Delay-Dependent Stability Criteria for Singular Time-Delay Systems

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Abstract This note studies the problem of singular time-delay systems. At first, the equivalence among several recent stability criteria is established, and a simplified version is derived. By using a delay decomposition method, a new stability criterion which is much less conservative than the existing ones is presented. A numerical example is given to illustrate the effectiveness and less conservatism of the new proposed stability criterion.

Key words Delay-dependent stability, singular systems, linear matrix inequalities (LMIs)

1 Introduction

Over the past decades, much attention has been focused on the stability analysis and controller synthesis for singular linear time-delay systems due to the fact that the singular system model is a natural presentation of dynamic systems and it can describe a large class of systems than regular ones, such as large-scale systems, power systems and constrained control systems. Similar to the state-space time-delay systems, the results on stability analysis and stabilization for singular time-delay systems can be classified into two categories, that is, delay-independent criteria[1, 2] and delay-dependent ones[3, 4]. Generally, the delay-dependent case is less conservative than delay-independent ones, especially when the delay is comparatively small.

Recently, many researchers have paid attention to stability analysis of singular systems with time-delay[3, 5, 6, 7]. The computational results in [3] show that its stability criterion is less conservative than the one in [5], see the Example 1 in [3]. In fact, the conclusion based on the computational results contains errors.

In this note, we will prove that the stability result proposed in [3] is equivalent to the ones in [5,6,7], and a simplified version of Theorem 1 in [3] will be derived. Furthermore, by using a delay composition method, a less conservative result will be presented.

2 Problem formulation

Consider the following continuous-time singular system with a time-varying delay in the state [3]:

$$(\Sigma):\quad E\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + A_\tau \boldsymbol{x}(t-\tau), \quad t>0 \tag{1}$$

$$\boldsymbol{x}(t) = \boldsymbol{\phi}(t) \quad t \in [-\tau,\ 0], \tag{2}$$

where $\boldsymbol{x}(t) \in \mathbf{R}^n$ is the state, $\boldsymbol{\phi}(t) \in \mathcal{C}_{n,\tau}$ is a compatible vector valued initial function. The matrix $E \in \mathbf{R}^{n\times n}$ may be singular and $\mathrm{rank}E = p \le n$. A, A_τ are constant matrices with appropriate dimensions. τ is an unknown but constant delay satisfying

$$0 < \tau \le \tau_m. \tag{3}$$

Without loss of generality, the matrices E, A and A_τ are assumed to have the forms:

$$E = \begin{bmatrix} I_p & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad A_\tau = \begin{bmatrix} A_{\tau 11} & A_{\tau 12} \\ A_{\tau 21} & A_{\tau 22} \end{bmatrix}. \tag{4}$$

For the system (Σ), [3] provided a stability criterion as follows.

Lemma 1. [3] The singular time-delay system (Σ) is regular, impulse free and asymptotically stable for any constant delay τ satisfying $0<\tau\le\tau_m$, if there exist matrices

$$P = \begin{bmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{bmatrix},\ P_{11}>0,\quad Q>0,$$
$$Z = \begin{bmatrix} Z_{11} & Z_{12} \\ * & Z_{22} \end{bmatrix} > 0,$$
$$Y = \begin{bmatrix} Y_{11} & 0 \\ Y_{21} & 0 \end{bmatrix},\quad W = \begin{bmatrix} W_{11} & 0 \\ W_{21} & 0 \end{bmatrix},$$
$$Y_1 = \begin{bmatrix} Y_{11} \\ Y_{21} \end{bmatrix},\quad W_1 = \begin{bmatrix} W_{11} \\ W_{21} \end{bmatrix}, \tag{5}$$

with appropriate dimensions and $P_{11} \in \mathbf{R}^{p\times p}$, $Z_{11} \in \mathbf{R}^{p\times p}$, $Y_{11} \in \mathbf{R}^{p\times p}$, $W_{11} \in \mathbf{R}^{p\times p}$ satisfying the following LMI:

$$\Phi < 0, \tag{6}$$

where

$$\Phi = \begin{bmatrix} \Phi_1 & PA_\tau - Y + W^{\mathrm T} + \tau_m A^{\mathrm T} Z A_\tau & -\tau_m Y_1 \\ * & -Q - W - W^{\mathrm T} + \tau_m A_\tau^{\mathrm T} Z A_\tau & -\tau_m W_1 \\ * & * & -\tau_m Z_{11} \end{bmatrix},$$
$$\Phi_1 = PA + A^{\mathrm T}P^{\mathrm T} + Y + Y^{\mathrm T} + Q + \tau_m A^{\mathrm T} Z A.$$

For convenience of comparison, the stability criteria in [5, 6, 7] are listed as the following lemmas.

