Squeezing Inequalities and Entanglement for Identical Particles

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Abstract

By identifying non-local effects in systems of identical Bosonic qubits through correlations of their commuting observables, we show that entanglement is not necessary to violate certain squeezing inequalities that hold for distinguishable qubits and that spin squeezing may not be necessary to achieve sub-shot noise accuracies in ultra-cold atom interferometry.

1 Introduction

A generic state of N distinguishable qubits is defined to be entangled if it is not fully separable [1], namely if it cannot be written as

$$\rho_{sep} = \sum_k p_k \rho_k^{(1)} \otimes \rho_k^{(2)} \cdots \otimes \rho_k^{(N)} \ , \tag{1}$$

where the $p_k \geq 0$ are weights, $\sum_k p_k = 1$, while each $\rho_k^{(j)}$ is a density matrix for the j-th qubit acting on the corresponding Hilbert space $\mathbb{H}_j$.

Entangled N-qubit states have been proposed as means to beat the so-called shot-noise limit in metrological applications [2, 3] and spin-squeezing techniques have been devised to generate them in systems of ultra-cold atoms [4, 5]. These states are then used in interferometric experiments where their states are rotated by means of collective spin components; in the case of distinguishable qubits, all these rotations are local; therefore, preliminary spin squeezing of separable states is necessary in order to introduce (non-local) quantum correlations.

However, in the case of ultra-cold trapped atoms, the qubits involved are identical, a fact that asks for a rethinking of the properties and behaviors valid for distinguishable qubits. Indeed, the notion of separability based on (1) is strictly associated with the tensor product structure of the Hilbert space, $\mathbb{H}_N = \bigotimes_{j=1}^{N} \mathbb{H}_j$, which is natural for N distinguishable particles. On the other hand, pure Bosonic states must be symmetric under exchange of particles and mixed states must be convex combinations of projections onto such states. This fact demands a different approach to the notions of non-locality and entanglement based not on a structure related to the particle aspect of first quantization, as in (1), rather on the behavior of correlation functions of commuting observables [6, 7, 8, 9], more generally related to the mode description typical of second quantization.

In the following, based on a generalized notion of entanglement [9] which reduces to the standard one for distinguishable qubits, we show that neither entangled states, nor spin-squeezing are necessary in order to achieve sub-shot noise accuracies. Though necessary, non-locality comes not from the states, rather from the rotations that are implemented in the interferometric experiments. Indeed, while all rotations are local for distinguishable qubits, certain collective spin observables become instead non-local for identical qubits and might permit sub-shot noise accuracies without the need of a preliminary squeezing of the input state as in [4, 5].

2   Entanglement for Identical Particles

In general, density matrices as in (1) are not allowed Bosonic states not even if $\rho_k^{(i)} = \rho_k$ for all i. Indeed, consider two qubits and fix an orthonormal basis $\mathbb{C}^4 \ni |ij\rangle$, $i,j = \uparrow, \downarrow$. If they are Bosons, their states cannot have non-vanishing components on the anti-symmetric state $|\Psi_-\rangle = \dfrac{|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle}{\sqrt{2}}$. Since $\langle\Psi_-|\rho \otimes \rho|\Psi_-\rangle = \mathrm{Det}(\rho)$, the density matrix $\rho \otimes \rho$ cannot correspond to a state of two Bosonic qubits unless ρ is a projection ($\mathrm{Det}(\rho) = 0$).

Therefore, the tensor product structure which is natural for distinguishable particles is not appropriate for discussing the entanglement properties of systems of indistinguishable particles. These should rather be investigated within the second quantization formalism whereby one introduces creation and annihilation operators $a_i^{\#}$ of single-particle orthonormal basis states $|i\rangle$, obeying the canonical commutation relations $[a_i, a_j^{\dagger}] = \delta_{ij}$.

Entanglement in such a context should correspond to whether, given a state ω of the system, there are non-classical correlations among commuting observables [7, 9]; for instance, between non-overlapping spatial regions $V_{1,2}$.

Specifically, this can be inspected by considering the structure of two-point functions of the form $\omega(P_{V_1} P_{V_2})$, where P_{V_α} is any polynomial in creation and annihilation operators $a^{\#}(\psi_\alpha)$ of states $|\psi_\alpha\rangle$ spatially localized in the region V_α. The relevant fact is that the spatially local algebras generated by polynomials P_{V_α} commute: $[a(\psi_1), a^{\dagger}(\psi_2)] = \langle\psi_1|\psi_2\rangle = 0$.

