IU-TH-6

Automatic Hermiticity

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We study the Hamiltonian that is not at first hermitian. Requirement that a measurement shall not change one Hamiltonian eigenstate into another one with a different eigenvalue imposes that an inner product must be defined so as to make the Hamiltonian normal with regard to it. After a long time development with the non-hermitian Hamiltonian, only a subspace of possible states will effectively survive. On this subspace the effect of the anti-hermitian part of the Hamiltonian is suppressed, and the Hamiltonian becomes hermitian. Thus hermiticity emerges automatically, and we *have no reason to maintain that at the fundamental level the Hamiltonian should be hermitian.* We also point out a possible misestimation of a past state by extrapolating back in time with the hermitian Hamiltonian. It is a seeming past state, not a true one.

PACS numbers: 03.65.Ta, 03.65.-w, 01.55.+b

Introduction In quantum theory the action S is real and thought to be more fundamental than the integrand $\exp(iS)$ of the Feynman Path Integral. But if we assume that the integrand is more fundamental than the action in quantum theory, then it is naturally thought that since the integrand is complex, the action also could be complex. Based on this assumption and other related works in some general relativity inspired backward causation developments [1] and the non-locality explanation of fine-tuning problems [2], the complex action theory has been studied intensively by one of the authors(H.B.N) and Ninomiya[3]. Indeed, many interesting suggestions have been made for Higgs mass[4], quantum mechanical philosophy[5], some fine-tuning problems[6, 7] and black holes[8]. In refs.[3–8] they studied a future-included version, that is to say, the theory including not only a past time but also a future time as an integration interval of time. In contrast to the above references, in this letter we consider a future-not-included version.

We shall study a system defined by the non-hermitian Hamiltonian H, which is correlated to the complex action, and look at the time-development of some state. However, as we know, the time development operator defined in terms of the non-hermitian Hamiltonian is non-unitary, and thus the probability conservation is not held. Furthermore, since the eigenstates of the Hamiltonian are not orthogonal, a transition that should not be possible could be measured. From these properties it does not look a physically reasonable theory. But, contrary to our naive expectation, we shall find that it could be a physically reasonable theory via two procedures.

The first procedure is to define a physically reasonable inner product I_Q such that the eigenstates of the Hamiltonian get orthogonal with regard to it, and thus it gives us a true probability for a transition from some state to another. As we shall see later, I_Q makes the Hamiltonian normal with regard to it. In other words I_Q has to be defined for consistency so that the Hamiltonian -even if it cannot be made hermitian - at least be normal. We explain how a reasonable physical assumption about the probabilities leads to the proper inner product I_Q, and define a hermiticity with regard to I_Q, Q-hermiticity.

The second procedure is to use a mechanism of suppressing the effect of the anti-hermitian part of the Hamiltonian H after a long time development. This is speculated in ref.[9]. In this letter we shall explicitly show the mechanism with the help of the proper inner product I_Q. For the states with high imaginary part of eigenvalues of H, the factor $\exp\left(-\frac{i}{\hbar}H(t-t_0)\right)$ will exponentially grow with t and faster the higher the eigenvalues are. After a long time the states with the highest imaginary part of eigenvalues of H get more favored to result than others. That is to say, the effect of the imaginary part, which shall be shown to correspond to the anti-Q-hermitian part of H, gets attenuated. Utilizing this effect to normalize the state, we can effectively obtain a Q-hermitian Hamiltonian.