Lemma 2. [5] Consider the descriptor system (Σ), for given scalars $\tau_m>0$, if there exist matrices $\tilde P_1>0$, $\tilde P_2$, $\tilde P_3$, $\tilde Q>0$, $\tilde R>0$, $\tilde T_i$ and $\tilde S_i$ of appropriate dimensions ($i = 1,\ 2,\ 3$) such that

$$\Gamma < 0, \tag{7}$$

where

$$\Gamma = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \tau_m \tilde T_1 \\ * & \Gamma_{22} & \Gamma_{23} & \tau_m \tilde T_2 \\ * & * & \Gamma_{33} & \tau_m \tilde T_3 \\ * & * & * & -\tau_m \tilde R \end{bmatrix},$$
$$\Gamma_{11} = \tilde Q + \tilde T_1 E + E^{\mathrm T}\tilde T_1^{\mathrm T} - \tilde S_1 A - A^{\mathrm T}\tilde S_1^{\mathrm T},$$
$$\Gamma_{12} = -\tilde T_1 E + E^{\mathrm T}\tilde T_2^{\mathrm T} - \tilde S_1 A_\tau - A^{\mathrm T}\tilde S_2^{\mathrm T},$$
$$\Gamma_{13} = \tilde P + \tilde S_1 + E^{\mathrm T}\tilde T_3^{\mathrm T} - A^{\mathrm T}\tilde S_3^{\mathrm T},$$
$$\Gamma_{22} = -\tilde Q - \tilde T_2 E - E^{\mathrm T}\tilde T_2^{\mathrm T} - \tilde S_2 A_\tau - A_\tau^{\mathrm T}\tilde S_2^{\mathrm T},$$
$$\Gamma_{23} = \tilde S_2 - E^{\mathrm T}\tilde T_3^{\mathrm T} - A_\tau^{\mathrm T}\tilde S_3^{\mathrm T},$$
$$\Gamma_{33} = \tau_m \tilde R + \tilde S_3 + \tilde S_3^{\mathrm T},$$
$$P = \begin{bmatrix} \tilde P_1 & \tilde P_2 \\ 0 & \tilde P_3 \end{bmatrix},$$

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then the system (Σ) is E–exponentially stable.

Lemma 3. [6] Given scalars $\tau_m > 0$. Then, for any delay $0 < \tau \leq \tau_m$, the singular delay system (Σ) is regular, impulse free and stable if there exist matrices $Q = Q^{\mathrm{T}} > 0$, $Z = Z^{\mathrm{T}} > 0$, P, Y and W, such that the following LMIs hold:

$$E^{\mathrm{T}}P = P^{\mathrm{T}}E \geq 0, \tag{8}$$

$$\Omega < 0, \tag{9}$$

where

$$\Omega = \begin{bmatrix} \Omega_{11} & \Omega_{12} & \tau_m Y^{\mathrm{T}} & \tau_m A^{\mathrm{T}}Z \\ * & \Gamma_{22} & \tau_m W^{\mathrm{T}} & \tau_m A_\tau^{\mathrm{T}}Z \\ * & * & -\tau_m Z & 0 \\ * & * & * & -\tau_m Z \end{bmatrix},$$

$$\Omega_{11} = P^{\mathrm{T}}A + A^{\mathrm{T}}P + Q - Y^{\mathrm{T}}E - E^{\mathrm{T}}Y,$$

