Turk J Elec Eng & Comp Sci, Vol.18, No.5, 2010, © TÜBİTAK
doi:10.3906/elk-1009-774

Robust feedback design for nonlinear systems: a survey

Alberto ISIDORI
Dipartimento di Informatica e Sistemistica Universita di Roma Via Ariosto 25
Rome - ITALY
e-mail: albisidori@dis.uniroma1.it

1. Introduction

A central problem in control theory is the design of feedback controllers so as to have certain outputs of a given plant *to track* prescribed reference trajectories. In any realistic scenario, this control goal has to be achieved in spite of a good number of phenomena which would cause the system to behave differently from that expected. These phenomena could be endogenous, for instance parameter variations, or exogenous, such as additional undesired inputs affecting the behaviour of the plant. For a time-invariant, finite-dimensional system, the problem in question can be formally cast as follows. Consider a controlled plant modelled by equations of the form

$$\begin{aligned} \dot{x} &= f(d, x, u) \\ z &= h(d, x) \\ y &= k(d, x) \end{aligned} \tag{1}$$

in which x is a vector of state variables, u is a vector of inputs to be used for *control* purposes, d is a vector of inputs which cannot be controlled and thus are viewed as undesired external *disturbances*, z is the vector of outputs that need to be *controlled* and y is a vector of outputs that are available for *measurement*, hence used to feed the device that supplies the control action. Let $z_{\rm ref}(t)$ denote the prescribed behavior, in time, that the controlled output $z(t)$ of (1) is required to reproduce. A way to address the design problem described above is to seek a controller, which receives $y(t)$ as input and produces $u(t)$ as output, able to guarantee that, in the resulting closed-loop system, $z(t)$ *asymptotically tracks* $z_{\rm ref}(t)$, i.e.,

$$\lim_{t\to\infty} \|z(t) - z_{\rm ref}(t)\| = 0\,. \tag{2}$$

Of course, as a generally accepted prerequisite to this specific design goal, as well as to any other design goal, the controller must also be able to secure a "proper behavior" of all the internal (state) variables which characterize the closed-loop system, not just the components of the controlled output z. A way to express this prerequisite is to impose that all these variables remain *bounded* when $d(t)$ and $z_{\rm ref}(t)$ are bounded.

The ability of successfully addressing this problem very much depends on how much the controller is allowed to know about the external stimuli $d(t)$ and $z_{\rm ref}(t)$ and on their specific shape. In the ideal situation in which $d(t)$ and $z_{\rm ref}(t)$ are exactly known, ahead of time, the design problem indeed looks much simpler. This is, though, only an extremely optimistic situation which does not represent, in any circumstance, a realistic scenario. The other extreme situation is the one in which nothing is known about these stimuli, but some loose bounds which they are known to satisfy. In this, pessimistic, scenario the best one could hope for is to guarantee certain ultimate bounds for the distance between $z(t)$ and $z_{\rm ref}(t)$, and not the fulfilment of a sharp goal such as (2). A more comfortable, intermediate, situation is the one in which $d(t)$ and $z_{\rm ref}(t)$ are only known *to belong to a fixed family* of functions of time, for instance the family of all solutions obtained from a fixed differential equation as the corresponding initial conditions are allowed to vary on a given set. This way of thinking of the external stimuli covers a number of cases of major practical relevance such as the classical problem of the set point control, the problem of active suppression of harmonic disturbances of unknown amplitude, phase and even frequency, the synchronization of nonlinear oscillations, and similar others. Once the components of $d(t)$ and $z_{\rm ref}(t)$ have been thought of in these terms, i.e., as members of a family of solutions obtained from a fixed differential equation, there is no reason to keep them separate in the model of the plant. In fact, they can be viewed as components of a larger vector of *exogenous inputs* $w = \mathrm{col}(d, z_{\rm ref})$. At the same time, in the model (1) the controlled output z can be replaced by the *tracking error*, i.e., by the difference $e(t) = z(t) - z_{\rm ref}(t)$ which, as the equations above show, can be expressed as a function of of the state x and of the exogenous input w. In this way, system (1) is replaced by a system modelled in the form

$$\begin{array}{rcl} \dot{x} &=& f_{\rm P}(w,x,u) \\ e &=& h_{\rm P}(w,x) \\ y &=& k_{\rm P}(w,x)\,. \end{array} \qquad (3)$$

In the setting described above, this model of the controlled plant is complemented by a model of all exogenous inputs, expressed in the form of a fixed, autonomous, system

$$\dot{w} = s(w)\,. \qquad (4)$$

As its initial condition $w(0)$ ranges on some prescribed set W, this system provides a model of all possible exogenous signals to be taken into account in the design problem: reference outputs required to be tracked, as well as disturbance inputs that need to be rejected. In this context, system (4) is referred to as the *exosystem*.

The design problem is to find a feedback controller

$$\begin{array}{rcl} \dot{\eta} &=& f_{\rm C}(\eta,y) \\ u &=& h_{\rm C}(\eta,y) \end{array} \qquad (5)$$

driven by the measured output y and producing the control input u, yielding a closed-loop system in which all trajectories are bounded and the regulated variable e asymptotically decays to 0 as $t \to \infty$. Given that the controlled plant (3) is nonlinear, the possibility of successfully handling this design problem is in general influenced by the specific choice of the sets of admissible initial conditions. Solving the problem for all possible values of $x(0), w(0), \eta(0)$ might be impractical and perhaps unnecessary. A reasonable scenario, though, is the one in which the sets X, W and H of admissible initial conditions $x(0), w(0)$ and $\eta(0)$ are *a priori* fixed, but otherwise arbitrary, *compact* sets. Cast in these terms, the design problem is to find a feedback controller such that, for all initial conditions in W, X, H the trajectories of the composite system (3) − (4) − (5) are bounded and $\lim_{t\to\infty} e(t) = 0$, uniformly in the initial condition (on $W \times X \times H$).

The problem thus defined encompasses a large number of major problems in feedback design (see e.g. [1, 2, 3]). If the model (5) is not explicitly dependent on a vector w of exogenous inputs, the problem reduces to the standard (possibly non-robust) problem of feedback stabilization of an equilibrium or, in general, of a compact invariant set (in which case e coincides with the distance of x from that equilibrium or, respectively, from that invariant set). If the vector w of exogenous inputs obeys the trivial dynamics $\dot{w} = 0$, it includes the problem of robust stabilization, in the presence of constant uncertain parameters, equilibrium or of a compact invariant set. In general, if some of the components of w have nontrivial dynamics, the setup is suitable to handle problems of asymptotic tracking and disturbance rejection of modelled exogenous input, robustly in the presence of parameter uncertainties. Finally, note in the case the exosystem (4) admits a decomposition of the form

$$\begin{aligned} \dot{w}_1 &= 0 \\ \dot{w}_2 &= s_2(w_1, w_2)\,, \end{aligned}$$

the setup include also the case in which the model of those exogenous inputs whose dynamic is nontrivial is itself affected by parameter uncertainties. This is the case, for instance, in the problem of rejecting harmonic disturbances of unknown frequency.

2. Tools for asymptotic analysis

2.1. Limit and steady-state behavior

2.1.1. Limit sets

In the analysis of dynamical systems, it is often important to characterize motions which possess some property of recurrence. Constant motions (which correspond to equilibria) or periodic motions are special cases of recurrent motions. When the variables which characterize the motions of a system are either constant or periodic, a system is usually said to be in *steady state*. In general, the steady state behavior of a dynamical system can be viewed as a kind of *limit* behavior, approached either as the *actual* time t tends to $+\infty$ or, alternatively, as the *initial* time t_0 tends to $-\infty$. Relevant, in this respect, are certain concepts introduced by G.D.Birkhoff in [4]. In particular, a fundamental role is played by the concept of ω-limit set of a given point, defined as follows. Consider an *autonomous* dynamical system

$$\dot{x} = f(x) \tag{6}$$

and let $x(t, x_0)$ denote its flow. A point x is said to be an ω-limit *point* of the motion $x(t, x_0)$ if there exists a sequence of times $\{t_k\}$, with $\lim_{k\to\infty} t_k = \infty$, such that

$$\lim_{k\to\infty} x(t_k, x_0) = x\,.$$

The ω-limit *set* of a point x_0, denoted $\omega(x_0)$, is *the union* of all ω-limit points of the motion $x(t, x_0)$.

