$ is a *càdlàg* adapted stochastic process and $E^W[\sup_{0\le t\le T} Y^W(t)] < \infty$. Denote by $\tilde{P}^W$ the unique martingale measure for the above model. Using standard arguments it follows that the restriction of the probability measure $\tilde{P}^W$ to the σ–algebra $\mathcal{F}_t^W$ satisfies

$$M(t) = \frac{d\tilde{P}^W}{dP^W}|\mathcal{F}_t^W = \exp(-\frac{1}{2}||\theta||^2 t - \langle\theta, W(t)\rangle) \tag{2.5}$$

where $\theta = b\sigma^*$. We denote by $||\cdot||$ and $\langle\cdot,\cdot\rangle$ the standard norm and the scalar product of $\mathbb{R}^d$, respectively.

A self financing strategy π with a horizon T and an initial capital x (see [15]) is a d–dimensional progressively measurable process $\pi = \{\gamma(t)\}_{t=0}^T$ which satisfies

$$\int_0^T \langle\gamma(t), S^W(t)\rangle^2 dt < \infty \ \text{ a.s.} \tag{2.6}$$

For a strategy π the portfolio value process $\{V^\pi(t)\}_{t=0}^T$ is given by

$$V^\pi(t) = x + \int_0^t \langle\gamma(u), dS^W(u)\rangle. \tag{2.7}$$

Recall, (see [11]) that stochastic integrals with respect to the Brownian motion has a continuous modification and so for any self financing strategy π the corresponding portfolio value process is a continuous one.

A self financing strategy π is called *admissible* if $V^\pi(t) \ge 0$ for all $t \in [0,T]$ and the set of such strategies with an initial capital no bigger than x will be denoted by $\mathcal{A}^W(x)$. We set $A^W = \bigcup_{x>0} A^W(x)$. For an *admissible* self financing strategy π the shortfall risk is given by (see [12]),

$$R(\pi) = \sup_{\tau\in\mathcal{T}_{[0,T]}^W} E^W[(Y^W(\tau) - V^\pi(\tau))^+], \tag{2.8}$$

which is the maximal possible expectation with respect to the probability measure P^W of the (discounted) shortfall. The shortfall risk for an initial capital x is given by

$$R(x) = \inf_{\pi\in\mathcal{A}^W(x)} R(\pi). \tag{2.9}$$

Next, we introduce the sequence of multinomial markets that we use in order to approximate the shortfall risk in the BS model. The same markets were used in [8] in order to approximate European option prices in the d– dimensional BS model. Let $A \in M_{d+1}(\mathbb{R})$ be an orthogonal matrix such that it last column equals to $(\frac{1}{\sqrt{d+1}}, ..., \frac{1}{\sqrt{d+1}})^*$. Let $\Omega_\xi = \{1, 2, ..., d+1\}^\infty$ be the space of finite sequences $\omega = (\omega_1, \omega_2, ...)$; $\omega_i \in \{1, 2, ..., d+1\}$ with the product probability $P^\xi = \{\frac{1}{d+1}, ..., \frac{1}{d+1}\}^\infty$. Define a sequence of i.i.d. random vectors $\xi^{(1)}, \xi^{(2)}, ...$ by

$$\xi^{(i)}(\omega) = \sqrt{d+1}(A_{\omega_i 1}, A_{\omega_i 2} ..., A_{\omega_i d}), \quad i \in \mathbb{N}. \tag{2.10}$$

Let $\mathcal{F}_m^\xi = \sigma\{\xi^{(k)}|k \le m\}$, $m \ge 0$ ($\mathcal{F}_0^\xi = \{\emptyset, \Omega_\xi\}$). Denote by $\mathcal{T}_m^\xi$ the set of all stopping times with respect to the filtration $\{\mathcal{F}_k^\xi\}_{k=0}^\infty$ with values from 0 to m.

For any n consider the n–step multinomial market which consists of a savings account $B^{(n)}(t)$ given by

$$B^{(n)}(t) = B(0) > 0 \tag{2.11}$$

and of d risky stocks $S^{\xi,n} = (S_1^{\xi,n}, ..., S_d^{\xi,n})$ given by the formulas $S_i^{\xi,n}(t) = S_i(0)$ for $t \in [0, T/n)$ and

(2.12)

$$S_i^{\xi,n}(t) = S_i(0) \prod_{m=1}^{k} (1 + \frac{b_i T}{n} + \sqrt{\frac{T}{n}} \sum_{j=1}^{d} \sigma_{ij} \xi_j^{(m)}),\ kT/n \le t < (k+1)T/n,\ k = 1, ..., n.$$

We assume that n is sufficiently large such that the terms in the above product are positive a.s. The market is active at the times $0, \frac{T}{n}, \frac{2T}{n}, ..., T$. It is well known that this market is complete and we denote by $\tilde{P}_n^{\xi}$ the unique martingale measure. Define the stochastic process $\{M^{(n)}(t)\}_{t=0}^{T}$

$$M^{(n)}(t) = \frac{d\tilde{P}_n^{\xi}}{dP^{\xi}} | \mathcal{F}_k^{\xi}, \quad kT/n \le t < (k+1)T/n,\ k = 0, 1, ..., n. \tag{2.13}$$

Clearly $\{M^{(n)}(\frac{kT}{n})\}_{k=0}^{n}$ is a martingale with respect to the probability measure P^{ξ} and the filtration $\{\mathcal{F}_k^{\xi}\}_{k=0}^{n}$. Explicit formulas for $M^{(n)}(t)$ were obtained in [8]. Consider an American option with the adapted payoff process

$$Y^{\xi,n}(k) = F(\frac{kT}{n}, S^{\xi,n}), \quad 0 \le k \le n. \tag{2.14}$$

A self financing strategy π with an initial capital x and a horizon n (see [15]) is a sequence $\pi = (\gamma(1), ..., \gamma(n))$ where $\gamma(k)$ are $\mathcal{F}_{k-1}^{\xi}$-measurable random vectors. The portfolio value $V^{\pi}(k)$, $k = 0, 1, ..., n$ is given by

$$V^{\pi}(k) = x + \sum_{i=0}^{k-1} \langle \gamma(i+1), (S^{\xi,n}((i+1)T/n) - S^{\xi,n}(iT/n)) \rangle. \tag{2.15}$$

We call a self financing strategy π *admissible* if $V^{\pi}(k) \ge 0$ for any $k \le n$. Denote by $\mathcal{A}^{\xi,n}(x)$ the set of all *admissible* self financing strategies with an initial capital no bigger than x, let $\mathcal{A}^{\xi,n} = \bigcup_{x>0} \mathcal{A}^{\xi,n}(x)$. The definitions for the shortfall risks in the multinomial markets are similar to the definitions in the BS model. Thus for the n–step multinomial market the shortfall risks are given by

$$R_n(\pi) = \max_{\tau \in \mathcal{T}_n^{\xi}} E^{\xi}[(Y^{\xi,n}(\tau) - V^{\pi}(\tau))^+] \ \text{ and } \ R_n(x) = \inf_{\pi \in \mathcal{A}^{\xi,n}(x)} R_n(\pi), \tag{2.16}$$

where E^{ξ} is the expectation with respect to P^{ξ}.