More in general, one can argue about the entanglement between observables belonging to two generic commuting sub-algebras $(\mathcal{A}, \mathcal{B})$ of the entire algebra generated by creation and annihilation operators, which we shall refer to as *algebraic bipartition* [9]. We shall call an operator $(\mathcal{A}, \mathcal{B})$-local if of the form AB, $A \in \mathcal{A}$ and $B \in \mathcal{B}$ and a state ω $(\mathcal{A}, \mathcal{B})$-separable if the expectations $\omega(AB)$ of local operators AB can be decomposed into a convex linear combination of product of expectations:

$$\omega(AB) = \sum_i \lambda_i \omega_i^a(A)\, \omega_i^b(B)\ , \quad \lambda_i > 0\ , \quad \sum_i \lambda_i = 1\ , \qquad (2)$$

in terms of other states $\omega_i^{a,b}$; otherwise, ω is $(\mathcal{A}, \mathcal{B})$-entangled.

Remark 1. In the case of two qubits, the above definitions reproduce the standard notions if one chooses the algebraic bipartition $\mathcal{A} = \mathcal{B} = M_2$, where M_2 the algebra of 2×2 matrices over $\mathbb{C}^2$, and $\omega(AB) = \mathrm{Tr}\,(\rho\, A \otimes B)$, with ρ a two-qubit density matrix. However, for identical particles, there is no a priori given bi-partition so that questions about entanglement and separability, non-locality and locality are meaningful only with reference to a specific class of (commuting) observables.

2.1 N Bosons in a Double-Well Potential

A concrete application of the previous considerations is the second quantization of a single-particle with Hilbert space $\mathbb{C}^2$ which, in the Bose-Hubbard approximation, effectively describes N ultra-cold atoms confined by a double-well potential. Then, the state $|\downarrow\rangle$ describes one atom located within the left well and the state $|\uparrow\rangle$ an atom localized within the right one. Let $|0\rangle$ be the vacuum state and $a^{\dagger}$, $b^{\dagger}$ the creation operators of a particle in the states $|\downarrow\rangle$ and $|\uparrow\rangle$, that is $a^{\dagger}|0\rangle = |\downarrow\rangle$, $b^{\dagger}|0\rangle = |\uparrow\rangle$.

When the total number N is conserved, the symmetric Fock space of this two-mode system is generated by $N+1$ orthonormal eigenvectors of the number operator $a^{\dagger}a + b^{\dagger}b$:

$$|k\rangle = \frac{(a^{\dagger})^k (b^{\dagger})^{N-k}}{\sqrt{k!(N-k)!}}|0\rangle, \quad 0 \le k \le N\ . \qquad (3)$$

Because of the orthogonality of the spatial modes, by considering the norm-closures of all polynomials P_a in $a, a^\dagger$, respectively P_b in $b, b^\dagger$, one obtains two commuting subalgebras $\mathcal{A}$ and $\mathcal{B}$.

According to (2), the states $|k\rangle$ are $(\mathcal{A}, \mathcal{B})$-separable; indeed, they are created by the $(\mathcal{A}, \mathcal{B})$-local operators $(a^\dagger)^k (b^\dagger)^{N-k}$. More in general, $(\mathcal{A}, \mathcal{B})$-separable states must be convex combinations of projections $|k\rangle\langle k|$ [9]:

$$\rho = \sum_{k=0}^{N} p_k |k\rangle\langle k| \qquad p_k > 0 \ , \ \sum_{k=0}^{N} p_k = 1 \ . \tag{4}$$

Consider instead the following operators

$$J_x = \frac{1}{2}(a^\dagger b + ab^\dagger) \ , \ J_y = \frac{1}{2i}(a^\dagger b - ab^\dagger) \ , \ J_z = \frac{1}{2}(a^\dagger a - b^\dagger b) \ , \tag{5}$$

that satisfy the $SU(2)$ algebraic relations $[J_x\,,\,J_y] = i\,J_z$. They are all non-local with respect to the algebraic bipartition $(\mathcal{A}, \mathcal{B})$ and such are the exponentials $e^{i\theta J_x}$ and $e^{i\theta J_y}$, while $e^{i\theta J_z} = e^{i\theta a^\dagger a} e^{-i\theta b^\dagger b}$ is $(\mathcal{A}, \mathcal{B})$-local.

By means of a Bogolubov transformation to other creation and annihilation operators $(c^\#, d^\#)$, such that $a = \dfrac{c+d}{\sqrt{2}}$ and $b = \dfrac{c-d}{\sqrt{2}}$, one obtains another bipartition $(\mathcal{C}, \mathcal{D})$ and rewrites

$$J_x = \frac{1}{2}(c^\dagger c - d^\dagger d) \ , \ J_y = \frac{1}{2i}(d^\dagger c - dc^\dagger) \ , \ J_z = \frac{1}{2}(c^\dagger d + cd^\dagger) \ .$$

Relatively to $\{\mathcal{C}, \mathcal{D}\}$, it is now $e^{i\theta\,J_x} = e^{i\theta\,c^\dagger c} e^{-i\theta\,d^\dagger d}$ which acts locally.

