Design of Robust Model Predictive Control Based on Multi-step Control Set

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Abstract By proposing the multi-step control set and using it as the terminal set, a new design method of robust model predictive control (MPC) is presented for the constrained polyhedral uncertain systems. Through designing a series of feedback control laws, the new robust controller can achieve large feasible region and high control performance. The characteristic property of the multi-step control set also makes it possible to derive an algorithm with low online computation burden. It could make a good trade off among the initial feasible region, the control performance, and the online computation burden. The numerical examples verify these results.

Key words Model predictive control (MPC), robust, multi-step control set

Model predictive control (MPC), also named as receding horizon control (RHC), is a kind of control algorithms characterized with receding horizon optimization. At each time instant, the MPC controller solves an optimization problem to get the current control input. So the online computation burden is always an important issue for MPC, especially for robust MPC. Since the last decade, robust MPC with low online computation burden, high control performance, and large feasible region has been a very attractive topic.

Based on the theory of control invariant set and using LMI tool, Kothare[1] proposed a design method of robust MPC controller for constrained polyhedral uncertain systems. Although the method in [1] is effective, if the number of models is large or the system dimension is high, the online computation burden will be heavy. To reduce its online computation burden, [2] proposed an offline design method. In addition, in order to release the conservativeness in [1], [3−4] suggested to use parameter dependent Lyapunov functions instead of a single Lyapunov function. Another spectrum of robust MPC based on the theory of control invariant set is efficient robust predictive control (ERPC)[5−7]. It designs offline an invariant set for an augment system and online gets the control inputs based on the unconstrained optimal feedback control law so as to reduce the online computation burden and achieve high control performance. However, all the above works are based on a control invariant set with a single feedback control law and assume that the current system state should be located in the control invariant set, so the design methods are conservative and the initial feasible region should be restricted.

In [8], Wan proposed an efficient robust MPC controller with low online computation burden, high control performance, and large feasible region, where free control moves followed by the terminal invariant set were introduced to release the conservativeness of [1]. The similar method was also adopted in [9]. But due to the uncertainty of systems, the feasibility and stability in [8−9] cannot be guaranteed. Reference [10] proposed a modified design to [8], but the online computation burden of the modified controller was very heavy.

For uncertain systems, Mayne[11] suggested to adopt the feedback MPC framework instead of free control moves to guarantee the feasibility of the MPC controller. This suggestion is important and instructive. But how to design it with acceptable online computation burden is still open. Recently, by using a series of feedback control laws, Lee[12] proposed the concept of periodic invariance. Although it is novel, the periodic invariance and the design methods based on it in [12] have two weaknesses. First, the control performance is not considered in the design. Second, the series of feedback control laws are mainly used to guarantee the invariance of the set and not to enlarge the feasible region of the controller.

In this paper, we propose a new concept, i.e., multi-step control set, which uses a series of feedback control laws to steer the system states from one ellipsoidal set to another and finally into a control invariant set. With this new concept and by introducing the index of the multi-step control set, a robust MPC controller with high control performance and large feasible region is designed. Furthermore, through offline designing the multi-step control sets, the online computation burden can be reduced such that we can reasonably balance the control performance, the feasible region, and the online computation burden.

This paper is organized as follows. In Section 1, the uncertain system model and some background are introduced. Section 2 gives the concept of multi-step control set and its design method. Then, two feedback robust MPC controllers are developed in Section 3. Numerical examples in Section 4 verify the effectiveness of the proposed robust MPC controllers.

Notations. Denote $\boldsymbol{u}(k+i|k)$ and $\boldsymbol{x}(k+i|k)$ as the control input and system state of time $k+i$, predicted at time k. $||\boldsymbol{x}||_W = \boldsymbol{x}^{\mathrm{T}}W\boldsymbol{x}$ and $\boldsymbol{x}(k|k) = \boldsymbol{x}(k)$.