The following theorem is the main result of the paper and it says that the shortfall risk $R(x)$ for an initial capital x of an American option in the multidimensional BS market can be approximated by a sequence of shortfall risks with an initial capital x of an American options in the multinomial markets defined above. This result has a practical value since for any n the shortfall risk $R_n(x)$ can be calculated by dynamical programming algorithm which is given in Section 5.

Theorem 2.1. *For any $x > 0$*

$$lim_{n\to\infty} R_n(x) = R(x). \tag{2.17}$$

The proof (which is given in Section 4) consists of two parts. In the first part we prove the inequality $R(x) \leq \liminf_{n\to\infty} R_n(x)$ and in the second part we prove that $R(x) \geq \limsup_{n\to\infty} R_n(x)$. In the first part we take a sequence of "almost" optimal portfolios $\{\pi_n\}_{n=1}^{\infty}$ for the multinomial markets and consider their limit in some sense that will be explained explicitly in Section 3. From the limit process we construct a portfolio π in the BS model such that $R(\pi) \leq \liminf_{n\to\infty} R_n(\pi_n) = \liminf_{n\to\infty} R_n(x)$. The second part is proved by a reversed operations. Namely, we take an "almost" optimal portfolio π in the BS model which has some smoothness properties. The existence of such portfolio will be proved by applying density arguments. From this portfolio we construct a sequence of portfolios $\{\pi'_n\}_{n=1}^{\infty}$ in the multinomial models which satisfy $\limsup_{n\to\infty} R_n(\pi'_n) \leq R(\pi)$.

3. Auxiliary lemmas

Let $I \subset [0,T]$ be a dense set in $[0,T]$ and let $\mathcal{T}_I \subset \mathcal{T}^W_{[0,T]}$ be the set of all stopping times with a finite number of values which belongs to I.

Lemma 3.1. *For any $\pi \in \mathcal{A}^W$,*

$$R(\pi) = \sup_{\tau\in\mathcal{T}_I} E[(Y^W(\tau) - V^{\pi}(\tau))^+]. \tag{3.1}$$

Proof. Choose $\epsilon > 0$. There exists $\tau \in \mathcal{T}^W_{[0,T]}$ such that

$$R(\pi) < E[(Y^W(\tau) - V^{\pi}(\tau))^+] + \epsilon. \tag{3.2}$$

For any n there exists a finite set $I_n \subset I$ for which $\bigcup_{z\in I_n}(z-\frac{1}{n}, z+\frac{1}{n}) \supseteq [0,T]$. Let a_n be the maximal element of I_n. Define $\tau_n = \min\{t \in I_n | t \geq \tau\}\mathbb{I}_{\tau_n \leq a_n} + a_n \mathbb{I}_{\tau_n > a_n}$, where $\mathbb{I}_D = 1$ if an event D occurs and $=0$ if not. Clearly, $\tau_n \leq a_n$ a.s. and for $t \in I_n \setminus \{a_n\}$ we have $\{\tau_n \leq t\} = \{\tau \leq t\} \in \mathcal{F}^W_t$. Thus $\tau_n \in \mathcal{T}_I$. Furthermore, $|\tau_n - \tau| \leq \frac{2}{n}$ and so $\tau_n \to \tau$ a.s. From (3.2) and the assumptions on F we obtain

$$\begin{aligned} R(\pi) &< \epsilon + E[\lim_{n\to\infty}(Y^W(\tau_n) - V^{\pi}(\tau_n))^+] = \epsilon + \\ &\lim_{n\to\infty} E[(Y^W(\tau_n) - V^{\pi}(\tau_n))^+] \leq \epsilon + \sup_{\tau\in\mathcal{T}_I} E[(Y^W(\tau) - V^{\pi}(\tau))^+] \end{aligned} \tag{3.3}$$

and the result follows by letting $\epsilon \downarrow 0$. □

The next lemma provides a general result for the shortfall risk measure.

Lemma 3.2. *Let $x > 0$. For any $\epsilon > 0$ there exists $\psi \in C((\mathbb{D}([0,T];\mathbb{R}^d), \mathcal{S}))$ such that the martingale which given by $Q(t) = E^W(\psi(S^W)|\mathcal{F}^W_t)$, $t \leq T$ is satisfying*

$$Q(0) < x \quad \text{and} \quad R(x) > \sup_{\tau\in\mathcal{T}^W_{[0,T]}} E^W\left(\left(Y^W(\tau) - \frac{Q(\tau)}{M(\tau)}\right)^+\right) - \epsilon. \tag{3.4}$$

Proof. Let $\epsilon > 0$. Set $K = E^W[\sup_{0\leq t\leq T} \frac{1}{M(t)}] < \infty$ and $\delta = \frac{\epsilon}{2(K+1)}$. There exists $\pi \in \mathcal{A}(x)$ such that $R(\pi) < R(x) + \delta$. The process $\Phi(t) := V^{\pi}(t)M(t)$, $t \leq T$ is a supermartingale with respect to P^W. Introduce the regular martingale $\Gamma(t) = E(\sup_{0\leq u\leq T} Y^W(u)M(u)|\mathcal{F}^W_t)$, $t \leq T$. The process $\Psi(t) := \Phi(t) \wedge \Gamma(t)$ is a supermartingale of class $\mathcal{D}$. By Doob's decomposition theorem there exists a continuous martingale $\{U(t)\}_{t=0}^{T}$ such that $U(0) = \Psi(0) \leq \Phi(0) = x$ and $U(t) \geq$

$\Psi(t)$ a.s. for all $t \leq T$. Observe that

$$(3.5)\quad \sup_{\tau\in\mathcal{T}^W_{[0,T]}} E^W\left(\left(Y^W(\tau)-\frac{U(\tau)}{M(\tau)}\right)^+\right) \leq \sup_{\tau\in\mathcal{T}^W_{[0,T]}} E^W\left(\left(Y^W(\tau)-\frac{\Psi(\tau)}{M(\tau)}\right)^+\right) = \sup_{\tau\in\mathcal{T}^W_{[0,T]}} E^W[(Y^W(\tau)-V^\pi(\tau))^+] < R(x)+\delta.$$

Next, choose a sequence $0 \leq \psi_n \in C((\mathbb{D}([0,T];\mathbb{R}^d),\mathcal{S}))$, $n \geq 1$ such that

$$(3.6)\quad \lim_{n\to\infty} E^W|\psi_n(S^W)-U(T)| = 0 \text{ and } E^W\psi_n(S^W) < E^W U(T) \leq x,\ n\in\mathbb{N}.$$