If x_e is an asymptotically stable equilibrium, then $x_\mathrm{e} = \omega(x_0)$ for all x_0 in a neighborhood of x_e. However, in general, an ω-limit point *is not* necessarily a limit of $x(t, x_0)$ as $t \to \infty$, because the function in question may not admit any limit as $t \to \infty$. It happens though, that if the motion $x(t, x_0)$ is *bounded*, then $x(t, x_0)$ asymptotically approaches *the set* $\omega(x_0)$.

Lemma 1 *Suppose there is a number M such that $|x(t, x_0)| \leq M$ for all $t \geq 0$. Then, $\omega(x_0)$ is a nonempty compact connected set, invariant under (6). Moreover, the distance of $x(t, x_0)$ from $\omega(x_0)$ tends to 0 as $t \to \infty$.*

A consequence of the definition and of this Lemma is that, if *all* motions issued from a set B are bounded, all such motions asymptotically approach the set

$$\ell^+(B) = \bigcup_{x_0 \in B} \omega(x_0)\,.$$

The set in question is filled by motions which are defined, and *bounded*, for all backward and forward times. However, the convergence of $x(t, x_0)$ to $\ell^+(B)$ is not guaranteed to be *uniform* in x_0, even if the set B is compact. There is a larger set, though, which does have this property of uniform convergence. This larger set, known as the ω-limit set *of the set* B, and denoted $\omega(B)$ is the set of all x for which there exists a sequence of pairs $\{x_k, t_k\}$, with $x_k \in B$ and $\lim_{k\to\infty} t_k = \infty$ such that

$$\lim_{k\to\infty} x(t_k, x_k) = x\,.$$

It is readily seen that

$$\ell^+(B) \subset \omega(B)\,.$$

but the converse inclusion is not true in general. The relevant properties of the ω-limit set of a set, which extend those presented earlier in Lemma 1, are as follows [5].

Lemma 2 *Let B be a nonempty bounded subset of $\mathbb{R}^n$ and suppose there is a number M such that $|x(t, x_0)| \leq M$ for all $t \geq 0$ and all $x_0 \in B$. Then $\omega(B)$ is a nonempty compact set, invariant under (6). Moreover, the distance of $x(t, x_0)$ from $\omega(B)$ tends to 0 as $t \to \infty$, uniformly in $x_0 \in B$. If B is connected, so is $\omega(B)$.*

Thus, as it is the case for the ω-limit set of a point, the ω-limit set of a bounded set B is filled with motions which exist for all $t \in (-\infty, +\infty)$ and are bounded backward and forward in time. But, above all, the set in question is *uniformly* approached by motions with initial state $x_0 \in B$. An important Corollary of the property of uniform convergence is that, if $\omega(B)$ is contained in the interior of B, then $\omega(B)$ is also *asymptotically stable.*

2.1.2. Steady-State Behavior

Consider system (6), with initial conditions in a closed subset $X \subset \mathbb{R}^n$. Suppose the set X is *positively invariant*, which means that for any initial condition $x_0 \in X$, the solution $x(t, x_0)$ exists for all $t \geq 0$ and $x(t, x_0) \in X$ for all $t \geq 0$. The motions of this system are said to be *ultimately bounded* if there is a bounded subset B with the property that, for every compact subset X_0 of X, there is a time $T > 0$ such that $x(t, x_0) \in B$ for all $t \geq T$ and all $x_0 \in X_0$.

If the motion of a system are ultimately bounded, all its motions asymptotically approach a compact invariant set, the set $\omega(B)$, which is filled by motions that exist for all $t \in (-\infty, +\infty)$ and are bounded. All such motions can be called *steady state* motions and the set $\omega(B)$ can be called the *steady state locus* of the system (see [6]). The *restriction* of (6) to $\omega(B)$ is the *steady state behavior* of (6).

The definition given in this way encompasses the classical notion of steady state response of a stable linear system and also provides a powerful extension to nonlinear systems. As an example, consider the case an asymptotically stable linear system

$$\dot{z} = Fz + Gu, \qquad z \in \mathbb{R}^n . \tag{7}$$

Let the input of this system be provided by a nonlinear "signal generator" of the form

$$\dot{w} = s(w) \qquad u = q(w) \tag{8}$$

whose initial condition $w(0)$ is allowed to range on compact set W, invariant for the dynamics of (8). It is easy to prove that the motions of the composite system (7) – (8), when initial conditions are taken in $W \times \mathbb{R}^n$, are ultimately bounded, and that the steady state locus of this composed system is the graph of a map $z = \pi(w)$ defined by

$$\pi(w) = \lim_{T\to\infty} \int_{-T}^{0} e^{-F\tau} Gq(w(\tau, w))d\tau . \tag{9}$$

As a consequence, the steady state response of (7) to an input $u(t) = q(w(t)$ is simply $x(t) = \pi(w(t))$.

There are various ways in which the result discussed in this example can be generalized. For instance, it can be extended to describe the steady state response of a nonlinear system

$$\dot{z} = f(z, u) \tag{10}$$

in the neighborhood of a locally exponentially stable equilibrium point $(z, u) = (0, 0)$, to an input u provided by a signal generator of the form (8). In general, the following property holds (see [6]).

Proposition 1 *Let W be a compact set, invariant under the flow of (8). Let Z be a closed set and suppose that the motions of (10)–(8) with initial conditions in $W \times Z$ are ultimately bounded. The steady state locus of the system is the graph of a possibly set-valued map defined on the whole of W.*

2.2. Stability of interconnected systems

2.2.1. Input-to-State Stability

A problem of paramount importance in analysis and design of feedback systems is the problem of determining the asymptotic properties of a system consisting of the interconnection of several parts, knowing the asymptotic properties of each individual component. The simplest interconnection to be considered in this setting is the *cascade connection* of two subsystems, which is written as

$$\begin{aligned} \dot{x} &= f(x, z) \\ \dot{z} &= g(z) , \end{aligned} \tag{11}$$

with $x \in \mathbb{R}^n$, $z \in \mathbb{R}^m$ and in which it is assumed that $f(0, 0) = 0$, $g(0) = 0$. If the equilibrium $x = 0$ of $\dot{x} = f(x, 0)$ is locally asymptotically stable and the equilibrium $z = 0$ of the lower subsystem is locally asymptotically stable then the equilibrium $(x, z) = (0, 0)$ of the cascade is locally asymptotically stable. However, in general, *global* asymptotic stability of the equilibrium $x = 0$ of $\dot{x} = f(x, 0)$ and *global* asymptotic stability of the equilibrium $z = 0$ of the lower subsystem *do not* imply *global* asymptotic stability of the equilibrium $(x, z) = (0, 0)$ of the cascade. To infer global asymptotic stability of the cascade, a stronger condition is needed, which expresses a property describing how – in the upper subsystem – the response $x(\cdot)$ is influenced by its *input* $z(\cdot)$.

The property in question requires that when $z(t)$ is bounded, over the semi-infinite time interval $[0, +\infty)$, then also $x(t)$ be bounded, and in particular that, if $z(t)$ asymptotically decays to 0 then also $x(t)$ decays to 0. These requirements altogether lead to the notion of *input-to-state stability*, introduced and thoroughly studied in [7, 8]. The notion in question is defined as follows. Consider a nonlinear system

$$\dot{x} = f(x, u) \tag{12}$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$, in which $f(0, 0) = 0$ and $f(x, u)$ is locally Lipschitz on $\mathbb{R}^n \times \mathbb{R}^m$. The input function $u : [0, \infty) \to \mathbb{R}^m$ of (12) can be any piecewise continuous bounded function. The set of all such functions is endowed with the supremum norm

$$\|u(\cdot)\|_\infty = \sup_{t \geq 0} |u(t)|$$

and denoted by L_∞^m.

System (12) is said to be *input-to-state stable* if there exist a class $\mathcal{KL}$ function $\beta(\cdot, \cdot)$ and a class $\mathcal{K}$ function $\gamma(\cdot)$, called a *gain function*, such that, for any input $u(\cdot) \in L_\infty^m$ and any $x_0 \in \mathbb{R}^n$, the response $x(t)$ of (12) in the initial state $x(0) = x_0$ satisfies

$$|x(t)| \leq \max\{\beta(|x_0|, t), \gamma(\|u(\cdot)\|_\infty)\} \qquad \text{for all } t \geq 0\,. \tag{13}$$

The property, for a given system, of being input-to-state stable, can be given a characterization which extends the classical criterion of Lyapunov for asymptotic stability. The following result, in fact, holds.