Physical significance of an inner product The Born rule of quantum mechanics is well-known in the form: When a quantum mechanical system prepared in a state $|i\rangle$ at time t_i time-develops into $|i(t_f)\rangle = e^{-\frac{i}{\hbar}H(t_f-t_i)}|i\rangle$ at time t_f, we will measure it in a state $|f\rangle$ with the probability $P_{f \text{ from } i} = |\langle f|i(t_f)\rangle|^2$. We note that the probability depends on how we define an inner product of the Hilbert space. A usual inner product is defined as a sesquilinear form. We denote it as $I(|f\rangle, |i(t_f)\rangle) = \langle f|i(t_f)\rangle$. It is $|I(|f\rangle, |i(t_f)\rangle)|^2$ that we measure by seeing how often we get $|f\rangle$ from $|i(t_f)\rangle$. Measuring the transition of superposition like $c_1|a\rangle + c_2|b\rangle$ repeatedly, we can extract the whole form of $I(|f\rangle, |i(t_f)\rangle)$ of any two states by using the sesquilinearity.

To consider an inner product in our theory with the non-hermitian Hamiltonian H, we first diagonalize H by using a non-unitary operator P as $H = PDP^{-1}$. We introduce an orthonormal basis $|e_i\rangle (i = 1, \ldots)$ satisfying $\langle e_i|e_j\rangle = \delta_{ij}$ by $D|e_i\rangle = \lambda_i|e_i\rangle$, where $\lambda_i (i = 1, \ldots)$ are generally complex. We also introduce the eigenstates $|\lambda_i\rangle$ of H by $|\lambda_i\rangle = P|e_i\rangle$, which obeys $H|\lambda_i\rangle = \lambda_i|\lambda_i\rangle$. We note that $|\lambda_i\rangle$ are not orthogonal to each other in a usual inner product I, $\langle\lambda_i|\lambda_j\rangle \neq \delta_{ij}$.

Since we are prepared, let us apply the usual inner product I to our theory with the non-hermitian Hamiltonian H, and consider a transition from an eigenstate $|\lambda_i\rangle$ to another $|\lambda_j\rangle$ $(i \neq j)$ fast in time Δt. Then, though H cannot bring the system from one eigenstate $|\lambda_i\rangle$ to another one $|\lambda_j\rangle$ $(i \neq j)$, the transition can be measured, that is to say, $|I(|\lambda_j\rangle, \exp\left(-\frac{i}{\hbar}H\Delta t\right)|\lambda_i\rangle)|^2 \neq 0$, since the two eigenstates are not orthogonal to each other. We believe that such a transition should be prohibited in a reasonable theory, based on the philosophy that a measurement - even performed in a short time - is fundamentally a physical development in time. Thus we think that the eigenstates have to be orthogonal to each other.

Since we are physically entitled to require that a truly functioning measurement procedure must necessarily have reasonable probabilistic results, we attempt to construct a proper inner product $I_Q(|f\rangle, |i\rangle) = \langle f|_Q\, i\rangle$ with the property that the eigenstates $|\lambda_i\rangle$ and $|\lambda_j\rangle$ get orthogonal to each other,

$$I_Q(|\lambda_i\rangle, |\lambda_j\rangle) = \delta_{ij}. \tag{1}$$

We believe that the true probability is given by such a proper inner product I_Q, based on which the Hamiltonian is conserved even if it is not hermitian and typically has complex eigenvalues. This condition applies to not only the eigenstates of the Hamiltonian but also those of any other conserved quantities. The transition from an eigenstate of such a conserved quantity to another eigenstate with a different eigenvalue should be prohibited in a reasonable theory.

A proper inner product and hermitian conjugate Let us define a proper inner product I_Q of some states $|\psi_1\rangle$ and $|\psi_2\rangle$ by

$$I_Q(|\psi_2\rangle, |\psi_1\rangle) = \langle\psi_2|_Q\psi_1\rangle \equiv \langle\psi_2|Q|\psi_1\rangle, \tag{2}$$

where Q is some operator chosen appropriately. Q has to correspond to a unit operator if the non-hermitian Hamiltonian is shifted to a hermitian one. In the usual real action theory the usual inner product I is defined to satisfy $\langle\psi_1(t)|\psi_2(t)\rangle = \langle\psi_2(t)|\psi_1(t)\rangle^*$. Hence we impose a similar relation on I_Q as

$$\langle\psi_1(t)|_Q\psi_2(t)\rangle = \langle\psi_2(t)|_Q\psi_1(t)\rangle^*. \tag{3}$$

Then we obtain a condition $Q^\dagger = Q$, namely, Q has to be hermitian.