Set $Q^{(n)}(t) = E^W(\psi_n(S^W)|\mathcal{F}^W_t)$, $t \leq T$ and introduce the set $C_n = \{\sup_{0\leq t\leq T}|U(t) -Q^{(n)}(t)| > \delta\}$. From (3.5) we obtain that for any n,

$$(3.7)\quad \sup_{\tau\in\mathcal{T}^W_{[0,T]}} E^W\left(\left(Y^W(\tau)-\frac{Q^{(n)}(\tau)}{M(\tau)}\right)^+\right) \leq \sup_{\tau\in\mathcal{T}^W_{[0,T]}} E^W\left(\left(Y^W(\tau)-\frac{U(\tau)}{M(\tau)}\right)^+\right) + \delta E^W[\sup_{0\leq t\leq T}\frac{1}{M(t)}] + E^W(\mathbb{I}_{C_n}\sup_{0\leq t\leq T} Y^W(t)) < R(x)+\frac{\epsilon}{2}+E^W(\mathbb{I}_{C_n}\sup_{0\leq t\leq T} Y^W(t)).$$

By using the Doob inequality for the continuous submartingale $\{|U(t)-Q^{(n)}(t)|\}_{t=0}^T$, it follows from (3.6) that $\lim_{n\to\infty} P(C_n) = 0$. This together with (3.7) gives that for sufficiently large n, $\sup_{\tau\in\mathcal{T}^W_{[0,T]}} E^W\left(\left(Y^W(\tau)-\frac{Q^{(n)}(\tau)}{M(\tau)}\right)^+\right) < R(x)+\epsilon$, as required. □

Given a probability space $(\Omega,\mathcal{F},P)$ consider a *càdlàg* stochastic process $S = \{S_t : \Omega \to \mathbb{R}^d\}_{t=0}^\Theta$, $(\Theta < \infty)$. Denote by $\mathcal{F}^S = \{\mathcal{F}^S_t\}_{t=0}^\Theta$ the usual filtration of S i.e. the smallest right continuous filtration with respect to which S is adapted, and such that the σ−algebras contain the null sets. Let $\mathcal{T}^S_{[0,\Theta]}$ be the set of all stopping times with respect to $\mathcal{F}^S$ which take values in $[0,\Theta]$.

In [13] the authors introduced the Meyer–Zheng (MZ) topology on the space $\mathbb{D}([0,\Theta];\mathbb{R})$. This topology will denoted by $(\mathbb{D}([0,\Theta];\mathbb{R}),MZ)$. The MZ topology is in fact the topology of convergence in measure, it is weaker than the Skorohod topology, but for the MZ topology any sequence of positive uniformly L^1−bounded supermartingales is relatively compact (see [13]). This fact together with the following lemma will be essential in the proof of Theorem 2.1.

Lemma 3.3. *Let $(\Omega,\mathcal{F},P)$ be a probability space and $S^{(n)} : \Omega \to (D[0,\Theta];\mathbb{R}^d)$ be a sequence of stochastic processes such that $S^{(n)} \to S$ a.s. on the space $(\mathbb{D}([0,\Theta];\mathbb{R}^d),\mathcal{S})$. Assume that for any n, $\{V^{(n)}(t)\}_{t=0}^\Theta$ is a (one dimensional) càdlàg positive supermartingale with respect to the filtration $\mathcal{F}^{S_n}_{[0,\Theta]}$ and $V^{(n)} \to V$ a.s. on the space $(\mathbb{D}([0,\Theta];\mathbb{R}),MZ)$ with respect to the MZ topology. Set*

$$(3.8)\qquad Q(t) = E(V(t)|\mathcal{F}^S_t),\quad t \leq \Theta.$$

Then the process $\{Q(t)\}_{0\leq t<\Theta}$ is a càdlàg positive supermartingale with respect to the filtration $\mathcal{F}^S$..

Proof. First, let us show that $\{Q(t)\}_{0\leq t<\Theta}$ is a supermartingale, i.e. for any $s < t < \Theta$ and $D \in \mathcal{F}^S_s$

$$E\mathbb{I}_D V(s) \geq E\mathbb{I}_D V(t). \tag{3.9}$$

Choose $s < s' < t$, $c > 0$ and $0 < \epsilon < \min(s'-s, \Theta-t)$. Let $\phi \in C((\mathbb{D}([0,\Theta];\mathbb{R}^d),\mathcal{S}))$ be a continuous bounded function such that $\phi(x)$ depends only on the restriction of x to the interval $[0,s']$. From the definition of the MZ topology we obtain

$$\limsup_{n\to\infty} E\int_{u=0}^{\epsilon}|\phi(S^{(n)})(V^{(n)}(s'+u)\wedge c)-\phi(S)(V(s'+u)\wedge c)|du$$
$$\leq ||\phi||_\infty \limsup_{n\to\infty} E\int_{u=0}^{\epsilon}(|V^{(n)}(s'+u)-V(s'+u)|\wedge c)du +$$
$$c\limsup_{n\to\infty} E\int_{u=0}^{\epsilon}|\phi(S)-\phi(S^{(n)})|du = 0.$$

Thus,

$$\lim_{n\to\infty}\frac{1}{\epsilon}E\int_{u=0}^{\epsilon}\phi(S^{(n)})(V^{(n)}(s'+u)\wedge c)du = \frac{1}{\epsilon}E\int_{u=0}^{\epsilon}\phi(S)(V(s'+u)\wedge c)du. \tag{3.10}$$

Similarly,

$$\lim_{n\to\infty}\frac{1}{\epsilon}E\int_{u=0}^{\epsilon}\phi(S^{(n)})(V^{(n)}(t+u)\wedge c)du = \frac{1}{\epsilon}E\int_{u=0}^{\epsilon}\phi(S)(V(t+u)\wedge c)du. \tag{3.11}$$

For any n, $\{V^{(n)}(\alpha)\wedge c\}_{\alpha=0}^{U}$ is a supermartingale with respect to $\mathcal{F}^{S^{(n)}}_{[0,\Theta]}$, this together with (3.10) and (3.11) gives

$$\frac{1}{\epsilon}E\int_{u=0}^{\epsilon}\phi(S)(V(t+u)\wedge c)du \leq \frac{1}{\epsilon}E\int_{u=0}^{\epsilon}\phi(S)(V(s'+u)\wedge c)du.$$

By taking $\epsilon \downarrow 0$ we obtain $E\phi(S)(V(t)\wedge c) \leq E\phi(S)(V(s')\wedge c)$. From density arguments and the fact that $D \in \sigma\{S_u | u < s'\}$ it follows that $E\mathbb{I}_D(V(s')\wedge c) \geq E\mathbb{I}_D(V(t)\wedge c)$ and by letting $s' \downarrow s$ and $c \uparrow \infty$ we obtain (3.9). Finally, since the map $t \to EQ(t) = EV(t)$ is right continuous we obtain (see [11]) that Q has a *càdlàg* modification. □

In [8] it was proved that

$$(S^{\xi,n}, M^{(n)}) \Rightarrow (S^W, M) \text{ on the space } (\mathbb{D}([0,T];\mathbb{R}^d),\mathcal{S})\times(\mathbb{D}([0,T];\mathbb{R}),\mathcal{S}). \tag{3.12}$$

We use the notation $S^{(n)} \Rightarrow S$ to indicate that the sequence $S^{(n)}$, $n \geq 1$ converges weakly to S (see [2]). We will use the concept "extended weak convergence" which was introduced in [1] by Aldous. The original definition was via prediction processes. For the case where the stochastic processes are considered with respect to their usual filtration he proved that extended weak convergence is equivalent to a more elementary condition which does not require the use of prediction processes (see [1] Proposition 16.15). We will use the above condition as the definition of extended weak convergence.