Remark 2. In first quantization, an $(\mathcal{A}, \mathcal{B})$-separable state for a $N = 2$ Bosonic qubits like $|11\rangle = a^\dagger b^\dagger |0\rangle$ corresponds to $\dfrac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}}$. Such state is surely entangled for distinguishable qubits, while, according to our definition, it is no longer so for identical Bosonic qubits; the reason is that its entanglement is only formal as it comes from the necessary symmetrization of the separable state $|\uparrow\downarrow\rangle$ [10].

A Bogolubov transformation as the one above corresponds to a change of basis in the single particle Hilbert space, from the one of spatially localized states, to the one of $c^\dagger |0\rangle = \dfrac{1}{\sqrt{2}}(|\downarrow\rangle + |\uparrow\rangle)$, $d^\dagger |0\rangle = \dfrac{1}{\sqrt{2}}(|\downarrow\rangle - |\uparrow\rangle)$. Physically speaking, such states are eigenstates of the single particle Hamiltonian in the Bose-Hubbard approximation with a highly penetrable barrier. The change to the *energy bipartition* $(\mathcal{C}, \mathcal{D})$ is non-local with respect to the *spatial bipartition* $(\mathcal{A}, \mathcal{B})$, though it corresponds to a local unitary rotation in first quantization.

2.2 Collective Spin Inequalities and Entanglement

In the case of a system of N distinguishable qubits, the collective angular momentum operators $J_{x,y,z}$ and the corresponding rotations are sums of single particle spin operators, $J_{x,y,z}^{(j)}$, i. e. $J_{x,y,z} = \sum_{j=1}^{N} J_{x,y,z}^{(j)}$. These operators are local with respect to the tensor product structure in (1).

Based on this, the variance $\Delta^2 J_{\vec{n}}$ of the collective spin $J_{\vec{n}} = n_x J_x + n_y J_y + n_z J_z$ along the unit spatial direction $\vec{n} = (n_x, n_y, n_z)$, with respect to separable vector states $|\Psi\rangle = \bigotimes_{j=1}^{N} |\psi_j\rangle$ results

$$
\begin{aligned}
\Delta^2 J_{\vec{n}} &= \langle \Psi | J_{\vec{n}}^2 | \Psi \rangle - \langle \Psi | J_{\vec{n}} | \Psi \rangle^2 \\
&= \frac{N}{4} - \sum_{j=1}^{N} \left(\langle \psi_j | J_n^{(j)} | \psi_j \rangle \right)^2 \leq \frac{N}{4} \, .
\end{aligned}
\tag{6}
$$

Therefore, $\Delta^2 J_{\vec{n}}$ is an entanglement witness for pure states, in the sense that if $\Delta^2 J_{\vec{n}} > N/4$ then the pure state $|\Psi\rangle$ cannot be fully separable.

This is no longer the case for N identical Bosonic qubits. Indeed, consider the number states $|k\rangle$ in (3); using (5), one gets

$$
\langle k | J_{\vec{n}} | k \rangle = \frac{n_z}{2} (2k - N)
\tag{7}
$$

$$
\langle k | J_{\vec{n}}^2 | k \rangle = \frac{N + 2k(N-k)}{4}
$$
$$
+ \; n_z^2 \frac{N(N-1) - 6k(N-k)}{4}
\tag{8}
$$

$$
\Delta^2 J_{\vec{n}} = \frac{1 - n_z^2}{4} \left(N + 2k(N-k) \right) \, .
\tag{9}
$$

Therefore, if $k \neq 0, N$, for all $\vec{n}$ that satisfy

$$
n_z^2 < \frac{2k(N-k)}{N + 2k(N-k)} \leq 1 \, ,
$$

the states $|k\rangle$, though $(\mathcal{A}, \mathcal{B})$-separable, nevertheless yield $\Delta^2 J_{\vec{n}} > N/4$; therefore, $\Delta^2 J_{\vec{n}}$ is not an entanglement witness for pure states of Bosonic qubits.