1 Background

Consider the following polyhedral uncertain system:

$$\boldsymbol{x}(k+1) = A(k)\boldsymbol{x}(k) + B(k)\boldsymbol{u}(k) \tag{1}$$

where $A(k) \in \mathbf{R}^{n\times n}$, $B(k) \in \mathbf{R}^{n\times m}$, and

$$(A(k), B(k)) \in \Omega = \left\{(A(k), B(k)) | (A(k), B(k)) = \sum_{i=1}^{n_p} \lambda_i (A_i, B_i), \lambda_i \geq 0, \sum_{i=1}^{n_p} \lambda_i = 1\right\} \tag{2}$$

The constraints on the system inputs and measurable states are given by

$$|\boldsymbol{u}_i(k)| \leq \bar{u}_i, \quad i = 1, \cdots, m \tag{3}$$

$$|\boldsymbol{F}_j\boldsymbol{x}(k)| \leq \bar{x}_j, \quad j = 1, \cdots, n \tag{4}$$

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The cost function of the MPC controller can be commonly chosen as

$$\min_{\boldsymbol{U}(k)} \max_{[A(k+i)|B(k+i)]\in\Omega,\, i\geq 0} J(k)$$

$$J(k) = \sum_{i=0}^{\infty}[||\boldsymbol{x}(k+i|k)||_{\mathscr{Q}} + ||\boldsymbol{u}(k+i|k)||_{\mathscr{R}}] \tag{5}$$

with notation $\boldsymbol{U}(k) = [\boldsymbol{u}(k|k)^{\rm T}, \boldsymbol{u}(k+1|k)^{\rm T}, \cdots]^{\rm T}$, $\mathscr{Q}$ and $\mathscr{R}$ are the weighted matrices of cost objective function, respectively.

For the uncertain system (1) ∼ (4), Kothare[1] adopted the following control invariant set in their design, which will be called as the classical invariant set to distinguish from the concepts in this paper: the set $S = \{\boldsymbol{x}|\boldsymbol{x}^{\rm T}Q^{-1}\boldsymbol{x} \leq 1\}$ is a classical invariant set if there exist matrices X, Y, and L, satisfying the following conditions:

$$\begin{aligned}
&\left[\begin{array}{cc} Q & (A_jQ+B_jY)^{\rm T} \\ A_jQ+B_jY & Q \end{array}\right] > 0, \quad j=1,\cdots,n_p \\
&\left[\begin{array}{cc} X & Y \\ Y^{\rm T} & Q \end{array}\right] > 0,\ X_{ii} \leq \bar{u}_i^2, \quad i=1,\cdots,m \\
&\left[\begin{array}{cc} L & FQ \\ QF^{\rm T} & Q \end{array}\right] > 0,\ L_{ii} \leq \bar{x}_i^2, \quad i=1,\cdots,n
\end{aligned} \tag{6}$$

where matrices X, Y, and L are matrix variables, $F = [\boldsymbol{F}_1^{\rm T}, \cdots, \boldsymbol{F}_n^{\rm T}]^{\rm T}$, $X > 0$, $L > 0$, and X_{ii} and L_{ii} represent the i-th diagonal elements of X and L.

Based on the classical invariant set, [8−9] proposed the controllers using free control moves followed by a terminal classical invariant set. In order to guarantee the feasibility, [11] suggested to adopt the feedback MPC framework to design robust MPC controllers, i.e., using a policy $\pi = \{\boldsymbol{u}(0), K_1, \cdots, K_{N-1}, K_N\}$ instead of the control series $\boldsymbol{U}(k)$, where K_i is the feedback control gain at the i-th step and after the N-th step the feedback control gain is always K_N. From the feedback MPC framework, the works of [1, 5−7] can be thought as the case of $N = 0$ and the works of [8−9, 12] as $N > 2$. As mentioned above, these design methods have some weaknesses and conservativeness. So in the following part, we will give a new method to design the robust MPC controller for constrained polyhedral uncertain systems.

2 Multi-step control set

In order to release the conservativeness of the above MPC controllers, we adopt a sequence of feedback control laws to design MPC controller. Considering the input and state constraints, we assume that the system states will stay in a sequence of ellipsoidal sets and the feedback control laws will steer the states to transfer among these sets and finally into a classical invariant set. Thus, we give the following concept of multi-step control set.

Definition 1 (Multi-step control set and s-step control set). If the system states in set S_0 can be steered into a classical invariant set by some constrained feedback control laws, the set S_0 is called a multi-step control set. If the system states in set S_0 can be steered into a classical invariant set by $s-1$ constrained feedback control laws, the set S_0 is called an s-step control set.

Obviously, from Definition 1, if we can find s sets S_i ($i = 0, \cdots, s-1$), where S_i is an $(s-i)$-step control set, such that the system states will be steered from $S_0 \rightarrow S_1 \rightarrow \cdots \rightarrow S_{s-1}$, we can find a way to design the multi-step control set. Referring to the periodically-invariant set in [12], the following theorem gives the synthesis method of the s-step control set.