Theorem 1 *Let $V : \mathbb{R}^n \to \mathbb{R}$ be a C^1 function satisfying*

$$\underline{\alpha}(|x|) \leq V(x) \leq \overline{\alpha}(|x|) \quad \text{for all } x \in \mathbb{R}^n \tag{14}$$

for some pair of class $\mathcal{K}_\infty$ functions $\underline{\alpha}(\cdot)$, $\overline{\alpha}(\cdot)$. Suppose there exists a class $\mathcal{K}_\infty$ function $\alpha(\cdot)$ and a class $\mathcal{K}$ function $\chi(\cdot)$ such that

$$|x| \geq \chi(|u|) \quad \Rightarrow \quad \frac{\partial V}{\partial x} f(x, u) \leq -\alpha(|x|) \quad \text{for all } x \in \mathbb{R}^n \text{ and } u \in \mathbb{R}^m\,. \tag{15}$$

Then, system (12) is input-to-state stable.

The comparison functions appearing in the estimates (14) and (15) are useful to obtain an estimate of the gain function $\gamma(\cdot)$ which characterizes the bound (13). In fact, it can be shown that $\gamma(\cdot)$ can be estimated as $\gamma(r) = \underline{\alpha}^{-1} \circ \overline{\alpha} \circ \chi(r)$.

2.2.2. Cascade and Feedback Connections

The property of input-to-state stability is of paramount importance in the analysis of stability of interconnected systems. The first application consists in the analysis of system (11).

Proposition 2 *Suppose that system $\dot{x} = f(x,z)$, viewed as a system with input z and state x is input-to-state stable and that system $\dot{z} = g(z)$ is globally asymptotically stable. Then, system (11) is globally asymptotically stable.*

The property of input-to-state stability also lends itself to a simple characterization of an important *sufficient condition* under which the feedback interconnection of two globally asymptotically stable systems remains globally asymptotically stable. Consider a composite system of the form

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2)\ , \end{aligned} \tag{16}$$

in which $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$. Suppose that the upper subsystem, viewed as a system with internal state x_1 and input x_2, is input-to-state stable, with gain function $\gamma_1(\cdot)$. Likewise, suppose that the lower subsystem, viewed as a system with internal state x_2 and input x_1 is input-to-state stable, with gain function $\gamma_2(\cdot)$.

Then, if the composite function $\gamma_1 \circ \gamma_2(\cdot)$ is a *simple contraction*, i.e. if

$$\gamma_1(\gamma_2(r)) < r \quad \text{for all } r > 0\,, \tag{17}$$

the system in question is globally asymptotically stable. This result is usually referred to as the *small-gain* theorem.

Theorem 2 *If the condition (17) holds, system (16) is globally asymptotically stable.*

3. Normal forms

3.1. Normal forms for control

3.1.1. Local and global normal forms

The design of feedback law for nonlinear systems can be rendered systematic and, to some extent, easier, if a system is represented in special *normal forms.* Consider a single-input single-output nonlinear system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{18}$$

with state $x \in \mathbb{R}^n$. Under appropriate assumptions (see e.g. [9, 10]) there exists a globally defined nonlinear change of coordinates changing (18) into a system of equations of the form

$$\begin{aligned} \dot{z} &= f_0(z,\xi) \\ \dot{\xi}_1 &= \xi_2 \\ &\cdots \\ \dot{\xi}_{r-1} &= \xi_r \\ \dot{\xi}_r &= q(z,\xi) + b(z,\xi)u\ . \\ y &= \xi_1 \end{aligned} \tag{19}$$

in which $z \in \mathbb{R}^{n-r}$. The equations thus defined are usually said to be a *normal form for control.* They are useful in understanding how certain control problems can be solved. Under additional assumptions (see again [10]), in the normal form (19) the function $f_0(z,\xi)$ can be rendered independent of $\xi_2,\ldots,\xi_r$, i.e. only dependent on z and ξ_1.

A concept of paramount importance associated with the normal form (19) is that of *zero dynamics.* Suppose the output of the system is identically zero for all $t \in \mathbb{R}$. This may occur if the state and the initial conditions are properly set. To figure out what this input and initial conditions should be is very easy. In fact, if $y(t) \equiv 0$, necessarily $\xi_i(t) \equiv 0$ for all $i = 1,\ldots,r$ and hence $z(t)$ is a motion of the autonomous system

$$\dot{z} = f_0(z,0)\,. \tag{20}$$

Moreover, the input necessarily coincides with $u(t) = -\,q(z(t),0)/b(z(t),0)$. Thus, system (20) is an autonomous dynamical system which describes the motions taking place in a system when input and initial conditions are chosen in such a way as to force the output to remain identically 0. These motions characterize what is called the *zero dynamics* of system (19).

3.1.2. Backstepping

Normal forms of a system lend themselves to the implementation of a powerful recursive method for global stabilization, known as *backstepping.* The method, introduced and thoroughly developed in [11] for a large set of feedback design problems, can be summarized as follows. Consider, to begin with, a system modelled by

$$\begin{aligned} \dot{z} &= f_0(z,\xi) \\ \dot{\xi} &= q(z,\xi) + b(z,\xi)u \end{aligned} \tag{21}$$

in which $z \in \mathbb{R}^n$, $\xi \in \mathbb{R}$, $f_0(0,0) = 0$ and $b(z,\xi) \neq 0$ for all (z,ξ). Suppose there exists a C^1 function $\xi^*(z)$, with $\xi^*(0) = 0$, such that equilibrium $z = 0$ of

$$\dot{z} = f_0(z,\xi^*(z)) \tag{22}$$

is globally asymptotically stable. In other words, suppose the equilibrium $z = 0$ of $\dot{z} = f_0(z,\xi)$, in which ξ is viewed as control is *stabilizable*, by means of a C^1 state-feedback control $\xi = \xi^*(z)$. The control in question, which is not actually implementable, because in the system (21) the real control is u, is called a *virtual control* for the state z. As consequence of this assumption, by the converse Theorem of Lyapunov, there exists a smooth positive definite and proper function $V(z)$ whose derivative along the trajectories of (22) is negative definite. Set $y = \xi - \xi^*(z)$ and rewrite system (21)

$$\begin{aligned} \dot{z} &= f_0(z,\xi^*(z)) + f_1(z,y)y \\ \dot{y} &= q(z,y+\xi^*(z)) + b(z,y+\xi^*(z))u - \frac{\partial \xi^*}{\partial z}[f_0(z,\xi^*(z)) + f_1(z,y)y] \end{aligned} \tag{23}$$

in which $f_1(z,y)$ is a continuous function. Since $b(z,y+\xi^*(x))$ is nowhere zero, the bottom equation can be drastically simplified, by appropriate choice of u, to obtain a new system with control u'

$$\begin{aligned} \dot{z} &= f_0(z,\xi^*(z)) + f_1(z,y)y \\ \dot{y} &= u'\,. \end{aligned} \tag{24}$$

The idea is now to choose the residual control u' to *force*, on system (24), the positive definite and proper function

$$W(z,y) = V(z) + y^2$$

to become a Lyapunov fuction. A simple algebra shows that, to this extent, it suffices to pick

$$u' = -\frac{\partial V}{\partial z} f_1(z, \xi^*(z)) - y\,.$$

This method can be recursively applied to stabilize systems expressed in triangular form as

$$\begin{array}{rcl} \dot{z} &=& f_0(z,\xi) \\ \dot{\xi}_1 &=& q_1(z,\xi_1) + b_1(z,\xi_1)\xi_2 \\ \dot{\xi}_2 &=& q_2(z,\xi_1,\xi_2) + b_2(z,\xi_1,\xi_2)\xi_3 \\ && \cdots \\ \dot{\xi}_r &=& q_r(z,\xi_1,\ldots,\xi_r) + b_r(z,\xi)(z,\xi_1,\ldots,\xi_r)u \end{array} \tag{25}$$

under the hypotheses that $b_1(\cdot), b_2(\cdot), \ldots, b_r(\cdot)$ are nowhere zero and that the equilibrium $z = 0$ of $\dot{z} = f_0(z,\xi_1)$ is *stabilizable*, by means of a C^1 virtual control $\xi_1 = \xi_1^*(z)$. By means of the arguments above, a *virtual control* $\xi_2^*(z,\xi_1)$ is initially found which stabilizes the subsystem composed by the first two equations. Then, using the same arguments (now for a system consisting of the first three equations, viewed as a system with state (z,ξ_1,ξ_2) and control ξ_3) a virtual stabilizing control $\xi_3^*(z,\xi_1,\xi_2)$ is found, and so on, until a control $u(z,\xi_1,\ldots,\xi_r)$ is designed which stabilizes the entire system (see also [12, 13]).