In greater generality, inequalities for mean values and variances of collective spin operators with respect to any (mixed) separable state of distinguishable qubits (1) have been derived in [11]; these are called *spin squeezing*

inequalities and read [1]

$$\langle J_{\vec{n}_1}^2 \rangle + \langle J_{\vec{n}_2}^2 \rangle + \langle J_{\vec{n}_3}^2 \rangle - \frac{N(N+2)}{4} \leq 0 \ , \tag{10}$$

$$\Delta^2 J_{\vec{n}_1} + \Delta^2 J_{\vec{n}_2} + \Delta^2 J_{\vec{n}_3} - \frac{N}{2} \geqslant 0 \ , \tag{11}$$

$$\langle J_{\vec{n}_1}^2 \rangle + \langle J_{\vec{n}_2}^2 \rangle - \frac{N}{2} - (N-1)\Delta^2 J_{\vec{n}_3} \leq 0 \ , \tag{12}$$

$$(N-1)(\Delta^2 J_{\vec{n}_1} + \Delta^2 J_{\vec{n}_2}) - \langle J_{\vec{n}_3}^2 \rangle - \frac{N(N-2)}{4} \geqslant 0 \ , \tag{13}$$

where $\vec{n}_{1,2,3}$ denotes any triplet of unit vectors corresponding to orthogonal spatial directions and $\langle X \rangle$ the mean value of an operator X.

It is thus interesting to study whether these inequalities are also satisfied by $(\mathcal{A}, \mathcal{B})$-separable states (4) of N identical qubits. Let $\langle k^a \rangle = \sum_{k=0}^{N} p_k \, k^a$, $a = 1, 2$, denote first and second moments of the $N + 1$-valued stochastic variable k with respect to the probability distribution $\pi = \{p_k\}_{k=0}^{N}$. Using (7), (9), mean-values and variances of collective spin operators $J_{\vec{n}}$ with respect to the states in (4) read

$$\langle J_{\vec{n}} \rangle = \frac{n_z}{2}\Big(2\langle k \rangle - N\Big) \tag{14}$$

$$\langle J_{\vec{n}}^2 \rangle = \frac{N(1 + 2\langle k \rangle) - 2\langle k^2 \rangle}{4}$$
$$+ \frac{n_z^2}{4}\Big(N(N-1) - 6N\langle k \rangle + 6\langle k^2 \rangle\Big) \tag{15}$$

$$\Delta^2 J_{\vec{n}} = \frac{N(1 + 2\langle k \rangle) - 2\langle k^2 \rangle}{4}$$
$$+ \frac{n_z^2}{4}\Big(6\langle k^2 \rangle - 2\langle k \rangle(N + 2\langle k \rangle) - N\Big) \ . \tag{16}$$

From the orthogonality of the triplet $\vec{n}_{1,2,3}$, it follows that $n_{1z}^2 + n_{2z}^2 + n_{3z}^2 = 1$; one can thus check that all inequalities but (12) are satisfied by $(\mathcal{A}, \mathcal{B})$-separable states. Concerning (12), its left hand side reads

$$\delta = \frac{N}{2}\Big(\Delta^2 k - \langle k \rangle(N - \langle k \rangle)\Big)$$
$$+ \frac{n_{3z}^2}{2}\Big((N+2)\langle k \rangle(N - \langle k \rangle) - 3N\Delta_k^2\Big) \ , \tag{17}$$

where $\Delta^2 k := \langle k^2 \rangle - \langle k \rangle^2$ is the variance of k with respect to $\pi = \{p_k\}_{k=0}^{N}$. If π is chosen such that $a := \langle k \rangle(N - \langle k \rangle) > N\Delta^2 k$, then δ becomes positive and

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[1]In [11], these inequalities are derived with respect to the standard triplet $\vec{n_1} = \hat{x}$, $\vec{n}_2 = \hat{y}$, $\vec{n}_3 = \hat{z}$. The result easily extends to more general triplets.

thus (12) results violated by the corresponding $(\mathcal{A}, \mathcal{B})$-separable states (4) for all orthogonal triplets with

$$1 \geq n_{3z}^2 \; > \; \frac{N(a - \Delta^2 k)}{(N+2)\, a - 3\, N\, \Delta^2 k} \; . \tag{18}$$

Consider the pure states $|\ell\rangle$ in (3) with $\ell \neq 0, N$; in such a case, $p_k = \delta_{k\ell}$, $\langle k \rangle = \ell$ and $\Delta^2 k = 0$, so that (12) is violated for $1 \geq n_{3z}^2 > N/(N+2)$.

3 Spin Squeezing and Metrology

The preceding results indicate that spin-squeezing inequalities that are derived for distinguishable qubits can not directly be used as entanglement witnesses in the context of identical qubits. Since the use of spin-squeezed states for metrological purposes have recently become the focus of much theoretical [2, 3, 12, 13] investigations, we now discuss the impact of particle indistinguishability on such an issue.