Theorem 1. Consider the uncertain system (1) ∼ (4). For an ellipsoidal set $S_0 = \{\boldsymbol{x}|\boldsymbol{x}^{\rm T}Q_0^{-1}\boldsymbol{x} \leq 1\}$, if there are $s-1$ ellipsoidal sets $S_i = \{\boldsymbol{x}|\boldsymbol{x}^{\rm T}Q_i^{-1}\boldsymbol{x} \leq 1\}(i = 1, \cdots, s-1)$ with corresponding matrices Y_i, X_i ($i = 0, \cdots, s-1$), satisfying the following conditions:

$$\left[\begin{array}{cc} Q_{i-1} & (A_jQ_{i-1}+B_jY_{i-1})^{\rm T} \\ A_jQ_{i-1}+B_jY_{i-1} & Q_i \end{array}\right] > 0$$
$$i=1,\cdots,s-1;\ j=1,\cdots,n_p \tag{7}$$

$$\left[\begin{array}{cccc} Q_{s-1} & \Phi_{(j,s-1)}^{\rm T} & Q_{s-1}\mathscr{Q}^{1/2} & Y_{s-1}^{\rm T}\mathscr{R}^{\frac{1}{2}} \\ \Phi_{(j,s-1)} & Q_{s-1} & 0 & 0 \\ \mathscr{Q}^{\frac{1}{2}}Q_{s-1} & 0 & \gamma I & 0 \\ \mathscr{R}^{\frac{1}{2}}Y_{s-1} & 0 & 0 & \gamma I \end{array}\right] > 0$$
$$\Phi_{(j,s-1)} = A_jQ_{s-1}+B_jY_{s-1}, \quad j=1,\cdots,n_p \tag{8}$$

$$\left[\begin{array}{cc} X_i & Y_i \\ Y_i^{\rm T} & Q_i \end{array}\right] > 0,\ X_{i,jj} \leq \bar{u}_j^2,$$
$$i=0,\cdots,s-1;\ j=1,\cdots,m \tag{9}$$

$$\left[\begin{array}{cc} L_i & FQ_i \\ Q_iF^{\rm T} & Q_i \end{array}\right] > 0,\ L_{i,jj} \leq \bar{x}_j^2,$$
$$i=0,\cdots,s-1;\ j=1,\cdots,n \tag{10}$$

then, the set S_0 is an s-step control set, the feedback control gain in S_i is $K_i = Y_iQ_i^{-1}$ and we call γ as the index of the s-step control set.

Proof. For $i < s$, from (7), we can get

$$Q_{i-1}^{-1} - (A_j + B_jY_{i-1}Q_{i-1}^{-1})^{\rm T}Q_i^{-1}(A_j + B_jY_{i-1}Q_{i-1}^{-1}) > 0 \tag{11}$$

That is, all the system states in set S_{i-1} will be steered into set S_i by the feedback control law with gain K_{i-1}. So the system states in set S_0 will be steered into set S_{s-1} by the control sequence with state feedback gains $K_0, \cdots, K_{s-2}$.

For $i \geq s$, from (8), it is clear that it is a classical invariant set, so the system states in set S_{s-1} will stay in it by the feedback control law with gain K_{s-1}. In addition, according to [1], we can get γ as the cost upper bound of the classical invariant set S_{s-1}, which is named as the index of the multi-step control set.

With respect to the input and state constraints, for a set S_i and its corresponding feedback control gain K_i, we can get

$$|\boldsymbol{u}_l|^2 = |\boldsymbol{\alpha}_lY_iQ_i^{-1}\boldsymbol{x}|^2 = |\boldsymbol{\alpha}_lY_iQ_i^{-\frac{1}{2}}Q_i^{-\frac{1}{2}}\boldsymbol{x}|^2 \leq ||\boldsymbol{\alpha}_lY_iQ_i^{-1/2}||,$$
$$l=1,\cdots,m$$

where $\boldsymbol{u}_l$ is the l-th element of the control input corresponding to the feedback control gain K_i and $\boldsymbol{\alpha}_l$ is the l-th row of m-dimensional identity matrix. Thus, condition (9) can be obtained.

Similarly handling constraint (4),

$$|\boldsymbol{F}_l\boldsymbol{x}|^2 = |\boldsymbol{F}_lQ_i^{\frac{1}{2}}Q_i^{-\frac{1}{2}}\boldsymbol{x}|^2 \leq ||\boldsymbol{F}_lQ_i^{\frac{1}{2}}|| \leq \bar{x}_l^2, \quad l=1,\cdots,n$$

then, we can get the condition (10). □

By Theorem 1 and Definition 1, the multi-step control set has the following properties.