3.1.3. Semiglobal Stabilization

Another setting in which normal forms are useful is a systematic method for stabilization *in the large* of certain classes of nonlinear systems, even in the presence of parameter uncertainties. To begin with, consider a system modelled by equations of the form (21). Assume that the equilibrium $z = 0$ of

$$\dot{z} = f_0(z,0)$$

is globally asymptotically stable. System that satisfy this assumption are usually called *minimum phase systems* in view of the fact that the assumption in question, in the case of a linear system, is equivalent to the assumption that all zeros of the transfer function have negative real part. Consider now for (21) a control law

$$u = -ky$$

in which $k > 0$, which yields the closed loop system

$$\begin{array}{rcl} \dot{z} &=& f_0(z,\xi) \\ \dot{\xi} &=& q(z,\xi) - b(z,\xi)k\xi\,. \end{array} \tag{26}$$

Then, the following property holds.

Proposition 3 *Consider system (26). Suppose that* $b(z,\xi) \geq b_0 > 0$ *for some* b_0 *and suppose that the equilibrium* $z = 0$ *of* $\dot{z} = f_0(z,0)$ *is globally asymptotically stable. Then, for every choice of two positive numbers* R, r*, with* $R \gg r$*, there is a number* k_1 *such that, for all* $k \geq k_1$ *there is a finite time* T *such that all trajectories of (26) with initial condition* $|x(0)| \leq R$ *remain bounded and satisfy* $|x(t)| \leq r$ *for all* $t \geq T$*.*

Thus, by increasing the gain parameter k, trajectories starting at any point of a ball of arbitrarily large radius R are steered to a ball of arbitrarily small radius r. Note, though, that the trajectories of the system are not steered to the point $(z,\xi) = (0,0)$. To indicate this property, it is said that the system is of *semiglobally practically stabilizable*, in the parameter k (see [14]). To obtain *asymptotic* stability, extra assumptions are needed. In addition to the obvious requirement that $q(0,0) = 0$, a simple sufficient condition under which asymptotic stability can be achieved is that equilibrium $z = 0$ of $\dot{z} = f_0(z,0)$ is not just asymptotically, but also *locally exponentially* stable.

A similar result holds for a general system of the form

$$\begin{aligned} \dot{z} &= f_0(z,\xi_1) \\ \dot{\xi}_1 &= \xi_2 \\ &\cdots \\ \dot{\xi}_{r-1} &= \xi_r \\ \dot{\xi}_r &= q(z,\xi_1,\ldots,\xi_r) + b(z,\xi_1,\ldots,\xi_r)u \end{aligned} \tag{27}$$

as shown, e.g. in [14]. For this result to hold, it is *essential* that $f_0(z,\xi)$ only depends on ξ_1 and not on the other components $\xi_2,\ldots,\xi_r$. The feedback law that provides semiglobal and practical (possibly asymptotic) stability is a linear law in the ξ_i's, of the form

$$u = -k[\xi_r + g^{r-1}a_0\xi_1 + g^{r-2}a_1\xi_2 + \cdots g a_{r-2}\xi_{r-1}]\,.$$

This is sometimes called a *partial–state* feedback. This stabilization result also covers the case in which the functions characterizing the system depend, in a smooth fashion, on a vector μ of constant uncertain parameters, so long as the latter ranges on a compact set.

3.2. Normal forms for state estimation

3.2.1. Uniformly observable systems

Observability can be defined in various ways for nonlinear systems. The concept best suited to the purpose of designing dynamic, output-feedback, stabilizing laws is that of *complete uniform observability* which is defined as follows. A system

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x,u) \end{aligned} \tag{28}$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$, output $y \in \mathbb{R}$, is completely uniformly observable if there exists a globally defined change of coordinates $z = \Phi(x)$ carrying system (28) into a system of the form

$$\begin{array}{rcl} \dot{z}_1 & = & \tilde{f}_1(z_1, z_2, u) \\ \dot{z}_2 & = & \tilde{f}_2(z_1, z_2, z_3, u) \\ & & \cdots \\ \dot{z}_{n-1} & = & \tilde{f}_{n-1}(z_1, z_2, \ldots, z_n, u) \\ \dot{z}_n & = & \tilde{f}_n(z_1, z_2, \ldots, z_n, u) \\ y & = & \tilde{h}(z_1, u) \end{array} \tag{29}$$

in which the $\tilde{h}(z_1, u)$ and $f_i(z_1, z_2, \ldots, z_{i+1}, u)$ satisfy

$$\frac{\partial \tilde{h}}{\partial z_1} \neq 0\,, \quad \text{and} \quad \frac{\partial \tilde{f}_i}{\partial z_{i+1}} \neq 0\,, \quad \text{for all} \quad i = 1, \ldots, n-1 \tag{30}$$

for all $z \in \mathbb{R}^n$, and all $u \in \mathbb{R}^m$. This, form, introduced and thoroughly studied in [15] is usually referred to as the *uniform observability normal form.* A simple sufficient condition under which a transformation yielding a system of the form (29) exists can be found in [15].

3.2.2. Nonlinear observers

The relevance of a uniform observability normal form is that it lends itself to the design of asymptotic observers. As shown in [15], an observer can be designed by picking a *copy* of the dynamics (29), corrected by and *output injection* term, namely as a system of the form

$$\begin{array}{rcl} \dot{\hat{z}}_1 & = & \tilde{f}_1(z_1, z_2, u) + \kappa c_{n-1}(y - \tilde{h}(z_1, u)) \\ \dot{\hat{z}}_2 & = & \tilde{f}_2(z_1, z_2, z_3, u) + \kappa^2 c_{n-2}(y - \tilde{h}(z_1, u)) \\ & & \cdots \\ \dot{\hat{z}}_{n-1} & = & \tilde{f}_{n-1}(z_1, \ldots, z_n, u) + \kappa^{n-1} c_1(y - \tilde{h}(z_1, u)) \\ \dot{\hat{z}}_n & = & \tilde{f}_n(z_1, \ldots, z_n, u) + \kappa^n c_0(y - \tilde{h}(z_1, u))\,, \end{array} \tag{31}$$

in which κ and $c_{n-1}, c_{n-2}, \ldots, c_0$ are design parameters. Under appropriate hypotheses, it can be shown that there is a choice of these design parameters such that this system behaves as a *global observer* (see [15]).

Proposition 4 *Suppose the maps $\tilde{f}_i(z_1, \ldots, z_i, z_{i+1}, u)$, for $i = 1, \ldots, n$, are globally Lipschitz with respect to $(z_1, \ldots, z_i)$, uniformly in z_{i+1} and u,. Suppose that $|\partial \tilde{h}/\partial z_1|$ and $|\partial \tilde{f}_i/\partial z_{i+1}|$, for $i = 1, \ldots, n-1$, are bounded from below and from above. Then, there exist a set of numbers $c_{n-1}, c_{n-2}, \ldots, c_0$ and a number κ^* such, that, for all kappa $\geq \kappa^*$,*

$$\lim_{t \to \infty} \|z(t) - \hat{z}(t)\| = 0$$

for all $z(0), \hat{z}(0)$ and all $u(\cdot)$.

Note that the assumptions of this proposition are automatically satisfied if it is known that $z(t)$ and $u(t)$ remain in a compact set.