$$\Omega_{12} = P^{\mathrm{T}}A_\tau + Y^{\mathrm{T}}E - E^{\mathrm{T}}W,$$

$$\Omega_{22} = W^{\mathrm{T}}E + E^{\mathrm{T}}W - Q.$$

Lemma 4. [7] Given scalars $\tau_m > 0$. Then for any delay $0 < \tau \leq \tau_m$, the singular delay system (Σ) is regular, impulse free and stable if there exist matrices $Q = Q^{\mathrm{T}} > 0$, $Z = Z^{\mathrm{T}} > 0$, and matrices P_1, P_2, P_3, X_{11}, X_{12}, X_{13}, X_{22}, X_{23}, X_{33}, Y_1, Y_2 and T_1, such that

$$E^{\mathrm{T}}P_1 = P_1^{\mathrm{T}}E \geq 0, \tag{10}$$

$$\Pi < 0, \tag{11}$$

$$X \geq 0, \tag{12}$$

where

$$\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} & -Y_1E + P_2^{\mathrm{T}}A_\tau + E^{\mathrm{T}}T_1^{\mathrm{T}} + \tau_m X_{13} \\ * & \Pi_{22} & -Y_2E + P_3^{\mathrm{T}}A_\tau + \tau_m X_{23} \\ * & * & -Q - T_1E - E^{\mathrm{T}}T_1^{\mathrm{T}} + \tau_m X_{33} \end{bmatrix},$$

$$\Pi_{11} = P_2^{\mathrm{T}}A + A^{\mathrm{T}}P_2 + Y_1E + E^{\mathrm{T}}Y_1^{\mathrm{T}} + \tau_m X_{11} + Q,$$

$$\Pi_{12} = P_1^{\mathrm{T}} - P_2^{\mathrm{T}} + A^{\mathrm{T}}P_3 + E^{\mathrm{T}}Y_2^{\mathrm{T}} + \tau_m X_{12},$$

$$\Pi_{22} = -P_3 - P_3^{\mathrm{T}} + \tau_m X_{22} + \tau_m Z,$$

$$X = \begin{bmatrix} X_{11} & X_{12} & X_{13} & Y_1 \\ * & X_{22} & X_{23} & Y_2 \\ * & * & X_{33} & T_1 \\ * & * & * & Z \end{bmatrix}.$$

In this note, we will prove the the equivalence among the above lemmas, and give a simplified version of these criteria. Furthermore, by using a delay decomposition method, an improved result is proposed.

3 The equivalence among several stability criteria

In this section, the equivalence among the existing stability criteria given in [3,5,6,7] will be established, which further shows that the computational results given in [3] are incorrect.

Now, we prove the equivalence among the stability conditions in Lemmas 1-4, and a new stability criterion which contains fewer decision variables is also derived.

Theorem 1. The following statements are equivalent:
i) inequality (6) is feasible.
ii) the following inequality is feasible:

$$\Psi < 0, \tag{13}$$

where

$$\Psi = \begin{bmatrix} \Psi_1 & PA_\tau + \tau_m A^{\mathrm{T}}ZA_\tau + \tau_m^{-1}H^{\mathrm{T}}Z_{11}H \\ * & -Q + \tau_m A_\tau^{\mathrm{T}}ZA_\tau - \tau_m^{-1}H^{\mathrm{T}}Z_{11}H \end{bmatrix},$$

$$\Psi_1 = PA + (PA)^{\mathrm{T}} + Q + \tau_m A^{\mathrm{T}}ZA - \tau_m^{-1}H^{\mathrm{T}}Z_{11}H,$$

$$H = \begin{bmatrix} I_p & 0 \end{bmatrix}.$$

iii) inequality (7) is feasible.
iv) inequality (9) with (8) is feasible.
v) inequalities (11) and (12) with (10) are feasible.

