Dynamical decoupling for a qubit in telegraph-like noises

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Based on the stochastic theory developed by Kubo and Anderson, we present an exact result of the decoherence function of a qubit in telegraph-like noises under dynamical decoupling control. We prove that for telegraph-like noises, the decoherence can be suppressed at most to the third order of the time and the periodic Carr-Purcell-Merboom-Gill sequences are the most efficient scheme in protecting the qubit coherence in the short-time limit.

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I. INTRODUCTION

Dynamical decoupling (DD) is a standard technique to suppress the spin decoherence, with a long tradition in magnetic resonance spectroscopy [1, 2]. Nowadays, DD is important in areas like quantum information processing, since prolonging the qubit coherence is a fundamental requirement to carry out effective operations on quantum states. The basic idea of DD is using a sequence of control pulses that frequently flip the spins to average out effects of random environmental field. DD is originated from Hahn's first spin echo experiment in 1950 [3]. After Hahn's work, more complex pulse sequences were introduced, among which a most famous example was the periodic Carr-Purcell-Merboom-Gill sequence (CPMG) [1, 4], which was initially widely used in magnetic resonance spectroscopy and, in recent years, was introduced to protect the qubit coherence in quantum information processing [5–8]. Concatenated DD (CDD) [9–14] is of special interest, since it recursively constructs pulse sequences to eliminate qubit decoherence to an arbitrary order of precision in the short-time expansion. The pulse number in CDD, however, exponentially increases. The first optimal pulse sequence in terms of the pulse number was introduced by Uhrig [15], which was a non-periodic DD. Uhrig's DD (UDD) was later shown to be universal [16, 17], in the sense that the leading n orders of the time expansion of the decoherence are eliminated by using n pulses for general finite quantum systems, i.e., systems with hard high-frequency cutoff in noise spectra. Recent experiments achieved remarkable progresses in prolonging the coherence via DD schemes [18–20].

An interesting question is how different DD schemes perform in suppressing spin or qubit decoherence caused by classical non-Gaussian noises, such as multi-state telegraph-like noises. For Gaussian noises, which are fully characterized by their second-order correlation functions, the DD control can be readily formulated as the integration of noise spectra modulated by a filtering function determined by the DD sequence [21]. For general non-Gaussian noises, however, such a formalism is not available and one has to rely on cumulant expansion and/or Gaussian approximation to numerically solve the problem. In Ref. [21], numerical calculation shows that CPMG is practically better than CDD and UDD for a telegraph noise. Later in Ref. [22], numerical search for optimal DD sequences finds solutions close to the CPMG for noises with soft cutoffs and to UDD for noises with hard cutoffs. These research indicates that CPMG may be optimal in combating certain non-Gaussian noises, but more research is still needed to reach the conclusive results.

In this paper, we present an exact result of the decoherence function under DD control, based on a stochastic theory [23–25] developed in 1950s mainly by Kubo and Anderson, when they studied the line shape of nuclear magnetic resonance (NMR) spectra. We prove that for a general multi-state telegraph-like noise, any DD schemes cannot fully eliminate the third order term in the short-time expansion of the decoherence, i.e., the decoherence function is at least of $O(t^3)$, and among all possible DD schemes, CPMG is the most optimal in suppressing the qubit decoherence in the short-time limit. Apart from the theoretical importance, these results are relevant to the spin decoherence problem in real systems such as spins in Si/SiO_2 interfaces, where the noises from coupling with dangling bonds can be approximately represented by a discrete multi-state telegraph-like noise [26].

The paper is organized as follows. In section II, we obtain the exact expression for the decoherence function under arbitrary DD control of a qubit in telegraph-like noises. In section III, we expand the decoherence function, solve the optimization problem and prove that CPMG is the global minimum solution.

II. EXACT DECOHERENCE FUNCTION

We consider a qubit (spin-1/2) under an external field and a random field $w(t)$. The Hamiltonian (in the rotating reference frame in which the external field is transformed to be zero) is

$$H = S_z w(t), \tag{1}$$

where $S_{x/y/z}$ is the spin operator along the $x/y/z$ direction. We assume $w(t)$ is a multi-state telegraph-like noise, i.e., it jumps suddenly and randomly among a set of discrete values $\{w_j\}$, with a transition rate matrix Γ, in the form

$$\frac{d}{dt} Y_j(t) = \Gamma_{jj'} Y_{j'}(t), \tag{2}$$

where $Y_j(t)$ is the probability for $w(t) = w_j$. Such a noise model is widely used in describing various stochastic physical

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process, such as the spectral diffusion of optical transitions due to atom collisions [27].