Definition 3.4. *A sequence $S^{(n)} : \Omega_n \to \mathbb{D}([0,T];\mathbb{R}^d)$, $n \geq 1$ extended weak converges to a stochastic process $S : \Omega \to \mathbb{D}([0,T];\mathbb{R}^d)$ if for any k and continuous bounded functions $\psi_1,...,\psi_k \in C((\mathbb{D}([0,T];\mathbb{R}^d),\mathcal{S}))$*

$$(S^{(n)}, H^{(n,1)},...,H^{(n,k)}) \Rightarrow (S, H^{(1)},...,H^{(k)}) \text{ on } (\mathbb{D}([0,T];\mathbb{R}^{d+k}),\mathcal{S}) \tag{3.13}$$

where for any $t \leq T$, $1 \leq i \leq k$ and $n \in \mathbb{N}$

$$H^{(n,i)}_t = E_n(\psi_i(S^{(n)})|\mathcal{F}^{S^{(n)}}_t), n\in\mathbb{N}, \text{ and } H^{(i)} = E(\psi_i(S)|\mathcal{F}^S_t) \tag{3.14}$$

E_n denotes the expectation with respect to the probability measure on Ω_n and E denotes the expectation with respect to the probability measure on Ω. We will denote extended weak convergence by $S^{(n)} \Rrightarrow S$.

Lemma 3.5. $S^{\xi,n} \Rrightarrow S^W$.

Proof. Define the map $G : (\mathbb{D}([0,T];\mathbb{R}^d), \mathcal{S}) \to (\mathbb{D}([0,T];\mathbb{R}^d), \mathcal{S})$ by $(G(x_1,...,x_d))(t) = (\exp(x_1(t),...,\exp(x_d(t)))$. Observe that G is a continuous map with continuous inverse (the inverse is defined only on functions $(x_1,...,x_d) \in \mathbb{D}([0,T];\mathbb{R}^d)$ which satisfy $\min_{1\le i\le d} \inf_{0\le t\le T} x_i(t) > 0$). Let $\{X(t) = (\ln S^W_1(t),...,\ln S^W_d(t))\}_{t=0}^T$ and $\{X^{(n)}(t) = (\ln S^{\xi,n}_1(t),...,\ln S^{\xi,n}_d(t))\}_{t=0}^T$, $n \in \mathbb{N}$. From (3.12) and the fact that G has a continuous inverse it follows that $X^{(n)} \Rightarrow X$. For any n the process $X^{(n)}$ has an independent increments and the process X is a continuous process with independent increments. From Corollary 2 in [9] we obtain $X^{(n)} \Rrightarrow X$ and so (since G is continuous) $S^{\xi,n} \Rrightarrow S^W$. □

4. Proof of main results

In this section we complete the proof of Theorem 2.1. Fix x. We start with the proof of the inequality $R(x) \le \lim_{n\to\infty} R_n(x)$. Here and in the sequel, for the sake of simplicity we will assume that indices have been renamed so that the whole sequence converges. Let $\pi_n \in \mathcal{A}^{\xi,n}(x)$, $n \in \mathbb{N}$ be a sequence such that

$$R_n(\pi_n) < R_n(x) + \frac{1}{n} \quad \forall n \in \mathbb{N}. \tag{4.1}$$

For any $n \in \mathbb{N}$ define the stochastic process $\{Z^{(n)}(t)\}_{t=0}^{2T}$ by $Z^{(n)}(t) = V^{\pi_n}_k M^{(n)}(t)$ for $\frac{kT}{n} \le t < \frac{(k+1)T}{n}$ and $k < n$, and $Z^{(n)}(t) = V^{\pi_n}(n) M^{(n)}(T)$ for $t \ge T$. From (2.13) it follows that $\{Z^{(n)}(t)\}_{t=0}^{2T}$ is a *càdlàg* martingale with respect to P^ξ and the filtration $\{\mathcal{F}^{S^{n,\xi}}_t\}_{t=0}^{2T}$, where we set $\mathcal{F}^{S^{n,\xi}}_t = \mathcal{F}^{S^{n,\xi}}_T$ for $t \ge T$. From [13] it follows that the sequence $Z^{(n)}$, $n \in \mathbb{N}$ in tight on the space $(\mathbb{D}([0,T];\mathbb{R}), MZ)$. We can extend all the processes in (3.12) to the interval $[0,2T]$ be letting their paths to be constants on the interval $[T,2T]$. From (3.12) we obtain that the sequence $(S^{\xi,n}, M^{(n)}, Z^{(n)})$, $n \in \mathbb{N}$ it tight on the space $(\mathbb{D}([0,2T];\mathbb{R}^d), \mathcal{S}) \times (\mathbb{D}([0,2T];\mathbb{R}), \mathcal{S}) \times (\mathbb{D}([0,2T];\mathbb{R}), MZ)$. Thus there exists a subsequence such that $(S^{\xi,n}, M^{(n)}, Z^{(n)}) \Rightarrow (S^W, M, Z)$, for some stochastic process Z which satisfies $Z(0) \le x$. Next, from the Skorohod representation theorem (see [4]) it follows that without loss of generality we can assume that there exists a probability space $(\Omega, \mathcal{F}, P)$ on which

$$(S^{\xi,n}, M^{(n)}, Z^{(n)}) \to (S^W, M, Z) \ \text{ a.s.} \tag{4.2}$$

on the space $(\mathbb{D}([0,2T];\mathbb{R}^d), \mathcal{S}) \times (\mathbb{D}([0,2T];\mathbb{R}), \mathcal{S}) \times (\mathbb{D}([0,2T];\mathbb{R}), MZ)$. From Lemma 3.3 it follows that the process $Q(t) := E(Z(t)|\mathcal{F}^{S^W}_t)$, $t \le T$ is a *càdlàg* supermartingale. The process $V(t) := \frac{Q(t)\wedge\Gamma(t)}{M(t)}$, $t \le T$ is a *càdlàg* supermartingale of class $\mathcal{D}$ with respect to the martingale measure $\tilde{P}^W$ ($\Gamma(t)$ was introduced after (3.4)). From Doob's decomposition theorem and the martingale representation theorem we obtain that there exists a portfolio $\pi \in \mathcal{A}(x)$ such that

$$V^\pi(0) = V(0) \le Q(0) = Z(0) \le x \ \text{ and } \ V^\pi(t) \ge V(t) \ \forall t \le T. \tag{4.3}$$