Via the inner product I_Q, we define the corresponding hermitian conjugate $\dagger_Q$ for some operator A by

$$\langle\psi_2|_Q A|\psi_1\rangle^* = \langle\psi_1|_Q\hat{A}^{\dagger_Q}|\psi_2\rangle. \tag{4}$$

Since the left-hand side can be expressed as $\langle\psi_2|_Q\hat{A}|\psi_1\rangle^* = \langle\psi_1|\hat{A}^\dagger Q^\dagger|\psi_2\rangle$, an explicit form of the Q-hermitian conjugate of A is given by

$$A^{\dagger_Q} = Q^{-1}A^\dagger Q. \tag{5}$$

$\dagger_Q$ is introduced for operators, but we can formally define $\dagger_Q$ for kets and bras, too. We define $\dagger_Q$ for kets and bras as $|\psi_1\rangle^{\dagger_Q} \equiv \langle\psi_1|_Q$ and $(\langle\psi_2|_Q)^{\dagger_Q} \equiv |\psi_2\rangle$. Then we can manipulate $\dagger_Q$ like a usual hermitian conjugate $\dagger$. When A satisfies $A^{\dagger_Q} = A$, we call A Q-hermitian. This is the definition of the Q-hermiticity with regard to the inner product I_Q. Since this relation can be expressed as $QA = (QA)^\dagger$, when A is Q-hermitian, QA is hermitian, and vice versa.

If some operator A can be diagonalized as $A = P_A D_A P_A^{-1}$, then Q-hermitian conjugate of A is expressed as $A^{\dagger_Q} = Q^{-1}(P_A^\dagger)^{-1}D_A^\dagger P_A^\dagger Q$. If we choose Q as $Q = (P_A^\dagger)^{-1}P_A^{-1}$, which satisfies $Q^\dagger = Q$, we have $A^{\dagger_Q} = P_A D_A^\dagger P_A^{-1}$. Therefore, if the diagonal components of D_A are real, namely, $D_A^\dagger = D_A$, then A is shown to be Q-hermitian. In the following we define Q by

$$Q = (P^\dagger)^{-1}P^{-1} \tag{6}$$

with the diagonalizing matrix P of the non-hermitian Hamiltonian H. Thus the inner product I_Q we shall use from now on depends on H via Q.

The Hamiltonian is Q-normal. To prove that the non-hermitian Hamiltonian H is Q-normal, i.e. normal with regard to the inner product I_Q, we first define

$$\text{“}P^{\dagger_Q}\text{”} \equiv \begin{pmatrix} \langle\lambda_1|_Q \\ \langle\lambda_2|_Q \\ \vdots \end{pmatrix} \tag{7}$$

by using the diagonalizing operator P of H, which has a structure as $P = (|\lambda_1\rangle, |\lambda_2\rangle, \ldots)$, where $|\lambda_i\rangle$ are eigenstates of H. We note that “$P^{\dagger_Q}$” is defined by using the Q-hermitian conjugate of kets, so “$P^{\dagger_Q}$” $\neq Q^{-1}P^\dagger Q$. Then we see that “$P^{\dagger_Q}$”$P = \mathbf{1}$, namely, “$P^{\dagger_Q}$” $= P^{-1}$. Hence we can say that P is Q-unitary.