Proof. i) $\Leftrightarrow$ ii):
Noticing that $Y = Y_1H$ and $W = W_1H$, pre- and post-multiplying

$$\begin{bmatrix} I & 0 & \tau_m^{-1}H^{\mathrm{T}} \\ 0 & I & -\tau_m^{-1}H^{\mathrm{T}} \\ 0 & 0 & I \end{bmatrix}$$

and its transpose on both sides of Φ in (6), and from the Schur complement, it follows that $\Phi < 0$ in Lemma 1 is equivalent to

$$\Psi + \begin{bmatrix} -\tau_m Y_1 - H^{\mathrm{T}}Z_{11} \\ -\tau_m W_1 + H^{\mathrm{T}}Z_{11} \end{bmatrix} (\tau_m Z_{11})^{-1} \begin{bmatrix} -\tau_m Y_1 - H^{\mathrm{T}}Z_{11} \\ -\tau_m W_1 + H^{\mathrm{T}}Z_{11} \end{bmatrix}^{\mathrm{T}} < 0. \tag{14}$$

So, $\Psi < 0$ holds if $\Phi < 0$ holds.
Conversely, if $\Psi < 0$ holds, by letting

$$Y_1 = -\tau_m^{-1}H^{\mathrm{T}}Z_{11}, \quad W_1 = \tau_m^{-1}H^{\mathrm{T}}Z_{11},$$

it yields that $\Phi < 0$ also holds.
Thus, $\Psi < 0$ is equivalent to $\Phi < 0$.
ii) $\Leftrightarrow$ iii):
Pre- and post-multiplying

$$\begin{bmatrix} I & 0 & 0 & -\tau_m^{-1}E^{\mathrm{T}} \\ 0 & I & 0 & \tau_m^{-1}E^{\mathrm{T}} \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$$

and its transpose on both sides of Γ in (7), it yields that

$$\begin{bmatrix} \Xi & \tilde{T} \\ * & -\tau_m \tilde{R} \end{bmatrix} < 0 \tag{15}$$

where

$$\Xi = \begin{bmatrix} \Xi_{11} & \Xi_{12} & \tilde{P} + \tilde{S}_1 - A^{\mathrm{T}}\tilde{S}_3^{\mathrm{T}} \\ * & \Xi_{22} & \tilde{S}_2 - A_\tau^{\mathrm{T}}\tilde{S}_3^{\mathrm{T}} \\ * & * & \tau_m\tilde{R} + \tilde{S}_3 + \tilde{S}_3^{\mathrm{T}} \end{bmatrix},$$

$$\Xi_{11} = \tilde{Q} - \tilde{S}_1A - (\tilde{S}_1A)^{\mathrm{T}} - \tau_m^{-1}E^{\mathrm{T}}\tilde{R}E,$$

$$\Xi_{12} = -\tilde{S}_1A_\tau - (\tilde{S}_2A)^{\mathrm{T}} + \tau_m^{-1}E^{\mathrm{T}}\tilde{R}E,$$

$$\Xi_{22} = -\tilde{Q} - \tilde{S}_2A_\tau - (\tilde{S}_2A_\tau)^{\mathrm{T}} - \tau_m^{-1}E^{\mathrm{T}}\tilde{R}E,$$

$$\tilde{T} = \begin{bmatrix} \tau_m\tilde{T}_1 + E^{\mathrm{T}}\tilde{R} \\ \tau_m\tilde{T}_2 - E^{\mathrm{T}}\tilde{R} \\ \tau_m\tilde{T}_3 \end{bmatrix}.$$

Similar to the proof of i) $\Leftrightarrow$ ii), it is clear that $\Gamma < 0$ is feasible if and only if $\Xi < 0$ is feasible.

Note that

$$\Xi = \bar{\Xi} + \tilde{S}\mathcal{A} + \mathcal{A}^{\mathrm{T}}\tilde{S}^{\mathrm{T}}, \tag{16}$$

where

$$\bar{\Xi} = \begin{bmatrix} \tilde{Q} - \tau_m^{-1}E^{\mathrm{T}}\tilde{R}E & \tau_m^{-1}E^{\mathrm{T}}\tilde{R}E & \tilde{P} \\ * & -\tilde{Q} - \tau_m^{-1}E^{\mathrm{T}}\tilde{R}E & 0 \\ * & * & \tau_m\tilde{R} \end{bmatrix},$$

$$\tilde{S} = \begin{bmatrix} \tilde{S}_1^{\mathrm{T}} & \tilde{S}_2^{\mathrm{T}} & \tilde{S}_3^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}},$$

$$\mathcal{A} = \begin{bmatrix} -A & -A_\tau & I \end{bmatrix},$$

from the elimination lemma ([8], p. 22), it is known that $\Xi < 0$ is equivalent to

$$\tilde{\Xi} := \mathcal{N}_{\mathcal{A}}^{\mathrm{T}}\bar{\Xi}\mathcal{N}_{\mathcal{A}} < 0, \tag{17}$$

where

$$\mathcal{N}_{\mathcal{A}} = \begin{bmatrix} I & 0 \\ 0 & I \\ A & A_\tau \end{bmatrix}.$$