3.2.3. A Nonlinear Separation Principle

Nonlinear observers can be used to the purpose of achieving asymptotic stability via dynamic output feedback. Consider a nonlinear system in observability canonical form (29), rewritten in compact form as

$$\begin{aligned} \dot{z} &= f(z,u) \\ y &= h(z,u)\,, \end{aligned} \tag{32}$$

with $f(0,0)=0$ and $h(0,0)=0$ and suppose there exists a feedback law $u=\alpha(z)$, with $\alpha(0)=0$, such that the equilibrium $z=0$ of

$$\dot{z} = f(z,\alpha(z))$$

is globally asymptotically stable. Consider an observer of the form (31), rewritten in compact form as

$$\dot{\hat{z}} = f(\hat{z},u) + G(y - h(\hat{z},u))\,. \tag{33}$$

An obvious choice to achieve asymptotic stability, suggested by the analogy with linear systems, would be to replace z by its estimate $\hat{z}$ in the map $\alpha(z)$. However, this simple choice may prove to be dangerous, for the following reason. A thorough analysis of the performances of the observer (31) reveals that the relation between the actual value of the state z and its estimate $\hat{z}$ has the following form

$$z(t) - \hat{z}(t) = D_\kappa e(t),$$

in which $e(t)$ vector which is guaranteed to decay to 0 and D_κ is the diagonal matrix

$$D_\kappa = \text{diag}(\kappa, \kappa^2, \ldots, \kappa^n)\,.$$

Thus, feeding the system (32) with a control $u=\alpha(\hat{z})$ would result in a system

$$\dot{z} = f(z, \alpha(z + D_\kappa e))\,.$$

which contains the possibly large parameter κ. Since the system is nonlinear, this may cause finite escape times. To overcome the problem, as a precautionary measure, it is appropriate to "saturate" the control, by choosing instead a law of the form

$$u = \sigma_L(\alpha(\hat{z}))) \tag{34}$$

in which $\sigma(r)$ is any function that coincides with r when $|r| \le L$, is strictly increasing and satisfies $|\sigma(r)| \le 2L$ for all $r \in \mathbb{R}$.

The consequence of this is that global asymptotic stability is no longer assured. However, it can be shown that, in the aggregate of (32), (33) and (34), *semiglobal stabilizability* is still possible. As a matter of fact, it can be shown (see [15]) that for every compact set K of initial conditions in the state space, there is a choice of the design parameters, κ, $c_{n-1}, c_{n-2}, \ldots, c_0$ and L such that the equilibrium $(z,\hat{z}) = (0,0)$ of the closed loop system is asymptotically stable, with a domain of attraction that contains K.

3.2.4. A Robust Nonlinear Separation Principle

We have seen earlier that a system in normal form (27) can be semiglobally stabilized by means of a feedback law which is a linear form of the states $\xi_1, \ldots, \xi_r$, that is a by means of a law of the form $u = H\xi$ in which H is a vector of "gain coefficients", which depend on the assigned compact set of initial conditions. Moreover, it has also been observed that a law of this type can be used to *robustly* stabilize the system, in case the functions which characterize the latter depend on a vector of uncertain parameters ranging on a compact set.

This design is based on the availability, for measurement, of the components $\xi_1, \xi_2, \ldots, \xi_r$ of the state of system. These variables coincide, by definition, with the measured output y and with its first $r-1$ derivatives with respect to time. If these variables are not directly available for feedback, one may attempt to replace them by means of estimates $\hat{\xi}_i$, $i = 1, \ldots, r$, provided by a "rough" observer of the form

$$\begin{array}{rcl} \dot{\hat{\xi}}_1 & = & \hat{\xi}_2 + \kappa c_{r-1}(y - \hat{\xi}_1) \\ \dot{\hat{\xi}}_2 & = & \hat{\xi}_3 + \kappa^2 c_{r-2}(y - \hat{\xi}_1) \\ & & \cdots \\ \dot{\hat{\xi}}_{r-1} & = & \hat{\xi}_r + \kappa^{r-1} c_1(y - \hat{\xi}_1) \\ \dot{\hat{\xi}}_r & = & \kappa^r c_0(y - \hat{\xi}_1)\,. \end{array} \tag{35}$$

This is not a real observer because the "perfect tracking" condition $\xi_i(t) = \tilde{\xi}_i(t)$ for all $i = 1, \ldots, r$ cannot be fulfilled. However, as shown originally in [16], it serves the purpose of providing variables, which, if used in a control law of the form $H\tilde{\xi}$, yield *semiglobal* asymptotic stability of the equilibrium of the closed loop system. As in the previous section, the occurrence of finite escape times can be avoided by saturating the control action, namely using an actual control law of the form $u = \sigma_L(H\hat{\xi})$.

4. Necessary conditions for asymptotic regulation

The design tools summarized in the previous section, namely the methods for semiglobal stabilization of nonlinear systems in control normal form and state estimation for system in uniform observability normal form, find their natural domain of application in the solution of the problem of asymptotic regulation considered at the beginning. In what follows, we consider a nonlinear system possessing a globally defined normal form, which for convenience is written as

$$\begin{array}{rcl} \dot{w} & = & s(w) \\ \dot{z} & = & f(w, z, \zeta, \xi_r) \\ \dot{\zeta} & = & A\zeta + B\xi_r \\ \dot{\xi}_r & = & q(w, z, \zeta, \xi_r) + b(w, z, \zeta, \xi_r)u \\ e & = & C\xi \end{array} \tag{36}$$

in which $z \in \mathbb{R}^m$, $\zeta \in \mathbb{R}^{r-1}$,

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ . & . & . & \cdots & . \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ 1 \end{pmatrix}.$$

The initial conditions $w(0), z(0), \zeta(0), \xi_r(0)$ of this system are in a compact set of the form $W \times Z \times \Xi \times E$, in which W is invariant for the dynamics of $\dot{w} = s(w)$. The functions characterizing the model (36) are assumed to be smooth functions of their arguments. In addition, we assume the existence of a pair of real numbers (b_0, b_1) such that

$$0 < b_0 \leq b(w, z, \zeta, \xi_r) \leq b_1 \,. \tag{37}$$

Note that

$$\zeta = \mathrm{col}(e, e^{(1)}, \ldots, e^{(r-2)}), \qquad \xi_r = e^{(r-1)} \,.$$

Motivated by the robust separation principle discussed in section 3.2.4, we assume throughout that the entire *partial state* $(\xi_1, \ldots, \xi_{r-1}, \xi_r)$ is available for measurement, i.e.

$$y = \mathrm{col}(\xi_1, \ldots, \xi_{r-1}, \xi_r) \,.$$

The states w and z, on the contrary, *are not* available for measurement.

In what follows we discuss the existence of a controller which solves the general design problem described at the beginning. Controlling the plant (36) by means of a controller of the form (5) yields an autonomous closed-loop system modelled by equations of the form

$$\begin{array}{rcl} \dot{w} &=& s(w) \\ \dot{z} &=& f(w, z, \zeta, \xi_r) \\ \dot{\zeta} &=& A\zeta + B\xi_r \\ \dot{\xi}_r &=& q(w, z, \zeta, \xi_r) + b(w, z, \zeta, \xi_r) h_{\mathrm{C}}(\eta, (\zeta, \xi_r)) \\ \dot{\eta} &=& f_{\mathrm{C}}(\eta, (\zeta, \xi_r)) \\ e &=& C\xi \,, \end{array} \tag{38}$$

whose initial conditions range in a compact set of the form

$$B = W \times Z \times \Xi \times E \times H \,,$$

in which H denotes the compact set where the initial condition $\eta(0)$ of the controller is allowed to range. Note that this system has a structure identical to the structure discussed earlier in section 2.1.2, namely that of a system with internal state (z, ζ, ξ_r, η) forced by the autonomous system $\dot{w} = s(w)$.

To say that the controller solves the design problem at issue is to say that, for any initial condition in $(w(0), z(0), \zeta(0), \xi_r(0), \eta(0)) \in B$, the integral curve $(w(t), z(t), \zeta(t), \xi_r(t), \eta(t))$ of (38):

(i) is bounded in forward time,

(ii) $\lim_{t\to\infty} e(t) = 0$, uniformly in $(w(0), z(0), \zeta(0), \xi_r(0), \eta(0))$.