Next we consider the relation “$P^{\dagger_Q}$”$HP = D$. The (i,j)-component of this relation in $|\lambda_i\rangle$ basis is written as $\langle\lambda_i|_Q H|\lambda_j\rangle = \lambda_i\delta_{ij}$. Taking the complex conjugate, we obtain $\langle\lambda_j|_Q H^{\dagger_Q}|\lambda_i\rangle = \lambda_i^*\delta_{ij}$, that is to say, $\langle\lambda_i|_Q H^{\dagger_Q}|\lambda_j\rangle = \lambda_i^*\delta_{ij}$. This is written in the operator form as “$P^{\dagger_Q}$”$H^{\dagger_Q}P = D^\dagger$. Therefore we obtain

$$[H, H^{\dagger_Q}] = P[D, D^\dagger]P^{-1} = 0. \tag{8}$$

Thus we see that H is Q-normal. In other words we can say that the inner product I_Q is defined so that H is normal with regard to it.

Furthermore for later convenience we decompose H as $H = H_{Qh} + H_{Qa}$, where $H_{Qh} = \frac{H+H^{\dagger_Q}}{2}$ and $H_{Qa} = \frac{H-H^{\dagger_Q}}{2}$ are Q-hermitian and anti-Q-hermitian parts of H respectively. If we decompose D as $D = D_R + iD_I$, where the diagonal components of D_R and D_I are the real and imaginary parts of the diagonal components of D respectively, H_{Qh} and H_{Qa} can be expressed as $H_{Qh} = PD_RP^{-1}$ and $H_{Qa} = iPD_IP^{-1}$.

Normalization of $|\psi\rangle$ and expectation value We consider some state $|\psi(t)\rangle$, which obeys the Schrödinger equation $i\hbar\frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle$. Normalizing it as $|\psi(t)\rangle_N \equiv \frac{1}{\sqrt{\langle\psi(t)|_Q\ \psi(t)\rangle}}|\psi(t)\rangle$, we define the expectation value of some operator $\mathcal{O}$ by

$$\begin{aligned}\bar{\mathcal{O}}_Q(t) &\equiv {}_N\langle\psi(t)|_Q\mathcal{O}|\psi(t)\rangle_N \\ &= {}_N\langle\psi(t_0)|_Q\mathcal{O}_{QH}(t-t_0)|\psi(t_0)\rangle_N, \qquad (9)\end{aligned}$$

where we have introduced the time-dependent operator in the Heisenberg picture, $\mathcal{O}_{QH}(t-t_0) \equiv e^{\frac{i}{\hbar}H^{\dagger_Q}(t-t_0)}\mathcal{O}e^{-\frac{i}{\hbar}H(t-t_0)}$. Since the normalization factor depends on time t, $|\psi(t)\rangle_N$ does not obeys the Schrödinger equation, but

$$\begin{aligned}&i\hbar\frac{d}{dt}|\psi(t)\rangle_N \\ &= H|\psi(t)\rangle_N - {}_N\langle\psi(t)|_Q H_{Qa}|\psi(t)\rangle_N|\psi(t)\rangle_N. \quad (10)\end{aligned}$$

In addition $\mathcal{O}_{QH}$ does not obey the Heisenberg equation, but

$$\begin{aligned}\frac{d}{dt}\mathcal{O}_{QH} &= \frac{i}{\hbar}\left(H^{\dagger_Q}\mathcal{O}_{QH} - \mathcal{O}_{QH}H\right) \\ &= \frac{1}{i\hbar}\left([\mathcal{O}_{QH}, H_{Qh}] + \{\mathcal{O}_{QH}, H_{Qa}\}\right). \quad (11)\end{aligned}$$

In both of the equations we find the effect of H_{Qa}, the anti-Q-hermitian part of the Hamiltonian H, though it seems to disappear in the classical limit. But with the second procedure we explain next, we shall find that in both of the equations the effect of H_{Qa} disappears.