From [13] there exists a subsequence $Z^{(n)}$ and a dense set $I \subset [0,T]$, such that for any $t \in J$

$$\lim_{n\to\infty} Z^{(n)}(t) = Z(t) \ \text{ a.s.} \tag{4.4}$$

Choose $\epsilon > 0$. From Lemma 3.1 we obtain that there exist a stopping time τ which excepts a finite number of values $\{t_1 < t_2 < ... < t_m\} \subset I$ such that

$$R(\pi) < \epsilon + E[(Y^W(\tau) - V^{\pi}(\tau))^+]. \tag{4.5}$$

From Lemma 3.2 in [3] and (4.2) it follows that there exists a sequence $\sigma_n \in \mathcal{F}^{S^{\xi,n}}_{[0,T]}$, $n \geq 1$ of stopping times with values in the set $\{t_1 < t_2 < ... < t_m\}$ which satisfy

$$\lim_{n\to\infty} \sigma_n = \tau \ \text{ a.s.} \tag{4.6}$$

Set $\tau_n = \max\{k|kT/n \leq \sigma_n\}$, $n \geq 1$. Observe that for any $k \leq n$, $\{\tau_n \leq k\} = \{\sigma_n < (k+1)T/n\} \in \mathcal{T}^{\xi}_k$ thus for any n, $\tau_n \in \mathcal{T}^{\xi}_n$. Furthermore,

$$|\frac{\tau_n T}{n} - \sigma_n| \leq \frac{1}{n} \ \text{ and } \ Z^{(n)}(\sigma_n) = Z^{(n)}(\tau_n T/n) \ \ \forall n. \tag{4.7}$$

From (2.3) it follows that the random variables $Y^{\xi,n}(\tau_n)$, $n \in \mathbb{N}$ are uniformly integrable. Thus, from Jensen's inequality and (4.1)–(4.7) it follows

$$\begin{aligned} R(x) &\leq R(\pi) \leq \epsilon + E[(Y^W(\tau) - V^{\pi}(\tau))^+] \leq \epsilon + \\ E&\left(\left(Y^W(\tau) - \tfrac{Q(\tau)}{M(\tau)}\right)^+\right) \leq \epsilon + E\left(E\left(Y^W(\tau) - \tfrac{Z(\tau)}{M(\tau)}\right)^+ \middle| \mathcal{F}^{S^W}_{\tau}\right)\Bigg) = \\ \epsilon + E&\left(\left(Y^W(\tau) - \tfrac{Z(\tau)}{M(\tau)}\right)^+\right) = \epsilon + E\left(\lim_{n\to\infty}\left(Y^{\xi,n}(\tau_n) - \tfrac{Z^{(n)}(\frac{\tau_n T}{n})}{M^{(n)}(\frac{\tau_n T}{n})}\right)^+\right) \\ &= \epsilon + \lim_{n\to\infty} E[(Y^{\xi,n}(\tau_n) - V^{\pi_n}(\tau_n))^+] \leq \epsilon + \lim_{n\to\infty} R_n(x). \end{aligned} \tag{4.8}$$

Since $\epsilon > 0$ was arbitrary we conclude that $R(x) \leq \lim_{n\to\infty} R_n(x)$.

Next, we show that $R(x) \geq \lim_{n\to\infty} R_n(x)$. Choose $\epsilon > 0$. From Lemma 3.2 it follows that there exists $\psi \in C((\mathbb{D}([0,T];\mathbb{R}^d), \mathcal{S}))$ such that the stochastic process $H(t) := E^W(\psi_i(S^W)|\mathcal{F}^{S^W}_t)$, $t \leq T$ satisfies $H(0) < x$ and

$$R(x) > \sup_{\tau \in \mathcal{T}^{S^W}_{[0,T]}} E^W\left(\left(Y^W(\tau) - \frac{H(\tau)}{M(\tau)}\right)^+\right) - \epsilon. \tag{4.9}$$

For any n define the stochastic process $H^{(n)}(t) = E^{\xi}(\psi(S^{n,\xi})|\mathcal{F}^{S^{n,\xi}}_t)$, $t \leq T$. From Lemma 3.5 we obtain

$$(S^{\xi,n}, H^{(n)}) \Rightarrow (S^W, H) \text{ on the space } (\mathbb{D}([0,T];\mathbb{R}^d), \mathcal{S}) \times (\mathbb{D}([0,T];\mathbb{R}), \mathcal{S}). \tag{4.10}$$

Since the process H is continuous then $\lim_{n\to\infty} H^{(n)}(0) = H(0)$. Thus, we will assume that n is sufficiently large such that $H^{(n)}(0) \leq x$. Observe that the process $\frac{H^{(n)}(kT/n)}{M^{(n)}(kT/n)}$, $0 \leq k \leq n$ is a martingale with respect to $\tilde{P}^{\xi}_n$ and the filtration $\{\mathcal{F}^{\xi}_k\}_{k=0}^n$, thus (since the multinomial markets are complete) there exists $\pi'_n \in \mathcal{A}^{\xi,n}(x)$ such that $V^{\pi'_n}(k) = \frac{H^{(n)}(kT/n)}{M^{(n)}(kT/n)}$, $k \leq n$. We obtain that for any n there exists a stopping time $\sigma_n \in \mathcal{T}_n^\xi$ which satisfies

$$(4.11)\quad E^\xi\left(\left(Y^{\xi,n}(\sigma_n) - \frac{Z^{(n)}(\frac{\sigma_n T}{n})}{M^{(n)}(\frac{\sigma_n T}{n})}\right)^+\right) > \sup_{\tau\in\mathcal{T}_n^\xi} E^\xi\left(\left(Y^{\xi,n}(\tau) - \frac{Z^{(n)}(\frac{\tau T}{n})}{M^{(n)}(\frac{\tau T}{n})}\right)^+\right) - \frac{1}{n} \geq R_n(\pi_n') - \frac{1}{n} \geq R_n(x) - \frac{1}{n}.$$

From (3.12) and (4.10) the sequence $(S^{\xi,n}, H^{(n)}, M^{(n)}, \sigma_n T/n)$ is tight on the space $(\mathbb{D}([0,T];\mathbb{R}^d),\mathcal{S}) \times (\mathbb{D}([0,T];\mathbb{R}),\mathcal{S}) \times (\mathbb{D}([0,T];\mathbb{R}),\mathcal{S}) \times [0,T]$. Thus there exists a subsequence such that $(S^{\xi,n}, H^{(n)}, H^{(n)}, \sigma_n T/n) \Rightarrow (S^W, H, M, \nu)$ for some random variable $\nu \leq T$. From the Skorohod representation theorem we can assume that there exists a probability space $(\Omega, \mathcal{F}, P)$ on which

$$(4.12)\qquad (S^{\xi,n}, H^{(n)}, M^{(n)}, \sigma_n T/n) \to (S^W, Z, M, \nu) \ \text{ a.s.}$$

on the space $(\mathbb{D}([0,T];\mathbb{R}^d),\mathcal{S}) \times (\mathbb{D}([0,T];\mathbb{R}),\mathcal{S}) \times (\mathbb{D}([0,T];\mathbb{R}),\mathcal{S}) \times [0,T]$. Observe that the joint distribution of (S^W, Z, M) in (4.12) remains as the original one. From Lemma 3.3 in [3] it follows that for any $t \leq T$, $\{\nu \leq t\}$ and $\mathcal{F}_T^{S^W}$ are conditionally independent given $\mathcal{F}_t^{S^W}$, and for any uniformly integrable *càdlàg* stochastic process $\{\Phi(t)\}_{t=0}^T$ adapted to the filtration $\mathcal{F}_{[0,T]}^{S^W}$

$$(4.13)\qquad E\Phi(\nu) \leq \sup_{\tau\in\mathcal{T}_{[0,T]}^S} E\Phi(\tau).$$