The spin coherence is characterized by the transverse polarization $x \equiv S_x + iS_y$. The equation of motion for x in the form of a so-called Kubo oscillator is

$$\dot{x}(t) = iw(t)x(t). \tag{3}$$

The ensemble average $\langle x(t)\rangle \equiv \int x(t)P[w(t)]D[w(t)]$ over a distribution of the random field $P[w(t)]$ gives the transverse polarization of the spin. Defining $Y_j(\varphi, t)$ as the probability density for finding $x(t) = e^{i\varphi}$ and $w(t) = w_j$, we can write the ensemble averages $\langle wx\rangle$ and $\langle x\rangle$ as

$$\langle wx\rangle = \sum_j \int_0^{2\pi} e^{i\varphi} w_j Y_j(\varphi, t) d\varphi, \tag{4a}$$

$$\langle x\rangle = \sum_j \int_0^{2\pi} e^{i\varphi} Y_j(\varphi, t) d\varphi. \tag{4b}$$

The probability density function satisfies the stochastic Liouville equation [24, 25]

$$\frac{\partial}{\partial t} Y_j(\varphi, t) = \sum_{j'} \Gamma_{jj'} Y_{j'}(\varphi, t) - w_j \frac{\partial}{\partial \varphi} Y_j(\varphi, t), \tag{5}$$

which contains both the sudden jumps of the random field and the precession of the spin polarization. By multiplying $e^{i\varphi}$ to both sides of Eq. (5) and integration over the phase angle, we obtain

$$\frac{d}{dt} y(t) = (\Gamma + iW)y(t), \tag{6}$$

where $y_j(t) \equiv \int_0^{2\pi} e^{i\varphi} Y_j(\varphi, t) d\varphi$, $W_{jj'} = \delta_{jj'} w_{j'}$. The spin polarization is $\langle x(t)\rangle = \sum_j y_j(t)$.

Under DD control, the spin is subjected to a sequence of flip operations between the $+z$ and $-z$ directions. In this paper, for the sake of simplicity, we assume the case of ideal π pulses. Equivalently, the flip control can be transformed to the flip of the field $w(t)$ in the reference frame rested on the flipped spin. Thus, we have the controlled Liouville equation as

$$\frac{d}{dt} y(t) = [\Gamma + iW f(t)]\, y(t), \tag{7}$$

where $f(t)$ is a step-like function jumping between $+1$ and -1 every time a control pulse is applied. The exact solution of the decoherence function under an N-pulse DD control is

$$\langle x(t)\rangle = \sum_j \left[e^{[\Gamma + (-1)^N iW] a_{N+1} t} \cdots e^{[\Gamma - iW] a_2 t} e^{[\Gamma + iW] a_1 t} y(0) \right]_j, \tag{8}$$

where $y(0)$ is the initial probability distribution of the random force, and $0 < a_n t < 1$ is the interval between the nth and $(n-1)$th pulses with $\sum a_n = 1$.

III. OPTIMAL DYNAMICAL DECOUPLING

To achieve a certain order of DD, we need to find the solution of the set of (normalized) pulse intervals $\{a_i\}$ to make the decoherence function $\langle x(t)\rangle$ equal to unity up to an error of a certain order of t in the short-time expansion. For this purpose, we expand the exponential functions in Eq. (8) into Taylor's series of t. The zeroth order term contains neither W nor Γ, and is explicitly $\sum_j y_j(0) = 1$ (the sum probability must be unity).

The higher order terms can be classified by the ordering of the matrices W and Γ. A few rules can be established to significantly simplify the expansion. First, any term starting with Γ must vanish, because of the probability conservation condition

$$\sum_j \Gamma_{jk} = 0. \tag{9}$$

Second, any term ending with Γ must vanish since

$$\Gamma y(0) = 0, \tag{10}$$

for the random force distribution is stationary. Third, the terms of the same order and containing no Γ sum to zero. This is because if Γ is set to zero (corresponding to the static inhomogeneous broadening condition), the decoherence function becomes

$$\langle x(t)\rangle = \sum_j \left[e^{iW(a_1 - a_2 + \ldots + (-1)^N a_{N+1})t} y(0) \right]_j = 1, \tag{11}$$

under the echo condition $a_1 - a_2 + ... + (-1)^N a_{N+1} = 0$. Using the three rules above, the only non-vanishing term of the decoherence up to the third order must have the form of $G_N(a_1, a_2, \ldots, a_N) W\Gamma W y(0)$. To minimize the decoherence in the third order, we just need to minimize the coefficient $G_N(a_1, a_2, \ldots, a_N)$.

With the conjecture that CPMG could be an optimal solution (as in the case of two-pulse control for Gaussian noises with hard cutoffs), we write the coefficient $G_N(a_1, a_2, \ldots, a_N)$ as a function of the deviations of the pulse positions from the CPMG timing,

$$\beta_n \equiv \alpha_n - \frac{2n-1}{2N}, \tag{12}$$

where $\alpha_n t$ is the position of the nth control pulse (i.e., $a_n \equiv \alpha_n - \alpha_{n-1}$). The echo condition is $\beta_1 - \beta_2 + \cdots + (-1)^{N+1}\beta_N = 0$, and another constraint is $1 > \alpha_N > \alpha_{N-1} > \cdots > \alpha_1 > 0$. These two conditions define the physical boundary for an N-pulse sequence. Then the coefficient of the third order term is