After some manipulation, one can get

$$\tilde{\Xi} = \begin{bmatrix} \tilde{P}A + A^{\mathrm{T}}\tilde{P}^{\mathrm{T}} + \tilde{Q} - \tau_m^{-1}E^{\mathrm{T}}\tilde{R}E + \tau_m A^{\mathrm{T}}\tilde{R}A & \tilde{P}A_\tau + \tau_m^{-1}E^{\mathrm{T}}\tilde{R}E + \tau_m A^{\mathrm{T}}\tilde{R}A_\tau \\ * & -\tilde{Q} - \tau_m^{-1}E^{\mathrm{T}}\tilde{R}E + \tau_m A_\tau^{\mathrm{T}}\tilde{R}A_\tau \end{bmatrix}.$$

By letting $P = \tilde{P}$, $Q = \tilde{Q}$ and $Z = \tilde{R}$, it is easy to know that Ψ in (13) is the same as $\tilde{\Xi}$, so $\Psi < 0$ if and only if $\tilde{\Xi} < 0$.

Thus, from the above analysis, one can get that $\Psi < 0$ if and only if $\Gamma < 0$.

ii) $\Leftrightarrow$ iv): Similar to the proof of (i) $\Leftrightarrow$ ii)), the equivalence between ii) and iv) can be easily obtained, and omitted here.

ii) $\Leftrightarrow$ v): Similar to the proofs of (i) $\Leftrightarrow$ ii)) and Theorem 2 in [9], the equivalence between ii) and v) can also be derived, and omitted here.

This completes the proof. □

Remark 1. Theorem 1 establishes the equivalence among several stability criteria reported in [3,5,6,7], which implies that the computation-based assertion in [3] that the stability criterion in [3] is less conservative than the one in [5], is incorrect. Compared with Lemma 1 [3], ii) of Theorem 1 involves less decision variables. Hence, from a mathematical point of view, ii) of Theorem 1 is more "powerful".

4 An improved stability criterion

In this section, an improved stability criterion will be proposed by using a delay decomposition method [10].

Theorem 2. The singular time-delay system (Σ) is regular, impulse free and asymptotically stable for a given positive integer N and any constant delay τ satisfying $0 < \tau \leq \tau_m$, if there exist matrices

$$P = \begin{bmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{bmatrix},\ P_{11} > 0,\ \ Q_i > 0,$$
$$Z_i > 0\ (i = 1,\ 2,\ \cdots,\ N), \tag{18}$$

with appropriate dimensions and $P_{11} \in \mathbf{R}^{p\times p}$ satisfying the following LMI:

$$\Theta < 0, \tag{19}$$

where

$$\Theta = \begin{bmatrix} \Theta_1 & \frac{N}{\tau_m}E^{\mathrm{T}}Z_1E & 0 & \cdots & 0 & PA_\tau + \frac{\tau_m}{N}A^{\mathrm{T}}\tilde{Z}A_\tau \\ * & \Theta_2 & \frac{N}{\tau_m}E^{\mathrm{T}}Z_2E & \cdots & 0 & 0 \\ * & * & \Theta_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & * & \cdots & \Theta_N & \frac{N}{\tau_m}E^{\mathrm{T}}Z_NE \\ * & * & * & \cdots & * & \Theta_{N+1} \end{bmatrix},$$

$$\Theta_1 = PA + A^{\mathrm{T}}P^{\mathrm{T}} + Q_1 + \frac{\tau_m}{N}A^{\mathrm{T}}\tilde{Z}A - \frac{N}{\tau_m}E^{\mathrm{T}}Z_1E,$$

$$\Theta_i = -Q_{i-1} + Q_i - \frac{N}{\tau_m}E^{\mathrm{T}}(Z_{i-1} + Z_i)E \quad (i = 2,\ 3,\ \cdots,\ N),$$