3.1 Spin Squeezing

For any orthogonal triplet of space-directions $\vec{n}_{1,2,3}$, the Heisenberg uncertainty relations for the SU(2) operators $J_{\vec{n}}$ read

$$\Delta^2 J_{\vec{n}_1} \Delta^2 J_{\vec{n}_2} \geqslant \frac{1}{4} \langle J_{\vec{n}_3} \rangle^2 \; . \tag{19}$$

One speaks of spin-squeezing when one of the variances can be made smaller than $\frac{1}{2} \left| \langle J_{\vec{n}_3} \rangle \right|$. The relevance of states satisfying this condition for achieving otherwise unavailable accuracies has been studied in relation to the measure of an angle θ by interferometric techniques. These are based on a rotation of an input state ρ into

$$\rho_\theta = \exp(-i\theta J_{\vec{n}_1}) \, \rho \, \exp(i\theta J_{\vec{n}_1}) \; , \tag{20}$$

and upon measuring on ρ_θ the collective spin $J_{\vec{n}_2}$, where $\vec{n}_2 \perp \vec{n}_1$. By choosing the remaining orthogonal unit vector $\vec{n}_3$ such that $\langle J_{\vec{n}_3} \rangle = \mathrm{Tr}(\rho \, J_{\vec{n}_3}) \neq 0$, by error propagation, the uncertainty $\delta\theta$ in the determination of θ can be estimated by [3]

$$\delta^2 \theta = \frac{\Delta^2 J_{\vec{n}_1}}{\left(\partial_\theta \langle J_{\vec{n}_2} \rangle_\theta \big|_{\theta=0} \right)^2} = \frac{\Delta^2 J_{\vec{n}_1}}{\langle J_{\vec{n}_3} \rangle^2} = \frac{\xi_W^2}{N} \; , \tag{21}$$

in terms of the spin-squeezing parameter

$$\xi_W^2 := \frac{N\Delta^2 J_{\vec{n}_1}}{\langle J_{\vec{n}_3}\rangle^2} \ . \tag{22}$$

The value $\delta^2\theta = \dfrac{1}{N}$ is called shot-noise limit; in the case of distinguishable qubits, it gives the lower bound to the attainable accuracies when the input state ρ is separable. Indeed, in such a case one finds $\xi_W^2 \leq 1$. This result follows from the inequality $\xi_W^2 \geq \xi_2^S$ where the new spin-squeezing parameter

$$\xi_S^2 := \frac{N\Delta^2 J_{\vec{n}_1}}{\langle J_{\vec{n}_2}\rangle^2 + \langle J_{\vec{n}_3}\rangle^2} \ , \tag{23}$$

has been introduced in [12]; by means of the local structure of the collective spin operators $J_{x,y,z}$, one can prove that ξ_S^2 is always ≥ 1 for separable states of distinguishable qubits. Therefore, using distinguishable qubits, the shot-noise limit can be beaten, namely accuracies better than $1/N$ can be achieved only if $\xi_S^2 < 1$, that is only by means of entangled states.

Let us instead consider N identical Bosonic qubits in the $(\mathcal{A}, \mathcal{B})$-separable pure states $|k\rangle$ and any triplet of orthogonal spatial directions $\vec{n}_{1,2,3}$ with $\vec{n}_1 \neq \hat{z}$. Using (7)-(9) and $n_{1z}^2 + n_{2z}^2 + n_{3z}^2 = 1$, one computes

$$\begin{aligned}
\xi_S^2 &= \frac{N\Delta^2 J_{\vec{n}_1}}{\langle J_{\vec{n}_2}\rangle^2 + \langle J_{\vec{n}_3}\rangle^2} = N\,\frac{1 - n_{1z}^2}{n_{2z}^2 + n_{3z}^2}\,\frac{N + 2k(N-k)}{(2k-N)^2} \\
&= \frac{N(N + 2k(N-k))}{(2k-N)^2} \geq 1 \ , \quad 0 \leq k \leq N \ . \tag{24}
\end{aligned}$$