From the *first* of these two requirements, in view of the results presented in section 2.1.2, we can assert that the closed-loop system has a well-defined *steady-state behavior*, characterized by the restriction of (38) to the invariant set $\omega(B)$. This being the case, it is easy to see that, the *second* requirement is met if and only if the invariant set $\omega(B)$ is a subset of the subspace

$$K := \{(w, z, \zeta, \xi_r, \eta) : C\zeta = 0\} \,.$$

Lemma 3 *Suppose the forward orbit the set B under the flow of (38) is bounded. Then,* $\lim_{t\to\infty} e(t) = 0$*, uniformly in the initial condition, if and only if* $\omega(B) \subset K$*.*

Having shown that $\omega(B) \subset K$, it is immediate to realize that $\omega(B)$ is necessarily a subset a smaller subspace, namely of the set

$$K' := \{(w, z, \zeta, \xi_r, \eta) : \zeta = 0, \xi_r = 0\}\,. \tag{39}$$

In fact, by the Lemma above, any trajectory entirely contained in $\omega(B)$ is such that $\xi_1(t)$ is identically zero. As a consequence, in view of the special structure of (38), any of such trajectories is also such that $\xi_2(t), \ldots, \xi_{r-1}, \xi_r(t)$ are identically zero. We see from this that $\omega(B) \subset K'$.

This observation bears a number of interesting consequences. The first one is a geometric characterization of *steady-state locus* of the closed loop system (38). The latter, in fact is the graph of a (possibly set-valued) map defined on W, entirely contained in the set K'. Hence, there exists two (possibly set-valued) maps

$$\begin{array}{rcccl} \pi & : & w \in W & \mapsto & \pi(w) \subset \mathbb{R}^m \\ \sigma & : & w \in W & \mapsto & \sigma(w) \subset \mathbb{R}^\nu \end{array}$$

such that

$$\omega(B) = \{(w, z, \zeta, \xi_r, \eta) : w \in W, z \in \sigma(w), \zeta = 0, \xi_r = 0, \eta \in \sigma(w)\}\,.$$

In general, $\pi(w)$ and $\sigma(w)$ do not simply consist of one single point, but can be *subsets* of $\mathbb{R}^m$ and, respectively, $\mathbb{R}^\nu$. However, it can be shown that these maps are upper-semicontinuous (see [17]). The fact that the maps in question might, or might not, be single-valued depends on certain properties of controlled system and also on the type of controller chosen. In the special case in which π and σ are *single-valued and continuously differentiable*, the maps in question can be given the rather expressive characterizations. In fact, in this case, since $\omega(B)$ is invariant under the flow of (38), the functions $z = \pi(w)$ and $\eta = \sigma(w)$ must satisfy

$$\frac{\partial \pi}{\partial w} s(w) = f(w, \pi(w), 0, 0) \tag{40}$$

$$\frac{\partial \sigma}{\partial w} s(w) = f_{\rm C}(\sigma(w), 0)\,. \tag{41}$$

The second consequence is the identification of the *steady state inputs* generated by any controller that solves the problem of output regulation. To this end, define the map

$$u_{\rm ss}(w, z) = -\frac{q(w, z, 0, 0)}{b(w, z, 0, 0)}\,. \tag{42}$$

In steady state, $\xi(t) \equiv 0$ and $\xi_r(t) \equiv 0$, and hence it follows from (38) that $w(t), z(t)$ and $\eta(t)$ are integral curves of

$$\begin{array}{rcl} \dot w & = & s(w) \\ \dot z & = & f(w, z, 0, 0)\,, \end{array} \tag{43}$$

and, respectively,

$$\dot\eta = f_{\rm C}(\eta, 0)\,, \tag{44}$$

satisfying

$$u_{\rm ss}(w(t), z(t)) = h_{\rm C}(\eta(t), 0)\,. \tag{45}$$

The controls $u_{\rm ss}(w(t), z(t))$ are called the *steady-state controls.* The property indicated in (45) expresses the ability, of the controller (5), to generate all those inputs which are required to keep the regulated output of the system (38) identically at zero. This property is usually referred to as the *internal model property.* Note that, if the map $\pi(w)$ is single-valued, in steady state $z(t) = \pi(w(t))$ and $\eta(t) = \sigma(w(t)$. Hence, the steady state controls can be expressed in the simpler form $u(t) = u^*(w(t))$ having set $u^*(w) = u_{\rm ss}(w, \pi(w))$, and the internal model property (45) assumes the simpler form

$$u^*(w(t)) = h_{\rm C}(\sigma(w(t)), 0)\,. \tag{46}$$

5. A unified design paradigm

5.1. The internal model

The properties deduced in the previous section can be taken as a point of departure for the design of a controller that solves the problem. For the sake of simplicity, we address the case in which there is a unique steady state, i.e. the maps $\pi(w)$ and $\sigma(w)$ are single-valued. While the map $\pi(w)$ should necessarily satisfy (40), the design should include steps guaranteeing the existence of a map $\sigma(w)$ satisfying an identity of the form (41) and the fulfillment of the internal model property (46). In general, this may require some extra hypotheses. In recent years, assumptions for the construction of a controller with the properties expressed by (41) and (46) have been progressively weakened, moving from the so-called assumption of "immersion into a linear observable system" (as in [18]) to "immersion into a nonlinear uniformly observable system" (as in [19]) to the recent results of [20], in which it was shown that no assumption is in fact needed for the construction of an internal model if only continuous (thus possibly not locally Lipschitz) controllers are acceptable. Assume, as in [19], the existence of an integer $d \in \mathbb{N}$ and of a smooth map $\phi(x_1, \ldots, x_d)$ such that

$$L_s^d u^*(w) = \phi(u^*(w), L_s u^*(w), \ldots, L_s^{d-1} u^*(w)) \qquad \forall w \in W\,, \tag{47}$$

and set

$$\varphi(x) = \begin{pmatrix} x_2 \\ \cdots \\ x_d \\ \phi(x_1, \ldots, x_d) \end{pmatrix}, \qquad \gamma(x) = x_1\,,$$

and

$$\tau(w) = \mathrm{col}(u^*(w), \ldots, L_s^{d-1} u^*(w))\,.$$

Then, it is easy to check that

$$\frac{\partial \tau}{\partial w} s(w) = \varphi(\tau(w)) \qquad \text{and} \qquad u^*(w) = \gamma(\tau(w)) \qquad \forall w \in W\,. \tag{48}$$

Setting

$$F(x) = \varphi(x) - G_0 x_1\,,$$

these identities can be rewritten in the form

$$\frac{\partial \tau}{\partial w} s(w) = F(\tau(w)) + G_0\gamma(\tau(w)) \qquad \text{and} \qquad u^*(w) = \gamma(\tau(w)) \qquad \forall w \in W\,. \tag{49}$$

In this expression, the coefficients of the d-dimensional column vector G_0 can be seen as free parameters, which can be exploited in the successive steps of the design. Note also that, as recent advances in the theory of nonlinear observers (see e.g. [20]) have shown, if d is large enough, and $F(x) = F_0 x$ with F_0 Hurwitz and (F_0, G_0) controllable, a C^1 map $\tau(\cdot)$ and a C^0 map $\gamma(\cdot)$ which do fulfill (48) always exist, regardless of assumption (47).