The mechanism for suppressing the anti-Q-hermitian part of the Hamiltonian To show the mechanism for suppressing the effect of H_{Qa}, we shall see the time development of $|\psi(t)\rangle$ explicitly. We introduce $|\psi'(t)\rangle$ by $|\psi'(t)\rangle = P^{-1}|\psi(t)\rangle$, and expand it as $|\psi'(t)\rangle = \sum_i a_i(t)|e_i\rangle$. Then $|\psi(t)\rangle$ can be written in an expanded form as $|\psi(t)\rangle = \sum_i a_i(t)|\lambda_i\rangle$. Since $|\psi'(t)\rangle$ obeys $i\hbar\frac{d}{dt}|\psi'(t)\rangle = D|\psi'(t)\rangle$, the time development of $|\psi(t)\rangle$ from some time t_0 is calculated as

$$\begin{aligned}|\psi(t)\rangle &= Pe^{-\frac{i}{\hbar}D(t-t_0)}|\psi'(t_0)\rangle \\ &= \sum_i a_i(t_0)e^{\frac{1}{\hbar}(\mathrm{Im}\lambda_i - i\mathrm{Re}\lambda_i)(t-t_0)}|\lambda_i\rangle. \quad (12)\end{aligned}$$

$\mathrm{Im}\lambda_i$ corresponds to the anti-Q-hermitian part of the Hamiltonian since $H_{Qa} = iPD_IP^{-1}$. As for the anti-Q-hermitian part H_{Qa}, we can crudely imagine that some of $\mathrm{Im}\lambda_i$ take the maximum value B. We denote the corresponding subset of $\{i\}$ as A. Then we can Taylor-expand H_{Qa} around its maximum and get a good approximation to the practical outcome of the model. In the Taylor-expansion we do not have the linear term because we expand it near the maximum, so we get only non-trivial terms of *second order*. In this way H_{Qa} gets a constant in the first approximation, and thus it is not so important observationally. Therefore, if a long time has passed, namely for large $t-t_0$, the states with $\mathrm{Im}\lambda_i|_{i\in A}$ survive and contribute most in the sum.

To show how $|\psi(t)\rangle$ is effectively described for large $t-t_0$, we introduce a diagonalized Hamiltonian $\tilde{D}_R$ as

$$\langle e_i|\tilde{D}_R|e_j\rangle \equiv \begin{cases}\langle e_i|D_R|e_j\rangle = \delta_{ij}\mathrm{Re}\lambda_i & \text{for} \quad i\in A, \\ 0 & \text{for} \quad i\notin A,\end{cases} \quad (13)$$

and define H_{eff} by $H_{\mathrm{eff}} \equiv P\tilde{D}_RP^{-1}$. H_{eff} is Q-hermitian, $H_{\mathrm{eff}}^{\dagger_Q} = H_{\mathrm{eff}}$, and satisfies $H_{\mathrm{eff}}|\lambda_i\rangle = \mathrm{Re}\lambda_i|\lambda_i\rangle$. Furthermore, we introduce $|\tilde{\psi}(t)\rangle \equiv \sum_{i\in A} a_i(t)|\lambda_i\rangle$. Then $|\psi(t)\rangle$ is approximately estimated as

$$\begin{aligned}|\psi(t)\rangle &\simeq e^{\frac{1}{\hbar}B(t-t_0)}\sum_{i\in A}a_i(t_0)e^{-\frac{i}{\hbar}\mathrm{Re}\lambda_i(t-t_0)}|\lambda_i\rangle \\ &= e^{\frac{1}{\hbar}B(t-t_0)}e^{-\frac{i}{\hbar}H_{\mathrm{eff}}(t-t_0)}|\tilde{\psi}(t_0)\rangle \\ &= |\tilde{\psi}(t)\rangle. \quad (14)\end{aligned}$$