In the case of the $(\mathcal{A}, \mathcal{B})$-separable density matrices (4), first observe that, thanks to the Cauchy-Schwartz inequality, one has $\sum_{k=0}^{N} p_k \langle J_{\vec{n}}\rangle_k^2 \geq \langle J_{\vec{n}}\rangle^2$, where $\langle X\rangle_k^2$ denotes the mean-value of X with respect to the number state $|k\rangle$. Then,

$$\begin{aligned}
\xi_S^2 &\geq N\,\frac{\sum_{k=0}^{N} p_k \Delta_k^2 J_{\vec{n}_1}}{\langle J_{\vec{n}_2}\rangle^2 + \langle J_{\vec{n}_3}\rangle^2} \\
&= \frac{1 - n_{1z}^2}{n_{2z}^2 + n_{3z}^2}\,\frac{\sum_{k=0}^{N} p_k\, N\left(N + 2k(N-k)\right)}{\left(\sum_{k=0}^{N} p_k\,(N-2k)\right)^2} \\
&\geq \frac{\sum_{k=0}^{N} p_k \left(k - \frac{N}{2}\right)^2}{\left(\sum_{k=0}^{N} p_k \left(k - \frac{N}{2}\right)\right)^2} \ ,
\end{aligned}$$

where the last inequality follow from the second line of (24). A further application of the Cauchy-Schwartz inequality to the right hand side of the last inequality yields $\xi_S^2 \geq 1$ for all $(\mathcal{A}, \mathcal{B})$-separable states when $n_1 \neq z$.

If one chooses $\vec{n}_1 = \hat{z}$, in the case of $(\mathcal{A}, \mathcal{B})$-separable mixed states, one finds $\Delta^2 J_z \neq 0$ and $\langle J_{\vec{n}_2, \vec{n}_3} \rangle = 0$; therefore, ξ_S^2 (and ξ_W^2) diverges. Instead, for $(\mathcal{A}, \mathcal{B})$-separable pure states $|k\rangle$ also $\Delta_k J_z = 0$ whence ξ_S^2 (and ξ_W^2) are not defined and must thus be computed by means of suitable limiting procedures.

Let us consider the $(\mathcal{A}, \mathcal{B})$-entangled vector state $|\Psi\rangle = \sum_{k=0}^{N} \sqrt{p_k} |k\rangle$, with real coefficients from a probability distribution $\pi = \{p_k\}_{k=0}^{N}$ over the stochastic variable k. Then, from (7)–(9) it follows that $\langle J_y \rangle = 0$ and $\Delta^2 J_z = \Delta^2 k$; therefore $\xi_S^2 = \xi_W^2$ and

$$\xi_W^2 = \frac{N\,\Delta^2 J_z}{\langle J_x \rangle^2} = \frac{N\,\Delta^2 k}{\left(\sum_{k=1}^{N} \sqrt{k(N-k+1)}\,\sqrt{p_k\,p_{k-1}} \right)^2} \ . \tag{25}$$

In the case of a Gaussian distribution peaked around $k = \ell \neq 0, N$,

$$p_k = \frac{1}{Z} \exp\left(-\frac{(k-\ell)^2}{\sigma^2} \right) \ , \quad Z = \sum_{k=0}^{N} \exp\left(-\frac{(k-\ell)^2}{\sigma^2} \right) \ , \tag{26}$$

one finds

$$\xi_W^2 = \frac{2N + \mathcal{O}\left(e^{-\frac{1}{2\sigma^2}} \right)}{\left(\sqrt{(\ell+1)(N-\ell)} + \sqrt{\ell(N-\ell+1)} \right)^2} \ . \tag{27}$$

Thus, for sufficiently small σ, $\xi_W^2 < 1$ for all $\ell \neq 0, N$.

On the other hand, by choosing

$$|\Psi\rangle = \frac{p}{N} \sum_{k=0,\,k\neq N/2}^{N} |k\rangle + (1-p)\,|N/2\rangle \ , \quad 0 < p < 1 \ , \tag{28}$$

it turns out that $\Delta^2 k = p(N+2)(N+1)/12$ so that (25) yields

$$\xi_W^2 = \frac{N(N+1)}{12(\sqrt{1-p}+q)^2} \ ,$$

where

$$q = \sqrt{\frac{p}{N^2(N+2)}} \sum_{k\neq N/2, N/2+1} \sqrt{k(N-k+1)} \ .$$

Letting $p \to 0$ one gets $|\psi\rangle \to |N/2\rangle$ and, if $N > 3$,

$$\xi_W^2 \to N(N+1)/12 > 1 \ .$$

Remark 3. The above two examples show that, when $\vec{n}_1 = \hat{z}$, the spin-squeezing parameters $\xi_{W,S}$ are not well-defined: different values for $\xi_{W,S}^2$ can be obtained by approaching a state $|k\rangle$ via different limit procedures. This fact is also of practical importance: indeed, in [14], approximations to Fock states $|k\rangle$ have been experimentally constructed that are characterized by spin-squeezing parameters $\xi_W^2 < 1$. This property arises from the fact that the approximations are $(\mathcal{A}, \mathcal{B})$-entangled states. The previous discussion shows that some care has to be taken in constructing the perturbations of $|k\rangle$; indeed, not all $(\mathcal{A}, \mathcal{B})$-entangled states arbitrarily close to it automatically have $\xi_{W,S}^2 \geq 1$. Therefore, the spin-squeezing parameters $\xi_{W,S}^2$ are not always useful for metrological applications, a better quantity is the so-called quantum Fisher information [15], which as we shall show below, is continuous and well defined for all Bosonic qubits.