Corollary 1. For the uncertain system (1) ~ (4), a classical invariant set is a multi-step control set whose number of steps is any positive integer, and an s-step control set is also an $(s+i)$-step control set, where i is any positive integer. Meanwhile, for an s-step control set with the series of ellipsoidal sets $S_0, \cdots, S_{s-1}$, each set in the series is also an s-step control set.

It is easy to get Corollary 1 from the definition of multi-step control set.

Corollary 2. For the uncertain system (1) ~ (4), sets $S_{(0,0)}, \cdots, S_{(h,0)}$ are its $h+1$ multi-step control sets and their indexes are $\gamma_i (i = 0, \cdots, h)$, respectively. Then, the set $\hat{S}_0 = \{\boldsymbol{x} | \boldsymbol{x}^{\mathrm{T}} \hat{Q}_0^{-1} \boldsymbol{x} \leq 1\}$, where $\hat{Q}_0 = \lambda_0 Q_{(0,0)} + \cdots + \lambda_h Q_{(h,0)}$ and $\lambda_0 + \cdots + \lambda_h \leq 1$, $\lambda_i \geq 0$ $(i = 0, \cdots, h)$, is also a multi-step control set and its index equals $\lambda_0\gamma_0 + \cdots + \lambda_h\gamma_h$.

Proof. Assume $S_{(0,0)}, \cdots, S_{(h,0)}$ are s-step control sets each with the series of ellipsoidal sets $S_{(i,0)}, \cdots, S_{(i,s-1)}$ $(i = 0, \cdots, h)$, respectively, where $S_{(i,j)} = \{\boldsymbol{x} | \boldsymbol{x}^{\mathrm{T}} Q_{(i,j)}^{-1} \boldsymbol{x} \leq 1\}$. And the step numbers of all the sets are the same, which is reasonable according to Corollary 1. From Theorem 1, all the sets satisfy conditions (7) ~ (10). Then, multiplying (7) ~ (8) by the coefficients $\lambda_0, \cdots, \lambda_h$, respectively, and summing up, we get

$$\begin{bmatrix} \hat{Q}_{i-1} & (A_j\hat{Q}_{i-1} + B_j\hat{Y}_{i-1})^{\mathrm{T}} \\ A_j\hat{Q}_{i-1} + B_j\hat{Y}_{i-1} & \hat{Q}_i \end{bmatrix} > 0,$$
$$i = 1, \cdots, s-1; \; j = 1, \cdots, n_p$$
$$\begin{bmatrix} \hat{Q}_{s-1} & \hat{\Phi}_{(j,s-1)}^{\mathrm{T}} & \hat{Q}_{s-1}\mathscr{Q}^{\frac{1}{2}} & \hat{Y}_{s-1}^{\mathrm{T}}\mathscr{R}^{\frac{1}{2}} \\ \hat{\Phi}_{(j,s-1)} & \hat{Q}_{s-1} & 0 & 0 \\ \mathscr{Q}^{\frac{1}{2}}\hat{Q}_{s-1} & 0 & \hat{\gamma}I & 0 \\ \mathscr{R}^{\frac{1}{2}}\hat{Y}_{s-1} & 0 & 0 & \hat{\gamma}I \end{bmatrix} > 0$$
$$\hat{\Phi}_{(j,s-1)} = A_j\hat{Q}_{s-1} + B_j\hat{Y}_{s-1}, \quad j = 1, \cdots, n_p$$

where $\hat{Q}_i = \lambda_0 Q_{(0,i)} + \cdots + \lambda_h Q_{(h,i)}$, $\hat{Y}_i = \lambda_0 Y_{(0,i)} + \cdots + \lambda_i Y_{(h,i)}$ $(i = 0, \cdots, s-1)$, and $\hat{\gamma} = \lambda_0\gamma_0 + \cdots + \lambda_h\gamma_h$.

That is to say, the set $\hat{S}_0 = \{\boldsymbol{x} | \boldsymbol{x}^{\mathrm{T}} (\lambda_0 Q_{(0,0)} + \cdots + \lambda_h Q_{(h,0)})^{-1} \boldsymbol{x} \leq 1\}$ and the series of ellipsoidal sets $\hat{S}_i = \{\boldsymbol{x} | \boldsymbol{x}^{\mathrm{T}} (\lambda_0 Q_{(0,i)} + \cdots + \lambda_h Q_{(h,i)})^{-1} \boldsymbol{x} \leq 1\}$ $(i = 1, \cdots, s-1)$ satisfy conditions (7) and (8) of Theorem 1 and the index of $\hat{S}$ is equal to $\lambda_0\gamma_0 + \cdots + \lambda_h\gamma_h$.