The factor $e^{\frac{1}{\hbar}B(t-t_0)}$ included in $|\tilde{\psi}(t)\rangle$ can be dropped out by normalization. Thus we have effectively obtained a Q-hermitian Hamiltonian H_{eff} after a long time development though our theory is described by the non-hermitian Hamiltonian H at first. Indeed the normalized state $|\psi(t)\rangle_N \simeq \frac{1}{\sqrt{\langle\tilde{\psi}(t)|_Q\ \tilde{\psi}(t)\rangle}}|\tilde{\psi}(t)\rangle \equiv |\tilde{\psi}(t)\rangle_N$ time-develops as $|\tilde{\psi}(t)\rangle_N = e^{-\frac{i}{\hbar}H_{\mathrm{eff}}(t-t_0)}|\tilde{\psi}(t_0)\rangle_N$. We see that the time dependence of the normalization factor has disappeared due to the Q-hermiticity of H_{eff}. Thus $|\tilde{\psi}(t)\rangle_N$, the normalized state by using the inner product I_Q, obeys the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}|\tilde{\psi}(t)\rangle_N = H_{\mathrm{eff}}|\tilde{\psi}(t)\rangle_N. \quad (15)$$

On the other hand, the expectation value is given by $\bar{\mathcal{O}}_Q(t) \simeq {}_N\langle\tilde{\psi}(t)|_Q\mathcal{O}|\tilde{\psi}(t)\rangle_N = {}_N\langle\tilde{\psi}(t_0)|_Q\tilde{\mathcal{O}}_{QH}(t-t_0)|\tilde{\psi}(t_0)\rangle_N$, where we have defined a time-dependent operator $\tilde{\mathcal{O}}_{QH}$ in the Heisenberg picture by $\tilde{\mathcal{O}}_{QH}(t-t_0) \equiv e^{\frac{i}{\hbar}H_{\mathrm{eff}}(t-t_0)}\mathcal{O}e^{-\frac{i}{\hbar}H_{\mathrm{eff}}(t-t_0)}$. We see that $\tilde{\mathcal{O}}_{QH}$ obeys the Heisenberg equation

$$\frac{d}{dt}\tilde{\mathcal{O}}_{QH}(t-t_0) = \frac{i}{\hbar}[H_{\mathrm{eff}}, \tilde{\mathcal{O}}_{QH}(t-t_0)]. \quad (16)$$

As we have seen above, the non-hermitian Hamiltonian H has become a hermitian one H_{eff} automatically with the proper inner product I_Q and the mechanism of suppressing the anti-hermitian part of H after a long time development. If H is written in a local form like $H = \frac{1}{2m}p^2 + V(q)$, does the locality remain even after H becomes hermitian? It is not clear, but for the moment let us assume that the hermitian Hamiltonian H_{eff} has a local expression like $H_{\mathrm{eff}} \simeq \frac{1}{2m_{\mathrm{eff}}}p_{\mathrm{eff}}^2 + V_{\mathrm{eff}}(q_{\mathrm{eff}})$, and see probability conservation. Besides a usual q_{eff}-representation of the state $|\tilde{\psi}(t)\rangle_N$, $\tilde{\psi}(q_{\mathrm{eff}}) \equiv \langle q_{\mathrm{eff}}|\tilde{\psi}(t)\rangle_N$, we introduce $\tilde{\psi}_Q(q_{\mathrm{eff}}) \equiv \langle q_{\mathrm{eff}}|_Q\tilde{\psi}(t)\rangle_N$, and define a probability density by

$$\rho_{\mathrm{eff}} = \tilde{\psi}_Q(q_{\mathrm{eff}})^*\tilde{\psi}(q_{\mathrm{eff}}) = {}_N\langle\tilde{\psi}(t)|_Q q_{\mathrm{eff}}\rangle\langle q_{\mathrm{eff}}|\tilde{\psi}(t)\rangle_N. \quad (17)$$