3.2 Quantum Fisher Information

In a measurement of the angle θ based on the state rotation (20), the error $\Delta\theta$ given by a locally unbiased estimator E of the angle θ is bounded by (see the Appendix)

$$\Delta^2\theta \geq \frac{1}{F[\rho, J_{\vec{n}_1}]} \, , \tag{29}$$

where $F[\rho, J_{\vec{n}_1}]$ is the so-called quantum Fisher information associated with the rotation of ρ around $\vec{n}_1$.

In order to overcome the shot-noise limit $\Delta^2\theta = 1/N$, the quantum Fisher information must then be strictly larger than N. In the appendix it is also showed that, in full generality,

$$F[\rho, J_{\vec{n}_1}] \, \Delta^2 J_{\vec{n}_2} \geq \langle J_{\vec{n}_3} \rangle^2 \, , \tag{30}$$

where $\vec{n}_{1,2,3}$ is a triplet of orthogonal spatial directions. Thus, if $\langle J_{\vec{n}_3} \rangle \neq 0$, one gets the following relation between the quantum Fisher information and the squeezing parameter ξ_W^2 in (22):

$$F^{-1}[\rho, J_{\vec{n}_1}] \leq \frac{\Delta^2 J_{\vec{n}_2}}{\langle J_{\vec{n}_3} \rangle^2} = \frac{\xi_W^2}{N} \, . \tag{31}$$

In the case of distinguishable qubits, from (29) and (31) it follows that spin-squeezing, namely $\xi_W^2 < 1$, opens the possibility of achieving $\Delta^2\theta < 1/N$, thus of beating the shot-noise limit.

In the case of identical qubits and of $(\mathcal{A}, \mathcal{B})$-separable states, the right hand side of the above inequality diverges if $\vec{n}_1 = \hat{z}$ as $\langle J_{\vec{n}_3} \rangle = 0$, while it does not make sense if $\vec{n}_2 = \hat{z}$ for then also $\Delta^2 J_{\vec{n}_2} = 0$ whence, as already

observed, ξ_W^2 is not defined. However, the quantum Fisher information is always well-defined. Indeed, using (38) and (9), one finds that, if $k \neq 0, N$,

$$F[|k\rangle\langle k|, J_{\vec{n}}] = 4\,\Delta^2 J_z = (1 - n_z^2)\Big(N + 2k(N - k)\Big) > N \qquad (32)$$

for $0 \le n_z^2 < \dfrac{2k(N - k)}{N + 2k(N - k)} < 1$. In particular,

$$F[|k\rangle\langle k|, J_y] = N + 2k(N - k) \ ,$$

so that, according to (29), for all $k \neq 0, N$ the $(\mathcal{A}, \mathcal{B})$-separable pure states $|k\rangle$ might overcome the shot-noise limit, with the twin Fock state $|N/2\rangle$ yielding $F[\rho, J_y] = \mathcal{O}(N^2)$ thus permitting to approach the so-called Heisenberg limit $\Delta^2\theta = 1/N^2$.

Remark 4. Notice that even approximating a number state $|\ell\rangle$ by an experimentally more realistic superposition $|\ell, \sigma\rangle$ of states $|k\rangle$ with coefficients as in (26), may beat the shot noise limit. Indeed, one computes

$$F[|\ell, \sigma\rangle\langle\ell, \sigma|, J_y] = 4\,\Delta^2 J_y = N + 2\ell(N - \ell) + \mathcal{O}\left(e^{-\frac{1}{\sigma^2}}\right) \ ,$$

which can be kept $> N$ by suitably small σ.

Instead, making the quantum Fisher information larger than N is impossible without $(\mathcal{A}, \mathcal{B})$-non-locality; indeed, $F[|\ell\rangle\langle\ell|, J_z] = 4\,\Delta^2 J_z = 0$. Even considering the $(\mathcal{A}, \mathcal{B})$-entangled perturbation $|\ell, \sigma\rangle$ does not help; indeed,

$$F[|\ell, \sigma\rangle\langle\ell, \sigma|, J_z] = 4\,\Delta^2 J_z = 8\mathrm{e}^{-\frac{1}{\sigma^2}} + \mathcal{O}\left(\mathrm{e}^{-\frac{2}{\sigma^2}}\right) \ .$$

Therefore, the lower bound to the error in (29) becomes arbitrarily large when $\sigma \to 0$.