Finally, by using (4.13) for the process $\Phi(t) := (Y^W(t) - \frac{H(t)}{M(t)})^+$, (4.9) and (4.11)–(4.12) we obtain

$$\lim_{n\to\infty} R_n(x) \leq \lim_{n\to\infty} E^\xi\left(\left(Y^{\xi,n}(\sigma_n) - \frac{Z^{(n)}(\frac{\sigma_n T}{n})}{M^{(n)}(\frac{\sigma_n T}{n})}\right)^+\right) =$$

$$E\left(\lim_{n\to\infty}\left(Y^{\xi,n}(\sigma_n) - \frac{Z^{(n)}(\frac{\sigma_n T}{n})}{M^{(n)}(\frac{\sigma_n T}{n})}\right)^+\right) = E^W\left(\left(Y^W(\nu) - \frac{H(\nu)}{M(\nu)}\right)^+\right) < R(x) + \epsilon$$

and the proof is completed.

Remark 4.1. *An interesting question is whether Theorem 2.1 is valid for game options which were introduced in* [10]*. Let* $F, G : [0,T] \times (\mathbb{D}([0,T];\mathbb{R}^d),\mathcal{S}) \to \mathbb{R}_+$ *such that* $F \leq G$ *satisfy the assumptions after (2.2). Set*

$$(4.14)\qquad H^W(t,s) = G(t,S^W)\mathbb{I}_{t<s} + F(s,S^W)\mathbb{I}_{s\leq t}, \quad t,s\in[0,T] \ \text{and}$$
$$H^{\xi,n}(k,l) = G(\tfrac{kT}{n}, S^{\xi,n})\mathbb{I}_{k<l} + F(\tfrac{lT}{n}, S^{\xi,n})\mathbb{I}_{l\leq k}, \quad n\in\mathbb{N},\ 0\leq k,l\leq n.$$

The terms $H^W(t,s)$ *and* $H^{\xi,n}(k,l)$ *are the payoff functions for the BS model and the n–step multinomial model, respectively. For game options the shortfall risk is defined by (see* [5]*)*

$$(4.15)\quad R^{(g)}(x) = \inf_{\pi\in\mathcal{A}^W(x)} \inf_{\sigma\in\mathcal{T}_{[0,T]}^W} \sup_{\tau\in\mathcal{T}_{[0,T]}^W} E^W[(H^W(\sigma,\tau) - V^\pi(\sigma\wedge\tau))^+]$$
$$\text{and } R_n^{(g)}(x) = \inf_{\pi\in\mathcal{A}^{\xi,n}(x)} \min_{\sigma\in\mathcal{T}_n^\xi} \max_{\tau\in\mathcal{T}_n^\xi} E^\xi[(H^{\xi,n}(\sigma,\tau) - V^\pi(\sigma\wedge\tau))^+].$$

The question is whether the equality $R^{(g)}(x) = \lim_{n\to\infty} R_n^{(g)}(x)$ *holds true. Following the proof above it can be shown that* $R^{(g)}(x) \geq \limsup_{n\to\infty} R_n^{(g)}(x)$*. The inequality* $R^{(g)}(x) \leq \liminf_{n\to\infty} R_n^{(g)}(x)$ *is more difficult to prove because of the additional* inf *(in formula (4.15)) which destroys the convexity that was used in (4.8) (by applying Jensen's inequality). At present it is not clear whether the weak convergence approach can be applied here.*

5. Analysis of the multinomial models

In this section we provide a dynamical programming algorithm for the shortfall risks and the corresponding optimal portfolios in the multinomial models. Similar analysis was done in [5] for game options in multinomial markets with one risky asset.

Definition 5.1. *A function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a piecewise linear function vanishing at ∞ if there exists a natural number n, such that*

$$\psi(y) = \sum_{i=1}^{n} \mathbb{I}_{[a_i, a_{i+1})}(c_i y + d_i) \tag{5.1}$$

where $c_1, ..., c_n, d_1, ..., d_n \in \mathbb{R}$ and $a_1 < a_2 < ... < a_{n+1} < \infty$.

Let $J = \{v^{(1)}, ..., v^{(d+1)}\} \subset \mathbb{R}^d$ such that

$$span\{v^{(1)}, ..., v^{(d+1)}\} = \mathbb{R}^d \text{ and } \exists p_1, ..., p_{d+1} > 0,\ \sum_{i=1}^{d+1} p_i v^{(i)} = 0. \tag{5.2}$$

Define the set $K_J = \{u \in \mathbb{R}^d | \langle u, v^{(i)} \rangle \geq -1,\ i = 1, ..., d+1\}$. Observe that K_J is a compact convex set.

Lemma 5.2. *Let $\psi_1, ..., \psi_{d+1} : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous, non increasing and piecewise linear functions vanishing at ∞. Define $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ by*

$$\psi(y) = \min_{u \in K_J} \sum_{i=1}^{d+1} \psi_i(y(1 + \langle u, v^{(i)} \rangle)). \tag{5.3}$$

Then ψ is continuous, non increasing and piecewise linear function vanishing at ∞.

Proof. Clearly ψ is a non increasing function. There exists a natural number n such that

$$\psi_i(y) = \sum_{j=1}^{n} \mathbb{I}_{[a_j, a_{j+1})}(c_j^{(i)} y + d_j^{(i)}),\ i = 1, ..., d+1 \tag{5.4}$$

where $c_j^{(i)}, d_j^{(i)} \in \mathbb{R}$ and $0 = a_1 < a_2 < ... < a_{n+1} < \infty$. Denote $I_k = [a_k, a_{k+1})$, $k = 1, ..., n$ and $I_{n+1} = [a_{n+1}, \infty)$. Set $\lambda_i = 1 + \sup_{u \in K} \langle u, v^{(i)} \rangle$, $i \leq d+1$. Notice that for any $y_1, y_2 \in \mathbb{R}_+$

$$\begin{aligned} |\psi(y_1) - \psi(y_2)| &\leq \textstyle\sum_{i=1}^{d+1} \sup_{u \in K} |\psi_i(y_1(1 + \langle u, v^{(i)} \rangle)) - \psi_i(y_2(1 + \langle u, v^{(i)} \rangle))| \\ &\leq |y_1 - y_2| \textstyle\sum_{i=1}^{d+1} \lambda_i \max_{1 \leq j \leq n} |c_j^{(i)}|. \end{aligned} \tag{5.5}$$