Similarly, dealing with conditions (9) and (10), since $\lambda_0 + \cdots + \lambda_h \leq 1, \lambda_i \geq 0$ $(i = 0, \cdots, h)$, we can get that the sets $\hat{S}_0, \cdots, \hat{S}_{s-1}$ satisfy conditions (9) and (10), too. Therefore, the set $\hat{S}_0$ is also a multi-step control set and its index equals $\lambda_0\gamma_0 + \cdots + \lambda_h\gamma_h$. □

From the proof of Corollary 2, it is obvious that the i-th step ellipsoidal set $\hat{S}_i$ of the new multi-step control set is also constructed by convex combination of the corresponding i-th step sets $S_{(0,i)}, \cdots, S_{(h,i)}$.

3 The robust MPC controller based on the multi-step control set

Based on the multi-step control set presented in Section 2, we can design a robust MPC controller, which adopts the current constrained input to steer the system state into a multi-step control set, and then the feedback control laws in the multi-step invariant set will steer the system state to the origin.

3.1 The robust MPC controller with online designing multi-step control set

For the uncertain system (1) ~ (4), similar to the method to approximate the infinite horizon min-max cost function in [1], we choose the cost function to make $\boldsymbol{x}(k+1|k)$ as close as possible to the origin as well as to make the index of the multi-step control set as small as possible. Since the index of the multi-step control set is the cost upper bound of its final classical invariant set, optimizing the index means approximately optimizing the control performance. Thus, the robust MPC controller can be designed as follows.

Algorithm 1.

$$\mathscr{P}_1: \min_{Q_i, Y_i, \boldsymbol{u}(k), X_i, L_i, r, \gamma} r + \gamma$$
$$\text{s.t.} \quad |\boldsymbol{u}_j(k)| \leq \bar{u}_j, \quad j = 1, \cdots, m$$
$$\begin{bmatrix} r & (A_j\boldsymbol{x}(k) + B_j\boldsymbol{u}(k))^{\mathrm{T}} \\ A_j\boldsymbol{x}(k) + B_j\boldsymbol{u}(k) & Q_0 \end{bmatrix} > 0$$
$$r \leq 1, \quad j = 1, \cdots, n_p$$
$$(7) \sim (10), \quad i = 0, \cdots, s-1 \tag{12}$$

where $\boldsymbol{u}(k)$ is the control input at time k.

For the stability of controller $\mathscr{P}_1$, we can get the following theorem.

Theorem 2. Consider the uncertain system (1) ~ (4). If controller $\mathscr{P}_1$ is feasible at time k for the current system state $\boldsymbol{x}(k)$, the closed loop system is asymptotically stable.

Proof. For the uncertain system (1) ~ (4), controller $\mathscr{P}_1$ is feasible at time k for the current state $\boldsymbol{x}(k)$. Let the optimal solution at time k as $(\boldsymbol{u}^*(k), Q_0^*, \cdots, Q_{s-1}^*, Y_0^*, \cdots, Y_{s-1}^*, X_0^*, \cdots, X_{s-1}^*, L_0^*, \cdots, L_{s-1}^*, r^*(k), \gamma^*(k))$ and $J^*(k) = r^*(k) + \gamma^*(k)$. That is, the system input $\boldsymbol{u}^*(k)$ will steer the system state $\boldsymbol{x}(k)$ into the set $S_0 = \{\boldsymbol{x} | \boldsymbol{x}^{\mathrm{T}} Q_0^{*-1} \boldsymbol{x} \leq 1\}$ and then the feedback control laws will steer it into the classical invariant set S_{s-1} along the sequence $S_0 \to S_1 \to \cdots \to S_{s-1}$.

