Delay-dependent Robust Stability for Uncertain Stochastic Systems with Interval Time-varying Delay

ZHANG Yan[1] HE Yong[1] WU Min[1]

Abstract This paper is concerned with the stability analysis for uncertain stochastic systems with interval time-varying delay. Improved delay-dependent robust stability criteria of uncertain stochastic systems with interval time-varying delay are proposed without ignoring any terms by considering the relationship among the time-varying delay, its upper bound, and their difference, and using both Itô's differential formula and Lyapunov stability theory. A numerical example is given to illustrate the effectiveness and the benefit of the proposed method.

Key words Uncertain stochastic systems, robust stability, interval time-varying delay, linear matrix inequality (LMI)

During the past decades, considerable attention has been devoted to the study of time delay systems due to the fact that time delay is often the main cause for instability and poor performance of a control system[1−22]. It is noted that stability criteria for delay systems can be classified into two categories according to their dependence on the information of delays, namely, delay-independent stability criteria and delay-dependent stability criteria. Since delay-dependent criteria make use of information on the length of delays, they are less conservative than delay-independent ones, particularly when the time delays are small. Therefore, increasing attention has recently been focused on delay-dependent stability analysis of delay systems[1, 4−8]. For example, delay-dependent stability criterion for systems with uncertain time-invariant delays was discussed in [1], delay-dependent robust stabilization of uncertain stochastic systems with time-varying delays was studied in [4], and delay-dependent stability criteria of time-varying delay system was obtained in [5].

Recently, a special type of time delay in practical engineering systems, i.e., interval time-varying delay, was identified and investigated[9−10, 12, 14, 17−18]. Interval time-varying delay is a time delay that varies in an interval, in which the lower bound is not restricted to 0. It is well known that there are systems that are stable with some nonzero delay, but are unstable without delay[2]. For such case, if there is a time-varying perturbation on the nonzero delay, it is of great significance to consider the stability of systems with interval time-varying delay. One typical example of dynamical system with interval time-varying delay is networked control systems (NCSs)[17].

On the other hand, stochastic systems have received much attention since stochastic modeling came to play an important role in many branches of science and engineering applications. In the past years, along with the development of science and technology, the bounding technology and the model transformation technique became more and more obviously conservative. To further improve the performance of delay-dependent stability criteria, much effort has been devoted recently to the development of the free weighting matrices method[5−7], in which neither the bounding technology nor model transformation is employed. However, when estimating the weak infinitesimal operator of Lyapunov-Krasovskii functional for systems with time-varying delay, some useful terms were ignored. For examples in [5, 8, 16], the derivative of $\int_{-h}^{0}\int_{t+\theta}^{t}\dot{\boldsymbol{x}}^{\mathrm{T}}(s)Z\dot{\boldsymbol{x}}(s)\mathrm{d}s\mathrm{d}\theta$ with, positive matrix Z was estimated as $h\dot{\boldsymbol{x}}^{\mathrm{T}}(t)Z\dot{\boldsymbol{x}}(t) - \int_{t-d(t)}^{t}\dot{\boldsymbol{x}}^{\mathrm{T}}(s)Z\dot{\boldsymbol{x}}(s)\mathrm{d}s$, where $0 \le d(t) \le h$, and the term $-\int_{t-h}^{t-d(t)}\dot{\boldsymbol{x}}^{\mathrm{T}}(s)Z\dot{\boldsymbol{x}}(s)\mathrm{d}s$ was ignored, it may lead to conservativeness. Although [13−14] retained these terms and proposed an improved delay-dependent stability criterion for systems with time-varying delay, there is room for further investigation. For instance, in [14], both terms $d(t)$ and $h - d(t)$ were enlarged as h. It is observed that $d(t)$ and $h - d(t)$ have an important relationship that their sum is h. So, the above may lead to conservativeness. The similar problem also existed in [22].

In this paper, the stochastic system is considered and an improved delay-dependent robust stability criterion is proposed without ignoring any terms by considering the relationship among the time-varying delay and its lower and upper bounds[15], and using both Itô's differential formula and the Lyapunov stability theory. A numerical example is given to illustrate the effectiveness and the benefits of the proposed method.