$$\Theta_{N+1} = -Q_N - \frac{N}{\tau_m}E^{\mathrm{T}}Z_NE + \frac{\tau_m}{N}A_\tau^{\mathrm{T}}\tilde{Z}A_\tau,$$

$$\tilde{Z} = \sum_{i=1}^{N} Z_i.$$

Proof. From (19), it follows that

$$PA + A^{\mathrm{T}}P^{\mathrm{T}} + Q_1 - \frac{N}{\tau_m}E^{\mathrm{T}}Z_1E < 0 \tag{20}$$

holds, which implies that

$$P_{22}A_{22} + A_{22}^{\mathrm{T}}P_{22}^{\mathrm{T}} < 0. \tag{21}$$

So, A_{22} is nonsingular. Pre- and post-multiplying $\begin{bmatrix} I & I & \cdots & I & I \end{bmatrix}$ and its transpose on the both sides of Θ in (19), it yields that

$$P(A + A_\tau) + (A + A_\tau)^{\mathrm{T}}P^{\mathrm{T}} - \frac{N}{\tau_m}\sum_{i=1}^{N}E^{\mathrm{T}}Z_iE < 0, \tag{22}$$

which implies that $A_{22} + A_{\tau 22}$ is also nonsingular. Thus, the pairs $(E,\ A)$ and $(E,\ A + A_\tau)$ are regular and impulse free.

Construct the Lyapunov-Krasovskii functional for system (Σ) as

$$V(\boldsymbol{x}_t) = \boldsymbol{x}^{\mathrm{T}}(t)PE\boldsymbol{x}(t) + \sum_{i=1}^{N}\Big(\int_{t-\tau_i}^{t-\tau_{i-1}}\boldsymbol{x}^{\mathrm{T}}(s)Q_i\boldsymbol{x}(s)\mathrm{d}s + \int_{-\tau_i}^{-\tau_{i-1}}\int_{t+\theta}^{t}\dot{\boldsymbol{x}}^{\mathrm{T}}(s)E^{\mathrm{T}}Z_iE\dot{\boldsymbol{x}}(s)\mathrm{d}s\mathrm{d}\theta\Big), \tag{23}$$

where $\boldsymbol{x}_t = \boldsymbol{x}(t+\theta)$ $(-\tau_m \leq \theta \leq 0)$ and $\tau_i = \frac{i}{N}\tau$ $(i = 0,\ 1,\ 2,\ \cdots,\ N)$.

Taking the time derivative of $V(\boldsymbol{x}_t)$ along with the solution of (Σ) yields

$$
\begin{aligned}
&\dot{V}(\boldsymbol{x}_t)\\
=&2\boldsymbol{x}^{\mathrm{T}}(t)PE\dot{\boldsymbol{x}}(t)\\
&+\sum_{i=1}^{N}\Big(\boldsymbol{x}^{\mathrm{T}}(t-\tau_{i-1})Q_i\boldsymbol{x}(t-\tau_{i-1})-\boldsymbol{x}^{\mathrm{T}}(t-\tau_i)Q_i\boldsymbol{x}(t-\tau_i)\Big)\\
&+\sum_{i=1}^{N}\Big(\frac{\tau}{N}\dot{\boldsymbol{x}}^{\mathrm{T}}(t)E^{\mathrm{T}}Z_iE\dot{\boldsymbol{x}}(t)-\int_{t-\tau_i}^{t-\tau_{i-1}}\dot{\boldsymbol{x}}^{\mathrm{T}}(s)E^{\mathrm{T}}Z_iE\dot{\boldsymbol{x}}(s)ds\Big)\\
\le&2\boldsymbol{x}^{\mathrm{T}}(t)P[A\boldsymbol{x}(t)+A_\tau\boldsymbol{x}(t-\tau)]\\
&+\sum_{i=1}^{N}\Big(\boldsymbol{x}^{\mathrm{T}}(t-\tau_{i-1})Q_i\boldsymbol{x}(t-\tau_{i-1})-\boldsymbol{x}^{\mathrm{T}}(t-\tau_i)Q_i\boldsymbol{x}(t-\tau_i)\Big)\\
&+\sum_{i=1}^{N}\Big(\frac{\tau_m}{N}[A\boldsymbol{x}(t)+A_\tau\boldsymbol{x}(t-\tau)]^{\mathrm{T}}Z_i[A\boldsymbol{x}(t)+A_\tau\boldsymbol{x}(t-\tau)]\Big)\\
&-\frac{N}{\tau_m}\sum_{i=1}^{N}([\boldsymbol{x}(t-\tau_{i-1})-\boldsymbol{x}(t-\tau_i)]^{\mathrm{T}}E^{\mathrm{T}}\\
&\qquad Z_iE[\boldsymbol{x}(t-\tau_{i-1})-\boldsymbol{x}(t-\tau_i)])\\
=&\boldsymbol{\xi}^{\mathrm{T}}(t)\Theta\boldsymbol{\xi}(t),
\end{aligned}
\tag{24}
$$