For the uncertain system model (1), we can get

$$\boldsymbol{x}(k+1) = A(k)\boldsymbol{x}(k) + B(k)\boldsymbol{u}^*(k)$$
$$\boldsymbol{x}(k+1)^{\mathrm{T}} Q_0^{*-1} \boldsymbol{x}(k+1) \leq r^*(k)$$

where $\boldsymbol{x}(k+1)$ is within the set S_0. Since the set S_0 is an s-step control set with the series of sets $S_1, \cdots, S_{s-1}$, the feedback control laws in these sets all satisfy the constraints on system states and inputs. Then, at time $k+1$, we can construct a solution by moving the optimal solution at time k one time instant ahead $(Y_0^* Q_0^{*-1} \boldsymbol{x}(k+1), Q_1^*, \cdots, Q_{s-1}^*, Q_{s-1}^*, Y_1^*, \cdots, Y_{s-1}^*, Y_{s-1}^*, X_1^*, \cdots, X_{s-1}^*, X_{s-1}^*, L_1^*, \cdots, L_{s-1}^*, L_{s-1}^*, r(k+1), \gamma^*(k))$, where $r(k+1) = \boldsymbol{x}(k+2)^{\mathrm{T}} Q_1^{*-1} \boldsymbol{x}(k+2)$. Since the optimal solution at time k is a feasible solution of controller $\mathscr{P}_1$, $\boldsymbol{u}(k+1) = Y_0^* Q_0^{*-1} \boldsymbol{x}(k+1)$ must satisfy the input constraints and by (7) we can get $r(k+1) < \boldsymbol{x}(k+1)^{\mathrm{T}} Q_0^{*-1} \boldsymbol{x}(k+1) \leq r^*(k)$. In addition, from Corollary 1, the set S_1 is also an s-step control set with the series of sets $S_2, \cdots, S_{s-1}, S_{s-1}$ and all the sets at time k are feasible. So the above solution at time $k+1$ is also feasible.

Since $r(k+1) < \boldsymbol{x}(k+1)^{\mathrm{T}} Q_0^{*-1} \boldsymbol{x}(k+1) < r^*(k)$ and $J(k+1) = r(k+1) + \gamma^*(k) < r^*(k) + \gamma^*(k)$, according to the optimization theorem, the optimal solution $J^*(k+1) \leq J(k+1)$, i.e., $J^*(k+1) \leq J^*(k)$. We can conclude that the closed-loop system is asymptotically stable. □

The controller $\mathscr{P}_1$ can achieve a larger initial feasible region and better control performance. But its online computation burden is still heavy. So in the following part, we will give an algorithm to reduce the online computation burden so as to balance the initial feasible region, the control performance, and the computation burden.

3.2 The robust MPC controller with low online computation burden

From Corollary 2, several multi-step control sets can construct a new multi-step control set. Thus, we can design offline some multi-step control sets and online construct a new multi-step control set to reduce online computation burden of the above controller. This is a common idea in MPC design[2, 5−8, 12]. Usually, most of the previous works emphasized the covered region of the sets designed offline. However, in order to improve the control performance at the same time, i.e., to decrease the index of the terminal multi-step control set of the robust MPC controller, we also need a multi-step invariant set with lower index according to Corollary 2. So, we give the following design algorithm.

Algorithm 2 (Offline design).

Step 1. Choose γ_1, and s_1 and solve the following optimization problem

$$\begin{aligned}&\min_{Q_{(1,i)},Y_{(1,i)},X_{(1,i)},L_{(1,i)}} -\log\left(\det(Q_{(1,0)})\right)\\ \text{s.t.}\quad &(7)\sim(10)\\ &\text{with } Q_i=Q_{(1,i)};\quad Y_i=Y_{(1,i)};\quad \gamma=\gamma_1;\\ &X_i=X_{(1,i)};\ L_i=L_{(1,i)},\quad i=0,\cdots,s_1-1\end{aligned} \tag{13}$$

Step 2. Choose γ_2 and solve the following optimization problem

$$\begin{aligned}&\min_{Q_2,Y_2,X_2,L_2} -\log\left(\det(Q_2)\right)\\ \text{s.t.}\quad &(8),\ (9),\ \text{and } (10)\\ &\text{with } Q_{s-1}=Q_2;\quad Y_{s-1}=Y_2;\quad \gamma=\gamma_2;\\ &X_{s-1}=X_2;\quad L_{s-1}=L_2\end{aligned} \tag{14}$$