Thus ψ is a continuous function. Next, we prove that ψ is a piecewise linear function. Fix $y > 0$ and introduce the set $L_y = \{\frac{a_1}{y} - 1, ..., \frac{a_{n+1}}{y} - 1\}$. For any $1 \leq \alpha \leq d+1$ and $\beta \in \{1, ..., n+1\}^{d+1}$ define the sets $L_\alpha^{(y)} = \{w \in \mathbb{R}^d | \langle v^{(i)}, w \rangle \in L_y, i \in \{1, ..., d+1\} \setminus \{\alpha\}\}$ and $K_\beta^{(y)} = \{u \in \mathbb{R}^d | y(1 + \langle v^{(j)}, u \rangle) \in I_{\beta_j},\ \forall j \leq d+1\}$. Set $L^{(y)} = \bigcup_{\alpha=1}^{d+1} L_\alpha^{(y)}$. There exists a finite sequence of real numbers $c_1, ..., c_m, e_1, ..., e_m$ (which does not depend on y) such that any $v \in L^{(y)}$ is of the form $v = (c_{k_1} + \frac{e_{r_1}}{y}, ..., c_{k_d} + \frac{e_{r_d}}{y})$, $k_1, ..., k_d, r_1, ..., r_d \in \{1, ..., m\}$. Notice that for any β, $K_\beta^{(y)} \subset K_J$ is a compact convex set. Furthermore, the extreme points of $K_\beta^{(y)}$ are in $L^{(y)}$. For each $\beta \in \{1,...,n+1\}^{d+1}$ the function $\psi^{(y)}: K_\beta^{(y)} \to \mathbb{R}_+$ which given by $\psi^{(y)}(u) = \sum_{i=1}^{d+1} \psi_i(y(1+\langle v^{(i)}, u\rangle))$, is a convex function. Since $\bigcup_{\beta\in\{1,...,n+1\}^{d+1}} K_\beta^{(y)} = K_J$, we obtain

$$\psi(y) = \min_{\beta\in\{1,...,n+1\}^{d+1}} \min_{u\in K_\beta^{(y)}} \psi^{(y)}(u) = \tag{5.6}$$
$$\min_{\beta\in\{1,...,n+1\}^{d+1}} \min_{u\in K_\beta^{(y)}\bigcap L^{(y)}} \psi^{(y)}(u) = \min_{u\in K_J\bigcap L^{(y)}} \psi^{(y)}(u).$$

Thus there exists a finite sequence of real numbers $f_1, ..., f_{\tilde{m}}, g_1, ..., g_{\tilde{m}}$ such that for any $y > 0$,

$$\psi(y) = f_i y + g_i \tag{5.7}$$

for some i (which depends on y). This together with the inequality $\psi(y) \le \sum_{i=1}^{d+1} \psi_i(y)$ and the fact that ψ is a continuous function gives that ψ is a piecewise linear function vanishing at ∞. □

Next, fix n and consider the n–step multinomial model. For any $\pi \in \mathcal{A}^{\xi,n}$ define a sequence of random variables $\{U^\pi(k)\}_{k=0}^n$ by

$$U^\pi(n) = (Y^{\xi,n}(n) - V^\pi(n))^+, \text{ and for } k < n \tag{5.8}$$
$$U^\pi(k) = \max(E^\xi(U^\pi(k+1)|\mathcal{F}_k^\xi), (Y^{\xi,n}(k) - V^\pi(k))^+).$$

Applying standard results for optimal stopping (see [14]) for the process $(Y^{\xi,n}(k) - V^\pi(k))^+$, $k = 0,1,...,n$ we obtain

$$U^\pi(0) = \max_{\tau\in\mathcal{T}_n^\xi} E^\xi[(Y^{\xi,n}(\tau) - V^\pi(\tau))^+] = R_n(\pi). \tag{5.9}$$

Set,

$$w^{(i)} = \sqrt{d+1}(A_{i1}, ..., A_{id}),\ w^{n,i} = \frac{T}{n}b + \sqrt{\frac{T}{n}}w^{(i)}\sigma^*, \quad i \le d+1 \tag{5.10}$$
$$J = \{w^{(1)}, ..., w^{(d+1)}\} \text{ and } J_n = \{w^{n,1}, ..., w^{n,d+1}\}$$

where the matrix A and the vector b were introduced in Section 2.

Definition 5.3. *Let $0 \le k < n$ and X be a nonnegative $\mathcal{F}_k^\xi$–measurable random variable. Define the set*

$$\mathcal{A}_k^{(n)}(X) = \left\{ Y | Y = X\left(1 + \left\langle \rho, \frac{T}{n}b + \sqrt{\frac{T}{n}}\xi^{(k+1)}\sigma^* \right\rangle\right), \right. \tag{5.11}$$
$$\left. \rho : \Omega_\xi \to K_{J_n} \textit{ is } \mathcal{F}_k^\xi - \textit{measurable} \right\}.$$

Notice that if $V^\pi(k) = X$ and $V^\pi(k+1) = Y$ for some $\pi = (\gamma(1), ..., \gamma(n)) \in \mathcal{A}^{\xi,n}$ and $k < n$ then from (2.12) and (2.15), $Y = X(1 + \langle \rho, \frac{T}{n}b + \sqrt{\frac{T}{n}}\xi^{(k+1)}\sigma^*\rangle)$ where $\rho = \frac{\mathbb{I}_{X>0}}{X}(\gamma_1(k+1)S_1^{\xi,n}(\frac{(k+1)T}{n}), ..., \gamma_d(k+1)S_d^{\xi,n}(\frac{(k+1)T}{n}))$. Clearly, if $X = 0$ then (π is *admissible*) $Y = 0$. Since we require $Y \ge 0$ to be satisfied for all possible values of $\xi^{(k+1)}$ then in view of independency of ρ and $\xi^{(k+1)}$ we conclude that $\mathcal{A}_k^{(n)}(X)$ is the set of all possible portfolio values at time $k+1$ provided the portfolio value at time k is X.

For any $0 \le k \le n$ let $\phi_k^{(n)} : J^k \to \mathbb{R}_+$ such that

$$\phi_k^{(n)}(\xi^{(1)}, ..., \xi^{(k)}) = Y^{\xi,n}(k). \tag{5.12}$$

Define a sequence of functions $H_k^{(n)}:\mathbb{R}_+\times J^k\to\mathbb{R}_+$, $k=0,1,...,n$ by the following backward relations. For any $u^{(1)},...,u^{(n)}\in J$ and $y\in\mathbb{R}_+$

$$H_n^{(n)}(y,u^{(1)},...,u^{(n)})=(\phi_n^{(n)}(u^{(1)},...,u^{(n)})-y)^+ \quad \text{and} \tag{5.13}$$

$$H_k^{(n)}(y,u^{(1)},...,u^{(k)})=\max\Bigg((\phi_n^{(n)}(u^{(1)},...,u^{(k)})-y)^+,\ \ \tfrac{1}{d+1}\inf_{u\in K_{J_n}}$$
$$\textstyle\sum_{i=1}^{d+1}H_{k+1}^{(n)}(y(1+\langle u,\frac{T}{n}b+\sqrt{\frac{T}{n}}w^{(i)}\sigma^*\rangle),u^{(1)},...,u^{(k)},w^{(i)})\Bigg) \text{ for } k<n.$$