where

$$\boldsymbol{\xi}(t)=\begin{bmatrix}\boldsymbol{x}^{\mathrm{T}}(t) & \boldsymbol{x}^{\mathrm{T}}(t-\tau_1) & \cdots & \boldsymbol{x}^{\mathrm{T}}(t-\tau)\end{bmatrix}^{\mathrm{T}}.$$

Therefore, by (19) it is easy to see that $\dot{V}(\boldsymbol{x}_t)<0$.

This completes the proof. □

Remark 2. In the proof of Theorem 2, the delay interval $[0,\ \tau_m]$ is divided into N segments of equal length $\frac{\tau_m}{N}$, such that the information of delayed states $\boldsymbol{x}(t-i\frac{\tau_m}{N})$ $(i=1,\ 2,\ \cdots,\ N)$ are all taken into account. It is clear that the Lyapunov function defined in Theorem 2 is more general than the ones in [3] and [5,6,7], etc.

The following theorem shows the relationship between Theorem 2 and ii) of Theorem 1.

Theorem 3. Inequality (19) is feasible if inequality (13) is feasible.

Proof. If inequality (13) is feasible, then there exists a scalar $\varepsilon>0$ such that

$$\tilde{\Psi}<0, \tag{25}$$

where

$$\tilde{\Psi}=\begin{bmatrix}
\tilde{\Psi}_1 & 0 & 0 & \cdots & 0 & PA_\tau+\tau_mA^{\mathrm{T}}ZA_\tau+\tau_m^{-1}H^{\mathrm{T}}Z_{11}H\\
* & -\varepsilon I & 0 & \cdots & 0 & 0\\
* & * & -\varepsilon I & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
* & * & * & \cdots & -\varepsilon I & 0\\
* & * & * & \cdots & * & -Q+\tau_mA_\tau^{\mathrm{T}}ZA_\tau-\tau_m^{-1}H^{\mathrm{T}}Z_{11}H
\end{bmatrix},$$

$$
\begin{aligned}
\tilde{\Psi}_1&=PA+(PA)^{\mathrm{T}}+Q+(N-1)\varepsilon I+\tau_mA^{\mathrm{T}}ZA\\
&\quad-\tau_m^{-1}H^{\mathrm{T}}Z_{11}H,\\
H&=\begin{bmatrix}I_p & 0\end{bmatrix}.
\end{aligned}
$$

Letting $Z_i=Z$ $(i=1,\ 2,\ \cdots,\ N)$, $Q_N=Q$, $Q_{N-1}=Q+\varepsilon I$, $\cdots$, $Q_1=Q+(N-1)\varepsilon I$, and denoting $\Delta=\Theta-\tilde{\Psi}$, it yields that

$$\Delta=\begin{bmatrix}
-\frac{N-1}{\tau_m}E^{\mathrm{T}}ZE & \frac{N}{\tau_m}E^{\mathrm{T}}ZE & 0 & \cdots & 0 & -\frac{1}{\tau_m}E^{\mathrm{T}}ZE\\
* & -\frac{2N}{\tau_m}E^{\mathrm{T}}ZE & \frac{N}{\tau_m}E^{\mathrm{T}}ZE & \cdots & 0 & 0\\
* & * & -\frac{2N}{\tau_m}E^{\mathrm{T}}ZE & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
* & * & * & \cdots & -\frac{2N}{\tau_m}E^{\mathrm{T}}ZE & \frac{N}{\tau_m}E^{\mathrm{T}}ZE\\
* & * & * & \cdots & * & -\frac{N-1}{\tau_m}E^{\mathrm{T}}ZE
\end{bmatrix}. \tag{26}$$

Next, we prove that $\Delta\le 0$ holds.