Remark 1. In the above algorithm, $\det(Q_{(1,0)})$ is the value of determinant of $Q_{(1,0)}$, then $\log(\det(Q_{(1,0)}))$ is proportional to the size of covered region of the ellipsoidal set $\{\boldsymbol{x}|\boldsymbol{x}^{\mathrm{T}}Q_{(1,0)}^{-1}\boldsymbol{x}\le 1\}$. It is a convex optimization problem and can be solved by efficient solvers. The same is for Step 2. γ_1 and γ_2 with $\gamma_1>\gamma_2>0$ are chosen in advance, and γ_2 is much smaller than γ_1. In Step 1, which is commonly used in the previous works, we get an s_1-step control set with its corresponding sets and index γ_1. In Step 2, we design a classical invariant set, which is also a multi-step control set from Corollary 1. By Corollary 2, we know that the index of the multi-step control set constructed by the sets obtained in Steps 1 and 2 can be combined by the indexes of these multi-step control sets. So the set obtained in Step 2, whose index is very small, can improve the index of the constructed multi-step control set.

Based on the sets obtained in Algorithm 2, we can get a robust MPC controller as follows.

Algorithm 3 (Online robust MPC controller).

$$\begin{aligned}\mathscr{P}_2:\ &\min_{\boldsymbol{u}(k),r,\lambda_0,\cdots,\lambda_{s_1}} r+(\lambda_0+\cdots+\lambda_{s_1-1})\gamma_1+\lambda_{s_1}\gamma_2\\ \text{s.t.}\quad &|\boldsymbol{u}_j(k)|\le\bar{u}_j,\quad j=1,\cdots,m\\ &\begin{bmatrix} r & (A_j\boldsymbol{x}(k)+B_j\boldsymbol{u}(k))^{\mathrm{T}}\\ A_j\boldsymbol{x}(k)+B_j\boldsymbol{u}(k) & \hat{Q}\end{bmatrix}>0\\ &r\le 1,\ j=1,\cdots,n_p\\ &\lambda_0+\cdots+\lambda_{s_1}\le 1;\ \lambda_i\ge 0,\quad i=0,\cdots,s_1\end{aligned} \tag{15}$$

where $\hat{Q}=\lambda_0 Q_{(1,0)}+\cdots+\lambda_{s_1-1}Q_{(1,s_1-1)}+\lambda_{s_1}Q_2$.

Since the multi-step control sets have been designed offline, the online optimization variables are only the combination coefficients and the current control input. So, the online computation burden of controller $\mathscr{P}_2$ is much lower than that of controller $\mathscr{P}_1$.

For the controller $\mathscr{P}_2$, we can get the following theorem.

Theorem 3. Consider the uncertain system (1) ∼ (4). If controller $\mathscr{P}_2$ is feasible at time k for the current system state $\boldsymbol{x}(k)$, the closed loop system is asymptotically stable.

We can prove Theorem 3 by taking the similar way as in Theorem 2.

Remark 2. Algorithms 2 and 3 provide a method to transfer part of online computation burden of robust MPC controller to offline design and furthermore reduce the online computation burden. With respect to the control performance, according to Corollary 2, we can repeat Step 1 or Step 2 in Algorithm 2 several times to design several sets with different indexes to further improve the control performance.

4 Numerical examples

In this section, we give an example to illustrate the initial feasible region and the control performance of the controllers developed above. In order to simplify the presentation, here we denote the controllers presented in [12] with $v=2$ as A1, [1] as A2, [4] as A3.

Consider the following uncertain system

$$\boldsymbol{x}(k+1)=\begin{bmatrix}1.15+0.35\sin(k) & 2+0.5\sin(k)\\ 0 & 1.15+0.35\sin(k)\end{bmatrix}\boldsymbol{x}(k)+\begin{bmatrix}0\\1\end{bmatrix}\boldsymbol{u}(k)$$

with $\bar{u}=1$, $\mathscr{Q}=\begin{bmatrix}1&0\\0&1\end{bmatrix}$, and $\mathscr{R}=1$. The system can be transferred to the formation with polytopic description with

$$A_1=\begin{bmatrix}1.5&2.5\\0&1.5\end{bmatrix},\quad A_2=\begin{bmatrix}0.8&1.5\\0&0.8\end{bmatrix},\quad \boldsymbol{B}=\begin{bmatrix}0\\1\end{bmatrix}$$

The initial feasible regions of controller $\mathscr{P}_1$ with $s=2$ and controller $\mathscr{P}_2$ with $s=5$ are shown in Fig. 1, along with A1 and A2. It is obvious that controllers $\mathscr{P}_1$ and $\mathscr{P}_2$ can get a larger feasible region than other controllers. The online computation burden is proportional to $n_1^3 n_2$ for an LMI optimization problem, where n_1 is the number of the optimization variables and n_2 is the number of rows in LMIs. Table 1 gives a comparison of n_1 and n_2 for these controllers.