5.2. The control

We consider, in what follows, a dynamic controller, with internal state (ψ, η), "driven" by the measured variables (ζ, ξ_r). The control in question is modelled by equations of the form

$$\begin{aligned} u &= \gamma(\eta) + \beta\dot{N}(\psi) + v \\ \dot{\psi} &= L(\psi) - Mv \\ \dot{\eta} &= F(\eta) + G_0[\gamma(\eta) + v] \\ v &= -k[\xi_r - K\zeta - N(\psi)] \end{aligned} \tag{50}$$

in which $F(\cdot), G_0, \gamma(\cdot)$ satisfy (48) for some $\tau(\cdot)$, while $L(\cdot), N(\cdot),\ M, K$ are smooth maps and, respectively, vectors of appropriate dimensions, and β, k are real numbers. It is assumed (without loss of generality) that

$$\frac{\partial N}{\partial \psi} M = 0 \tag{51}$$

in which case

$$\dot{N}(\psi) = \frac{\partial N}{\partial \psi} L(\psi)\,.$$

Changing ξ_r into

$$\theta = \xi_r - K\zeta - N(\psi)$$

yields a closed-loop system of the form

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{z} &= f(w, z, \zeta, K\zeta + N(\psi) + \theta) \\ \dot{\zeta} &= A\zeta + B[K\zeta + N(\psi) + \theta] \\ \dot{\psi} &= L(\psi) - Mv \\ \dot{\eta} &= F(\eta) + G_0[\gamma(\eta) + v] \\ \dot{\theta} &= Q(w, z, \zeta, K\zeta + N(\psi) + \theta) + b(w, z, \zeta, K\zeta + N(\psi) + \theta)[\gamma(\eta) + v] \\ &\quad + \Delta(w, z, \zeta, K\zeta + N(\psi) + \theta)\dot{N}(\psi)\,, \end{aligned}$$

in which we have set

$$\begin{aligned} Q(w,z,\zeta,\xi_r) &= q(w,z,\zeta,\xi_r) - K(A\zeta + B\xi_r) \\ \Delta(w,z,\zeta,\xi_r) &= b(w,z,\zeta,\xi_r)\beta - 1\,, \end{aligned}$$

with control

$$v = -k\theta\,.$$

Note that, in case the coefficient $b(w,z,\zeta,\xi_r)$ only depends on the measured variables (ζ,ξ_r), one can choose $\beta = 1/b$, obtaining in this way $\Delta(w,z,\zeta,\xi_r) = 0$.

This system can be regarded as a system with input v and output θ, having relative degree 1, with input v to be chosen as $v = -k\theta$, that is as a negative output feedback. To facilitate the analysis, this system can be brought in normal form. As a matter of fact, there exists globally defined changes of coordinates

$$\begin{aligned} Y \,:\, \psi &\mapsto \chi = Y(w,z,\zeta,\psi,\theta) \\ X \,:\, \eta &\mapsto x = X(w,z,\zeta,\psi,\eta,\theta) \end{aligned}$$

yielding a system of the form

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{z} &= f(w,z,\zeta,K\zeta + N(\hat{Y}) + \theta) \\ \dot{\zeta} &= A\zeta + B[K\zeta + N(\hat{Y}) + \theta] \\ \dot{\chi} &= \frac{\partial K}{\partial \psi}\Big[L(\hat{Y}) + M\Big(\frac{Q}{b}(w,z,\zeta,\theta + N(\hat{Y}) + K\zeta) + \frac{\Delta}{b}\,\dot{N}(\hat{Y}) + \gamma(\hat{X})\Big)\Big] + R_\chi \\ \dot{x} &= F(\hat{X}) - G_0\Big(\frac{Q}{b}(w,z,\zeta,\theta + N(\hat{Y}) + K\zeta) + \frac{\Delta}{b}\,\dot{N}(\hat{Y})\Big) + R_x \\ \dot{\theta} &= Q(w,z,\zeta,K\zeta + N(\hat{Y}) + \theta) + b(w,z,\zeta,K\zeta + N(\hat{Y}) + \theta)[\gamma(\hat{X}) + v] \\ &\quad + \Delta(w,z,\zeta,K\zeta + N(\hat{Y}) + \theta)\dot{N}(\hat{Y})\,, \end{aligned} \tag{52}$$

in which $\hat{Y}, \hat{X}$ are the inverses of the maps Y, X and R_χ, R_x denote appropriate residual functions which, as an appropriate calculation shows, vanish at $\theta = 0$.

The system obtained in this way can be seen as feedback interconnection of a system Σ_1 with input θ and state (w,z,ζ,χ,x) and of a system Σ_2 with input (w,z,ζ,χ,x) and state θ. Appealing to the semiglobal stabilization results discussed in section 3.1.3, one can arrive at the following (intermediate) conclusion. Set $\theta = 0$ in the upper subsystem, to obtain

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{z} &= f(w,z,\zeta,K\zeta + N(\chi)) \\ \dot{\zeta} &= A\zeta + B(K\zeta + N(\chi)) \\ \dot{\chi} &= L(\chi) + M\Big(\frac{Q}{b}(w,z,\zeta,K\zeta + N(\chi)) + \frac{\Delta}{b}\dot{N}(\chi) + \gamma(x)\Big) \\ \dot{x} &= F(x) - G_0\Big(\frac{Q}{b}(w,z,\zeta,K\zeta + N(\chi)) + \frac{\Delta}{b}\dot{N}(\chi)\Big)\,, \end{aligned} \tag{53}$$

and suppose that this system possesses a compact invariant set $\mathcal{A}$ which is locally exponentially stable, with a domain of attraction that contains the set of admissible initial conditions. Suppose also that the "coupling term" by means of which the state (w,z,ζ,χ,x) of Σ_1 "drives" the dynamics of the state θ of Σ_2, i.e. the function

$$Q(w,z,\zeta,K\zeta+N(\chi))+b(w,z,\zeta,K\zeta+N(\chi))\gamma(x)+\Delta(w,z,\zeta,K\zeta+N(\chi))\dot{N}(\chi)\,, \tag{54}$$

vanishes on the set $\mathcal{A}$. Then, there is a number k^* such that, for all $k>k^*$, all trajectories of system (52) with control $v=-k\theta$ are bounded in forward time and, moreover, the state (w,z,ζ,χ,x) of Σ_1 asymptotically converges to $\mathcal{A}$, while the state θ of Σ_2 asymptotically converges to 0. If, in addition, the regulated variable $e=\zeta_1$ vanishes $\mathcal{A}$, the proposed controller solves the problem of asymptotic regulation.

5.3. The structure of the core subsystem

All of the above suggests the use of the degrees of freedom in the choice of the parameters of the controller in order to fulfill the hypotheses outlined above. The goal is to shape the internal model $\{F(x),G_0,\gamma(x)\}$ and find, if possible, a triplet $\{L(\chi),M,N(\chi)\}$ in such a way that system (53) possesses a compact invariant set $\mathcal{A}$ which is locally exponentially stable and attracts all admissible initial conditions, and that both ζ_1 and the map (54) vanish on this set.

By assumption, there exists $\pi(w)$ and $\tau(w)$ satisfying (40) and (48). Hence, if $L(0)=0$ and $N(0)=0$, the set

$$\mathcal{A}=\{(w,z,\zeta,\chi,x): w\in W, z=\pi(w),\zeta=0,\chi=0,x=\tau(w)\}$$

is a compact invariant set of (53). Moreover, by construction, the map (54) vanishes on this set. Trivially, also ζ_1 vanishes on this set. Thus, it is concluded that if this set $\mathcal{A}$ can be made locally exponentially stable, with a domain of attraction that contains the compact set of all admissible initial conditions, the proposed controller, with large k, solves the problem of asymptotic tracking.

System (53) is not terribly difficult to handle. As a matter of fact, it can be regarded as interconnection of three much simpler subsystems. To see this, set

$$z_{\rm a}=z-\pi(w)\qquad\qquad \tilde{x}=x-\tau(w)$$

and define

$$\begin{aligned}
f_{\rm a}(w,z_{\rm a},\zeta,\xi_r)&=f(w,z_{\rm a}+\pi(w),\zeta,\xi_r)-f(w,\pi(w),0,0)\\
h_{\rm a}(w,z_{\rm a},\zeta,\xi_r)&=\frac{Q}{b}(w,z_{\rm a}+\pi(w),\zeta,\xi_r)-\frac{Q}{b}(w,\pi(w),0,0)\\
\Delta_{\rm a}(w,z_{\rm a},\zeta,\xi_r)&=\frac{\Delta}{b}\,(w,z_{\rm a}+\pi(w),\zeta,\xi_r)=\beta-\frac{1}{b(w,z_{\rm a}+\pi(w),\zeta,\xi_r)}\;.
\end{aligned}$$