Observe that J_n (for sufficiently large n) satisfies (5.2). Thus from Lemma 5.2 it follows (by backward induction) that for any $k\leq n$ and $u^{(1)},...,u^{(k)}\in J$, $H_k^{(n)}(\cdot,u^{(1)},...,u^{(k)})$ is continuous, non increasing and piecewise linear function vanishing at ∞. These facts allow us to define the functions $\{h_k^{(n)}:\mathbb{R}_+\times J^k\to K_{J_n}\}_{k=0}^{n-1}$ by

$$h_k^{(n)}(y,u^{(1)},...,u^{(k)})=argmin_{u\in K_{J_n}}\sum_{i=1}^{d+1}H_{k+1}^{(n)}(y(1+\langle u, \tag{5.14}$$
$$\frac{T}{n}b+\sqrt{\frac{T}{n}}w^{(i)}\sigma^*\rangle),u^{(1)},...,u^{(k)},w^{(i)}).$$

Namely,

$$\min_{u\in K_{J_n}}\textstyle\sum_{i=1}^{d+1}H_{k+1}^{(n)}(y(1+\langle u,\frac{T}{n}b+\sqrt{\frac{T}{n}}w^{(i)}\sigma^*\rangle),u^{(1)},...,u^{(k)},w^{(i)})= \tag{5.15}$$
$$\textstyle\sum_{i=1}^{d+1}H_{k+1}^{(n)}(y(1+\langle h_k^{(n)}(y,u^{(1)},...,u^{(k)}),\frac{T}{n}b+\sqrt{\frac{T}{n}}w^{(i)}\sigma^*\rangle),u^{(1)},...,u^{(k)},w^{(i)})$$

for any $y\in\mathbb{R}_+$ and $u^{(1)},...,u^{(k)}\in J$.

Let $x>0$ be an initial capital. Define $\tilde{\pi}=\tilde{\pi}(n,x)\in\mathcal{A}^{\xi,n}(x)$ by

$$V^{\tilde{\pi}}(0)=x, \text{ and for } 0\leq k<n \tag{5.16}$$
$$V^{\tilde{\pi}}(k+1)=V^{\tilde{\pi}}(k)(1+\langle h_k^{(n)}(V^{\tilde{\pi}}(k),\xi^{(1)},...,\xi^{(k)}),\tfrac{T}{n}b+\sqrt{\tfrac{T}{n}}\xi^{(k+1)}\sigma^*\rangle).$$

Theorem 5.4. *For any $n\in\mathbb{N}$ and $x>0$*

$$R_n(x)=R_n(\tilde{\pi}(n,x))=H_0^{(n)}(x). \tag{5.17}$$

Proof. Fix $n\in\mathbb{N}$ and $x>0$. Let $\pi\in\mathcal{A}^{\xi,n}(x)$ an arbitrary portfolio. Denote $\tilde{\pi}=\tilde{\pi}(n,x)$. First we prove by backward induction that for any $k\leq n$,

$$H_k^{(n)}(V^{\pi}(k),\xi^{(1)},...,\xi^{(k)})\leq U^{\pi}(k) \text{ and } H_k^{(n)}(V^{\tilde{\pi}}(k),\xi^{(1)},...,\xi^{(k)})=U^{\tilde{\pi}}(k). \tag{5.18}$$

For $k=n$, we obtain from (5.8) and (5.12)–(5.13) that the relations (5.18) hold with equality. Suppose that (5.18) holds true for $k+1$ and prove them for k. Let $\rho:\Omega_\xi\to K_{J_n}$ be a $\mathcal{F}_k^{\xi}$ measurable random vector such that $V^{\pi}(k+1)=V^{\pi}(k)(1+\langle\rho,\frac{T}{n}b+\sqrt{\frac{T}{n}}\xi^{(k+1)}\sigma^*\rangle)$. From the induction assumption we obtain

$$E^{\xi}(U^{\pi}(k+1)|\mathcal{F}_k^{\xi})\geq E^{\xi}(H_k^{(n)}(V^{\pi}(k+1),\xi^{(1)},...,\xi^{(k)},\xi^{(k+1)})|\mathcal{F}_k^{\xi}) \tag{5.19}$$
$$=\tfrac{1}{d+1}\textstyle\sum_{i=1}^{d+1}H_{k+1}^{(n)}(V^{\pi}(k)(1+\langle\rho,\frac{T}{n}b+\sqrt{\frac{T}{n}}w^{(i)}\sigma^*\rangle),\xi^{(1)},...,\xi^{(k)},w^{(i)})\geq$$
$$\tfrac{1}{d+1}\inf_{u\in K_{J_n}}\textstyle\sum_{i=1}^{d+1}H_{k+1}^{(n)}(V^{\pi}(k)(1+\langle u,\frac{T}{n}b+\sqrt{\frac{T}{n}}w^{(i)}\sigma^*\rangle),\xi^{(1)},...,\xi^{(k)},w^{(i)}).$$

Denote $\tilde{\rho} = h_k^{(n)}(V^{\tilde{\pi}}(k), \xi^{(1)}, ..., \xi^{(k)})$. From (5.15)–(5.16) and the induction assumption it follows

$$(5.20) \quad E^{\xi}(U^{\tilde{\pi}}(k+1)|\mathcal{F}_k^{\xi}) = E^{\xi}(H_k^{(n)}(V^{\tilde{\pi}}(k+1), \xi^{(1)}, ..., \xi^{(k)}, \xi^{(k+1)})|\mathcal{F}_k^{\xi})$$

$$= \frac{1}{d+1}\sum_{i=1}^{d+1} H_{k+1}^{(n)}(V^{\tilde{\pi}}(k)(1+\langle \tilde{\rho}, \frac{T}{n}b + \sqrt{\frac{T}{n}}w^{(i)}\sigma^*\rangle), \xi^{(1)}, ..., \xi^{(k)}, w^{(i)}) =$$

$$\frac{1}{d+1}\inf_{u\in K_{J_n}} \sum_{i=1}^{d+1} H_{k+1}^{(n)}(V^{\tilde{\pi}}(k)(1+\langle u, \frac{T}{n}b + \sqrt{\frac{T}{n}}w^{(i)}\sigma^*\rangle), \xi^{(1)}, ..., \xi^{(k)}, w^{(i)}).$$

Combining (5.8), (5.12)–(5.13) and (5.19)-(5.20) we obtain that (5.18) holds true. Next, by using (5.18) for $k = 0$ and (5.9) it follows that for any $\pi \in \mathcal{A}^{\xi,n}(x)$

$$R_n(\pi) = U^{\pi}(0) \geq H_0^{(n)}(V^{\pi}(0)) \geq H_0^{(n)}(x) = U^{\tilde{\pi}}(0) = R_n(\tilde{\pi}).$$

Thus $R_n(x) = R_n(\tilde{\pi}) = H_0^{(n)}(x)$, as required. □

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