Table 1 Comparison of n_1 and n_2 for several RMPC controllers

	$\mathscr{P}_1$	$\mathscr{P}_2$	A1	A2	A3
n_1	13	7	12	6	14
n_2	37	15	31	20	34

Choose the initial system state $\boldsymbol{x}(0)=[8,-1.5]^{\mathrm{T}}$ and $\boldsymbol{x}(0)=[1.5,0.3]^{\mathrm{T}}$. Respectively, adopting controller $\mathscr{P}_1$ with $s=2$, controller $\mathscr{P}_2$ with $s_1=3$, $\gamma_1=2\,000$, $\gamma_2=50$, and controllers A1, A2, A3, the control costs are listed in Table 2 and the trajectories of controllers $\mathscr{P}_1$ and $\mathscr{P}_2$ are shown in Fig. 2 with $\boldsymbol{x}(0)=[8,-1.5]^{\mathrm{T}}$.

Table 2 Comparison of closed-loop costs for several RMPC controllers

$\boldsymbol{x}(0)$	$\mathscr{P}_1$	$\mathscr{P}_2$	A1	A2	A3
$[8, -1.5]^{\mathrm{T}}$	229.91	243.6	258.42	261.66	261.66
$[1.5, 0.3]^{\mathrm{T}}$	24.04	26.09	32.72	37.73	37.73

Fig. 2 shows the closed loop system is stable. From Table 2, the cost values of the controllers presented in this paper are obviously smaller than those of others. That is, the controllers in this paper can get high control performance and large feasible region with low online computation, especially the controller $\mathscr{P}_2$.

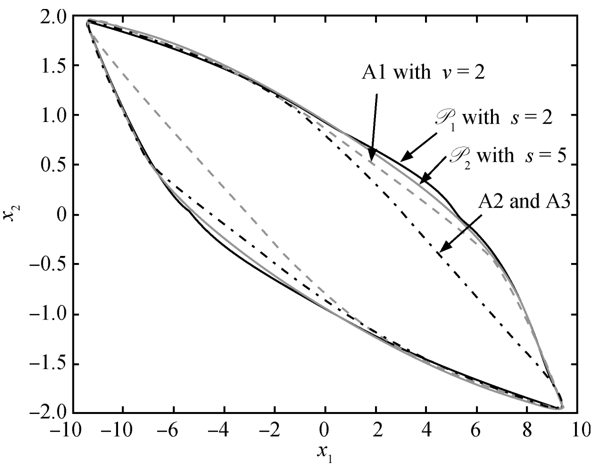

Fig. 1 Comparison of initial feasible regions for several RMPC controllers

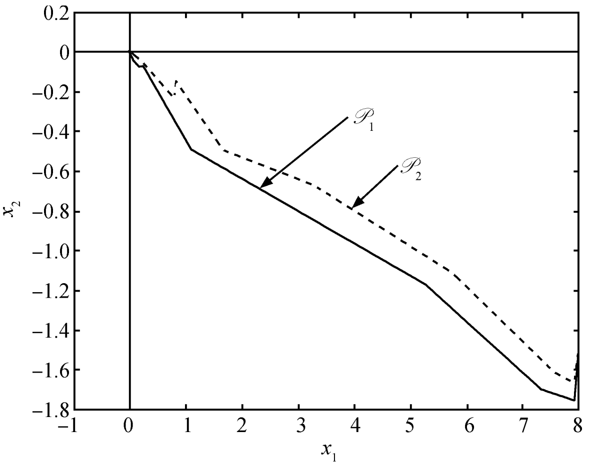

Fig. 2 System state trajectories of controllers $\mathscr{P}_1$ and $\mathscr{P}_2$ with $\boldsymbol{x}_0 = [8, -1.5]^{\mathrm{T}}$

5 Conclusions

In this paper, a design method of robust MPC controller for the constrained systems with polyhedral uncertainty is proposed based on the multi-step control set. By adopting a series of feedback control laws and optimizing the index of the multi-step control set, the robust MPC controller can get high control performance and large feasible region. By making use of some useful properties of the multi-step control set, a robust MPC controller with low online computation burden is developed, which can balance the control performance, initial feasible region, and online computation burden reasonably.

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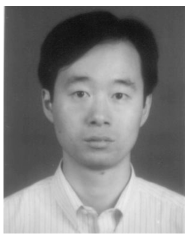

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