In view of this, the core subsystem (53) can be seen as a system with input $\bar{u}$ and output $\bar{y}$ defined as

$$\begin{array}{rcl}
\dot{w}&=&s(w)\\
\dot{z_{\rm a}}&=&f_{\rm a}(w,z_{\rm a},\zeta,K\zeta+N(\chi))\\
\dot{\zeta}&=&A\zeta+B(K\zeta+N(\chi))\\
\dot{\chi}&=&L(\chi)+M[h_{\rm a}(w,z_{\rm a},\zeta,K\zeta+N(\chi))+\Delta_{\rm a}(w,z_{\rm a},\zeta,K\zeta+N(\chi))\dot{N}(\chi)+\bar{u}]\\
\dot{\tilde{x}}&=&F(\tilde{x}+\tau(w))-F(\tau(w))-G_0[h_{\rm a}(w,z_{\rm a},\zeta,K\zeta+N(\chi))+\Delta_{\rm a}(w,z_{\rm a},\zeta,K\zeta+N(\chi))\dot{N}(\chi)]\\
\bar{y}&=&\gamma(\tilde{x}+\tau(w))-\gamma(\tau(w))
\end{array} \tag{55}$$

subject to unitary output feedback

$$\bar{u} = \bar{y}\,.$$

System (55), in turn, can be seen as the cascade of an "inner loop" consisting of a subsystem, which we call the "auxiliary plant", modelled by equations of the form

$$\begin{array}{rcl} \dot{w} & = & s(w) \\ \dot{z}_{\rm a} & = & f_{\rm a}(w, z_{\rm a}, \zeta, K\zeta + u_{\rm a}) \\ \dot{\zeta} & = & (A + BK)\zeta + Bu_{\rm a} \\ y_{\rm a} & = & h_{\rm a}(w, z_{\rm a}, \zeta, K\zeta + u_{\rm a}) + \Delta_{\rm a}(w, z_{\rm a}, \zeta, K\zeta + u_{\rm a})v_{\rm a}\,, \end{array} \tag{56}$$

controlled by

$$\begin{array}{rcl} \dot{\chi} & = & L(\chi) + M[y_{\rm a} + \bar{u}] \\ u_{\rm a} & = & N(\chi) \\ v_{\rm a} & = & \dot{N}(\chi)\,, \end{array} \tag{57}$$

cascaded with a system, which we call a "weighting filter", modelled by equations of the form

$$\begin{array}{rcl} \dot{\tilde{x}} & = & F(\tilde{x} + \tau(w)) - F(\tau(w)) - G_0 y_{\rm a} \\ \bar{y} & = & \gamma(\tilde{x} + \tau(w)) - \gamma(\tau(w))\,. \end{array} \tag{58}$$

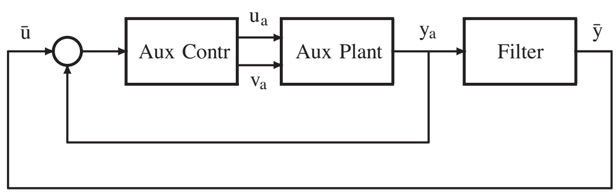

Figure 1. The feedback structure of system (53)

All of this is depicted in Figure 1. Having interpreted system (53) as the system resulting from a unitary output feedback on system (55), the idea is now to use the degrees of freedom in the design to make the latter a stable system and to force its gain to be a simple contraction.

5.4. The asymptotic properties of the core subsystem

System (55) is the cascade of two subsystems: the "inner loop", consisting of (56) and (57), and the "filter" (58). An obvious prerequisite for stability is the stability of both subsystems of the cascade. Stability of the filter (58) is not an issue. As a matter of fact, appealing to the results of [15], it is not difficult to prove the existence of a filter which is globally input-to-state stable, actually with a linear gain function.

As far as the inner loop is concerned, the simplest situation in which the design paradigm outlined above can be successfully implemented is the case in which the controlled plant is globally asympotically and locally exponentially *minimum phase*. In this case, in fact, setting $M = 0$, $N = 0$, and letting $\dot{\chi} = L(\chi)$ to be any arbitrary globally stable system, it is always possible, by known methods, to find a vector K that makes the inner loop stable (in a semiglobal sense) with an arbitrarily small linear gain function.

If, on the contrary, the plant *is not* minimum-phase, a more sophisticated design is necessary, seeking $L(\cdot), M, N(\cdot)$ and K in such a way as to obtain – whenever possible – a stable inner loop, with a gain function which, composed with the gain function of the filter (58), would respect the small gain condition required for the stability of (53). A number of relevant cases in which this is possible have been recently presented in the literature (see [21] and [22]). They include the complete solution of the problem in the case of a (linear) controlled plant having an arbitrary number of zeros at the origin and a discussion of the case in which the controlled plant has a zero with positive real part. In the latter case, the method is applicable if the frequencies which characterize the harmonic components of the exogenous input exceed a minimal value determined by the gain needed to make the inner loop stable. This shows that, in a non-minimum phase system, a tradeoff exists between stability and performance. In fact, the minimal gain needed to stabilize the unstable zero dynamics of the original plant determines a *lower limit* on the frequencies of the exogenous inputs for which the desired *tracking properties* can be achieved.

References

[1] A. Isidori, L. Marconi, A. Serrani, *Robust Autonomous Guidance: and Internal-Model Approach*, Springer Verlag, London (2003).

[2] J. Huang, *Nonlinear Output Regulation: Theory and Applications.* SIAM, Philadephia (2004)

[3] A. Pavlov, N. van de Wouw, H. Nijmeijer, *Uniform Output Regulation of Nonlinear Systems: a Convergent Dynamics Approach.* Birkhauser, Boston (2006)

[4] G.D. Birkhoff, *Dynamical Systems*, American Mathematical Society (1927).

[5] J.K. Hale, L.T. Magalhães, W.M. Oliva, *Dynamics in Infinite Dimensions*, Springer Verlag, New York (2002).

[6] A. Isidori, C.I. Byrnes, Steady-state behaviors in nonlinear systems with an application to robust risturbance rejection, *Annual Reviews in Control*, **32**, pp. 1-16, 2008).

[7] E.D. Sontag, On the input-to-state stability property, *European Journal of Control*, **1**, pp. 24–36 1995.

[8] E.D. Sontag and Yang Y., New characterizations of input-to-state stability, *IEEE Trans. Automatic Control*, **AC-41**, pp. 1283–1294, 1996.

[9] C.I. Byrnes, A. Isidori, Asymptotic stabilization of minimum phase nonlinear systems, *IEEE Trans. Automatic Control*, **AC-36**, pp. 1122-1137, 1991.

[10] A. Isidori, *Nonlinear Control Systems*, *3rd ed.*, Springer Verlag, London (1995).

[11] M. Kristic, I. Kanellakopoulos, P. Kokotovic, *Nonlinear Adaptive Control Design*, J.Wiley (New York), 1995.

[12] R. Marino, P.Tomei, *Nonlinear Control Design: adaptive, robust*, Prentice Hall (New York), 1995.

[13] A. Isidori, *Nonlinear Control Systems: volume II*, Springer Verlag, London (1999).

[14] A.R. Teel and L. Praly, Tools for semiglobal stabilization by partial state and output feedback. *SIAM J. Control Optim.*, **33**, pp. 1443–1485, 1995.

[15] J.P. Gauthier, I. Kupka, *Deterministic Observation Theory and Applications*, Cambridge University Press, Cambridge (2001).

[16] F. Esfandiari, H. Khalil, Output feedback stabilization of fully linearizable systems. *International J. of Control* **56**, pp. 1007–1037, 1992.

[17] C.I. Byrnes, A. Isidori, Limit sets, zero dynamics and internal models in the problem of nonlinear output regulation, *IEEE Trans. Automatic Control*, **AC-48**, pp. 1712–1723, 2003.

[18] J. Huang, C.F. Lin, On a Robust Nonlinear Multivariable Servomechanism Problem, IEEE Transaction on Automatic Control, **AC-39**, pp. 1510–1513, 1994.

[19] C.I. Byrnes, A. Isidori, Nonlinear internal models for output regulation, *IEEE Trans. on Automatic Control*, **AC-49**, pp. 2244–2247, 2004.

[20] L. Marconi, L. Praly, A. Isidori A, Output stabilization via nNonlinear Luenberger observers, SIAM J. Control and Optimization, **45**, pp. 2277–2298, 2006.

[21] F. Delli Priscoli, L. Marconi, A. Isidori, A dissipativity-based approach to output regulation of non-minimum-phase systems, *Systems & Control Lett.*, **58**, pp. 584–591, 2009

[22] F. Delli Priscoli, L. Marconi, A. Isidori, A method for robust regulation of ron-mimimum-phase linear systems, Proc. 6th IFAC Symp on *Robust Control Design* (Haifa, Israel), pp. 255–260, 2009.