When $N=1$, it is obvious that $\Delta=0$, so $\Theta<0$ is also feasible.

If $N=2$, then Δ becomes to

$$\Lambda:=\begin{bmatrix}
-\frac{1}{\tau_m}E^{\mathrm{T}}ZE & \frac{2}{\tau_m}E^{\mathrm{T}}ZE & -\frac{1}{\tau_m}E^{\mathrm{T}}ZE\\
* & -\frac{4}{\tau_m}E^{\mathrm{T}}ZE & \frac{2}{\tau_m}E^{\mathrm{T}}ZE\\
* & * & -\frac{1}{\tau_m}E^{\mathrm{T}}ZE
\end{bmatrix}. \tag{27}$$

Pre- and post-multiplying $\begin{bmatrix} I & I & I\\ 0 & I & 0\\ 0 & \frac{1}{2}I & I\end{bmatrix}$ and its transpose on the both sides of Λ, it gets

$$\tilde{\Lambda}:=\begin{bmatrix}
0 & 0 & 0\\
* & -\frac{4}{\tau_m}E^{\mathrm{T}}ZE & 0\\
* & * & 0
\end{bmatrix}. \tag{28}$$

It is obvious that $\tilde{\Lambda}\le 0$, which implies that $\Delta\le 0$ holds.

For the case of $N>2$, the proof is similar to the case of $N=2$, and omitted here.

The result is established. □

Remark 3. From Theorem 3, it is easy to see that Theorem 2 is less conservative than ii) of Theorem 1. As N increasing, the conservatism of Theorem 2 decreases. An example in the next section will verify this fact.

5 Example

Example 1. [3] Consider a singular delay system which is in the form of (1) with

$$E=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix},\quad A=\begin{bmatrix}0.5 & 0\\ -1 & -1\end{bmatrix},\quad A_\tau=\begin{bmatrix}-1 & 0\\ 0 & 0\end{bmatrix}.$$

Table 1 lists the comparison of the calculating results obtained by the stability criteria in [3,5,6,7,11] and this note.

It is worth pointing out that the maximum τ_m obtained by Theorem 3.5 in [11] should be 1.1547, and not 1.1612 which was given in [3].

Certainly, the maximum τ_m obtained by Theorem 1 in [3] should be 1.1547, and not 1.2011 which was listed in [3].

From Table 1, it is clear that Theorem 1 in [3] may not be less conservative than Theorem 3.5 in [11]. Fortunately, Example 2 in [6] showed that the calculating results obtained by Theorem 1 in [6] may be less conservative than the ones obtained by Theorem 3.5 in [11], and no theoretical proof had been provided in [6].

Summarily, ii) of Theorem 1 in this note contains the fewest variables and Theorem 2 in this note is less conservative than those in [3,5,6,7].

Table 1 Comparisons of delay-dependent stability conditions of Example 1

Methods	Maximum τ_m allowed	Number of variables
Theorem 1 [7]	1.1547	53
Theorem 1 [5]	1.1547	33
Theorem 3.5 [11]	1.1547	24
Theorem 1 [6]	1.1547	17
Theorem 1 [3]	1.1547	13
ii) of Theorem 1	1.1547	9
Theorem 2 $N = 2$	1.1954	15
Theorem 2 $N = 3$	1.2025	21
Theorem 2 $N = 4$	1.2044	27
Theorem 2 $N = 5$	1.2052	33

6 Conclusion

This note studies the stability of singular systems with state delay, and proves theoretically the equivalence among several recent results via the techenique for eliminating redundant variables. By making use of a delay decomposition method, a result which is much less conservative than the previous relevant ones is obtained, which has been shown by a numerical example.

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