$$
\begin{aligned}
G_N(\beta) =& \frac{1}{12N^2} + \frac{1}{N}\left\{\left[-\beta_N + 2\beta_{N-1} - 2\beta_{N-2} + ... + (-1)^N\beta_1\right]^2 + \left[-\beta_{N-1} + 2\beta_{N-2} + \cdots + (-1)^{N-1}\beta_1\right]^2 + \cdots + \left[-\beta_1\right]^2\right\} \\
&+ \left\{2\beta_N^2\left[-\beta_N + 2\beta_{N-1} - 2\beta_{N-2} + ... + (-1)^N\beta_1\right] + 2\beta_{N-1}^2\left[-\beta_{N-1} + 2\beta_{N-2} + \cdots + (-1)^{N-1}\beta_1\right] + \cdots + 2\beta_1^2(-\beta_1)\right\} \\
\equiv& \frac{1}{12N^2} + h_N + g_N,
\end{aligned} \tag{13}
$$

where h_N denotes the second-order term, and g_N the third-order term. The first order term vanishes because of the echo condition. Since the second order is always positive ($h_N > 0$), CPMG ($\beta = 0$) must be at least a local minimum solution of the decoherence function.

If CPMG is not the global minimum, we can find a set of $\{\beta \neq 0\}$ so that $h_N(\beta) + g_N(\beta) < 0$. Since $h_N(\beta) > 0$, we must have $g_N(\beta) < 0$. Using a real number λ to scale the deviation, we get a function of the scaling factor as

$$f_N(\lambda) \equiv g_N(\lambda\beta) + h_N(\lambda\beta) = \lambda^3 g_N(\beta) + \lambda^2 h_N(\beta). \tag{14}$$

The function $f_N(\lambda)$ monotonically decreases with λ for $\lambda \geq 1$. Thus when λ is increased from 1, the DD sequence suppresses the decoherence better and better. But λ cannot be infinitely increased under the physical conditions. When λ is increased to a boundary value λ_B, the deviation $\lambda\beta$ will reach the physical boundary, at which either two adjacent pulses coincide (and become a null operation) or a pulse reaches the boundary time at 0 or t. This means a new pulse sequence with fewer pulses is obtained. Suppose the new pulse sequence has N' pulses (with $N' < N$), the coefficient can be written as

$$\frac{1}{12N^2} + f_N(\lambda_B) = \frac{1}{12N'^2} + g_{N'}(\beta') + h_{N'}(\beta'), \tag{15}$$

where β' is the deviation from the N'-pulse CPMG, and $g_{N'}$ and $h_{N'}$ are defined as in Eq. (13). Obviously,

$$g_{N'}(\beta') + h_{N'}(\beta') < 0. \tag{16}$$

Then following the same procedure as in the N-pulse sequence, we can do the scaling $\lambda'\beta'$ again to find pulse sequences performing better in suppressing the decoherence by increasing λ' from 1 to a new boundary value. Then we would find a new sequence which performs better with fewer pulses than N'. So on and so forth, we will have to conclude that the one-pulse sequence (Hahn echo) is the optimal solution among all sequences (including the multiple-pulse ones), which can be easily checked to be wrong by comparing the performance of the two-pulse CPMG and the Hahn echo. Thus, to avoid contradictions, we conclude with the following theorem:

Theorem 1 *For an arbitrary multi-state telegraph-like noise, CPMG is the global minimum solution of the decoherence function among all DD sequences of the same number of pulses.*

A corollary which can be directly derived from Eq. (13) is
Corollary 1 *The decoherence of a qubit in a multi-state telegraph-like noise can not be suppressed by DD beyond the third order of the short-time expansion.*

This conclusion is consistent with previous numerical solutions in Ref. [22], which give pulse sequences close to CPMG for boson baths with power-law high-frequency cutoffs. Actually, the second order correlation function of the telegraph-like noise

$$\langle w(t)w(t')\rangle = \sum_j \left[W e^{-\Gamma(t-t')} W y(0)\right]_j, \tag{17}$$

has the form of exponential functions. The Fourier transformation to the frequency domain then has a power-law decay profile in the high frequency end. Similarly, the higher order correlation functions (which in general cannot be factorized into the second order correlation functions as for Gaussian noises) also have the form of exponential functions. Such exponential function form of the correlation functions results from the sudden jumps in the telegraph-like noises, which, in the physical nature, is induced by instantaneous (Markovian) collisions in the bath. It is the existence of a very short timescale in the system (the collision memory time), or in other words, a very large energy scale, that limits the performance of any DD schemes to the third order of precision.

IV. CONCLUSIONS

In summary, we have derived an exact expression for the decoherence function of a qubit in arbitrary multi-state telegraph-like noises, based on the stochastic theory. We prove that CPMG is the globally optimal solutions among all possible dynamical decoupling sequences of the same number of pulses to suppress the decoherence in the short time limit. Because of the instantaneous random jumps in the noises, the decoherence cannot be eliminated beyond the third order of the short time.

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