When dealing with $(\mathcal{A}, \mathcal{B})$-separable mixed states (4), by means of equation (37) in the Appendix, one computes [9]

$$F[\rho, J_{\vec{n}}] = (1 - n_z^2)\Big(N + 2N\langle k\rangle - \langle k^2\rangle \qquad (33)$$

$$- 4\sum_{k=0}^{N} \frac{p_k p_{k+1}}{p_k + p_{k+1}}(k + 1)(N - k)\Big) \ . \qquad (34)$$

Thus, if (32) holds for a certain $|\ell\rangle$, then, by continuity, $F[\rho, J_{\vec{n}}] > N$ for a probability distribution $\pi = \{p_k\}_{k=0}^{N}$ suitably peaked around $k = \ell$, hence able to overcome the shot-noise limit.

On the other hand, from the previous section we know that for all such mixed states $\xi_W^2 \ge \xi_S^2 \ge 1$; therefore, based on this lower bound to the squeezing parameter, we would wrongly discard such states as not useful for metrological applications.

4   Discussion

Unlike for distinguishable qubits, in the case of identical Bosonic qubits, en-
tangled states are not necessary to reach sub-shot noise accuracies in param-
eter estimation. This phenomenon is surely due to the non-local character
of the system; however, the non-locality is not in the states $|k\rangle$ which in fact
are $(\mathcal{A}, \mathcal{B})$-separable, rather in the $(\mathcal{A}, \mathcal{B})$-non-local character of the state-
rotation (20) generated by $J_{\vec{n}} \neq J_z$. Were the qubits distinguishable, neither
the state nor the rotation would carry elements of non-locality so that in or-
der to beat the shot-noise limit, the state should be turned into an entangled
one before feeding the interferometric apparatus practically implementing
the state-rotation (20). This is exactly what is done via a spin-squeezing
technique in the experiments reported in [4, 5]. Instead, the main point we
make here is that in experiments involving identical qubits, no preliminary
squeezing is needed before rotating the state. One might as well do with,
say, a state $|N/2\rangle$, the rotation around $\vec{n} \neq \hat{z}$ taking care of introducing the
necessary non-locality.

5   Appendix

A most used quantum Fisher information $F[\rho, J_{\vec{n}}]$ is given by [16]

$$F[\rho, J_{\vec{n}}] := \mathrm{tr}\left(\rho\, L^2\right) , \tag{35}$$

where L, known as symmetric logarithmic derivative, is a Hermitean operator
such that

$$\partial_\theta \rho_\theta\Big|_{\theta=0} = \frac{\rho\, L + L\, \rho}{2} = -i\,[J_{\vec{n}}\,,\,\rho] . \tag{36}$$

Given a spectral decomposition $\rho = \sum_j r_j\,|r_j\rangle\langle r_j|$, one computes

$$F[\rho, J_{\vec{n}}] = 2 \sum_{i,j\,:\,r_i \neq r_j} \frac{(r_i - r_j)^2}{r_i + r_j}\, \left|\langle r_i|J_{\vec{n}}\,|r_j\rangle\right|^2 . \tag{37}$$

From such an expression one sees that the quantum Fisher information is a
continuous function of the state ρ and that, for pure states,

$$F[|\psi\rangle\langle\psi|, J_{\vec{n}}] = 4\,\Delta_\psi^2 J_{\vec{n}} . \tag{38}$$

An estimator E is locally unbiased if $\partial_\theta \mathrm{Tr}(\rho_\theta\, E)\Big|_{\theta=0} = 1$; then, inequality (29)
follows from applying to this relation the Cauchy-Schwartz inequality for
matrices

$$\left|\mathrm{Tr}(AB)\right|^2 \leq \mathrm{Tr}(A^\dagger A)\mathrm{Tr}(B^\dagger B) .$$

Analogously, inequality (30) follows from the fact that $\mathrm{Tr}(\rho_\theta(J_{\vec{n}_2} - \langle J_{\vec{n}_2}\rangle_\theta) = 0$ implies

$$\left| \mathrm{Tr}\Big(\partial_\theta \rho_\theta (J_{\vec{n}_2} - \langle J_{\vec{n}_2}\rangle_\theta) \Big) \right| = \left| \partial_\theta \langle J_{\vec{n}_2}\rangle_\theta \right| = \left| \langle J_{\vec{n}_3}\rangle_\theta \right| \,.$$

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