Then, since we have $i\hbar\frac{\partial}{\partial t}\tilde{\psi}(q_{\rm eff}) = H_{\rm eff}\tilde{\psi}(q_{\rm eff})$ and $i\hbar\frac{\partial}{\partial t}\tilde{\psi}_Q(q_{\rm eff}) = H_{\rm eff}^*\tilde{\psi}_Q(q_{\rm eff})$, we obtain a continuity equation $\frac{\partial\rho_{\rm eff}}{\partial t} + \frac{\partial}{\partial q_{\rm eff}} j_{\rm eff}(q_{\rm eff}, t) = 0$, where $j_{\rm eff}(q_{\rm eff}, t)$ is a probability current density defined by $j_{\rm eff}(q_{\rm eff}, t) = \frac{i\hbar}{2m_{\rm eff}}\left(\frac{\partial}{\partial q_{\rm eff}}\tilde{\psi}_Q^*\tilde{\psi} - \tilde{\psi}_Q^*\frac{\partial}{\partial q_{\rm eff}}\tilde{\psi}\right)$. Thus we see that if $H_{\rm eff}$ has a local expression, we have the probability conservation $\frac{d}{dt}\int\rho_{\rm eff}dq_{\rm eff} = 0$.

Discussion In this letter we have studied a system described by the non-hermitian Hamiltonian H. For a measurement to be physically reasonable, we have introduced the proper inner product I_Q so that H gets normal with regard to it, and defined Q-hermiticity, i.e. hermiticity with regard to I_Q. Next we have explicitly presented the mechanism for suppressing the effect of the anti-Q-hermitian part of H after the long time development, and thus effectively obtained the hermitian Hamiltonian $H_{\rm eff}$. This result suggests that we have no reason to maintain that at the fundamental level the Hamiltonian should be hermitian. Furthermore we have seen that if $H_{\rm eff}$ is written in a local form, we obtain the continuity equation leading to probability conservation.

Finally let us discuss an estimation of a state at an early time t_1. It is expressed as

$$|\psi_{true}(t_1)\rangle_N = e^{-\frac{i}{\hbar}H(t_1-t_0)}|\psi(t_0)\rangle_N. \tag{18}$$

But if a historian who lives at a late time t was asked about the state at t_1, he would extrapolate back in time from his own time t by using the phenomenological Hamiltonian $H_{\rm eff} = H_{\rm eff}^{\dagger_Q}$ rather than the fundamental one H, because at the late time he would only know the hermitian Hamiltonian. Thus he would specify an early state at time t_1 as

$$|\psi_{historian}(t_1)\rangle_N = e^{-\frac{i}{\hbar}H_{\rm eff}(t_1-t)}e^{-\frac{i}{\hbar}H(t-t_0)}|\psi(t_0)\rangle_N. \tag{19}$$

This is a false picture and different from the true state (18) because $H_{\rm eff} \neq H$. Actually, the seeming past state $|\psi_{historian}(t_1)\rangle_N$ will be mainly a superposition of the eigenstates correlated to the subset A. Since the set of the eigenstates correlated to the subset A is much smaller than that of all the eigenstates of H, the seeming state $|\psi_{historian}(t_1)\rangle_N$ would look like necessarily having come from a special rather tiny part of the full Hilbert space in the fundamental theory. In other words, it would look to the historian that the universe necessarily had begun in a state inside the rather tiny subspace of the fundamental Hilbert space with the highest imaginary part of the eigenvalues of the Hamiltonian. That is to say, the fundamentally true initial state $|\psi(t_0)\rangle_N$ tends to be hidden from the historian at the late time more and more as the time t gets later and later. This story implies that if our universe had begun with a non-hermitian Hamiltonian at first in some fundamental theory, then we could misestimate the early state at the time t_1 by using the hermitian Hamiltonian to extrapolate back in time.

This work was supported by Danish Natural Science Research Council (FNU, Denmark) and Grant-in-Aid for Scientific Research (Nos.18740127 and 21740157) from the Ministry of Education, Culture, Sports, Science and Technology (MEXT, Japan), and begun during the stay of one of the authors (K.N.) at NBI by using the JSPS Researcher Exchange Program FY2008. K.N. would like to thank Jan Ambjorn, who accepted his stay initially, Klara Pavicic and all the other members and visitors of NBI for their kind hospitality.

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