Notations. Throughout this paper, $\mathbf{R}^n$ and $\mathbf{R}^{n\times m}$ denote, respectively, the n-dimensional Euclidean space and the set of all $n \times m$ real matrices. $\|\cdot\|$ stands for the usual $\boldsymbol{L}_2[0,\infty)$ norm. The notation $P > 0$ (≥ 0) means that P is a real symmetric and positive (semi-positive) definite matrix. $(\boldsymbol{\Omega}, \mathbf{F}, \{\mathbf{F}_t\}_{t\ge 0}, \mathbf{P})$ is a complete probability space with a filtration $\{\mathbf{F}_t\}_{t\ge 0}$ satisfying the conditions that it is right continuous and $\mathbf{F}_0$ contains all $\mathbf{P}$-null sets. $\mathbf{L}_{F_0}^2([-\tau',0],\mathbf{R}^n)$ denotes the family of all bounded $\mathbf{F}_0$-measurable $\mathbf{C}([-\tau',0],\mathbf{R}^n)$-valued random variables $\boldsymbol{\xi} = \{\boldsymbol{\xi}(\theta) : -\tau' \le \theta \le 0\}$ such that $\sup_{-\tau'\le\theta\le 0}\mathrm{E}\{|\boldsymbol{\xi}(\theta)|^2\} < \infty$. $\mathrm{E}\{\cdot\}$ stands for the mathematical expectation. The symbol "$*$" within a matrix represents the symmetric terms of the matrix, e.g. $\begin{bmatrix} X & Y \\ * & Z \end{bmatrix} = \begin{bmatrix} X & Y \\ Y^{\mathrm{T}} & Z \end{bmatrix}$. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible with algebraic operations.

1 Preliminaries and problem formulation

Consider the following uncertain linear stochastic system Σ with interval time-varying delay

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1. School of Information Science and Engineering, Central South University, Changsha 410083, P. R. China
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$$\begin{aligned}\mathrm{d}\boldsymbol{x}(t) =& [(A+\triangle A(t))\boldsymbol{x}(t)+(A_d+\triangle A_d(t))\times\\ &\boldsymbol{x}(t-\tau(t))]\mathrm{d}t+[(E+\triangle E(t))\boldsymbol{x}(t)+\\ &(E_d+\triangle E_d(t))\boldsymbol{x}(t-\tau(t))]\mathrm{d}w(t)\end{aligned} \quad (1)$$

$$\boldsymbol{x}(t) = \boldsymbol{\phi}(t), \quad t\in[-h_2, 0] \quad (2)$$

where $\boldsymbol{x}(t) \in \mathbf{R}^n$ is the state vector, $w(t)$ is a scalar Brownian motion defined on a complete probability space $(\boldsymbol{\Omega}, \mathbf{F}, \mathbf{P})$ with a filtration $\{\mathbf{F}_t\}_{t\geq 0}$, and $\boldsymbol{\phi}(t)$ is any given initial condition in $\mathbf{L}^2_{F_0}([-\tau, 0]; \mathbf{R}^n)$. A, A_d, E, and E_d are known real constant matrices with appropriate dimensions, $\triangle A(t)$, $\triangle A_d(t)$ and $\triangle E(t)$, $\triangle E_d(t)$ are all unknown time-varying matrices with appropriate dimensions which represent the system uncertainty and stochastic perturbation uncertainty, respectively, and are assumed to be of the following form

$$\begin{aligned}&[\triangle A(t) \quad \triangle A_d(t) \quad \triangle E(t) \quad \triangle E_d(t)] =\\ &DF(t)\,[G_1 \quad G_2 \quad G_3 \quad G_4]\end{aligned} \quad (3)$$

where D, G_1, G_2, G_3, and G_4 are known real constant matrices with appropriate dimensions. $F(t)$ is unknown real time-varying matrix with Lebesgue measurable elements bounded by

$$F^{\mathrm{T}}(t)F(t) \leq I, \quad \forall t \quad (4)$$

The time delay, $\tau(t)$, is a time-varying differentiable function that satisfies

$$0 \leq h_1 \leq \tau(t) \leq h_2, \quad \dot{\tau}(t) \leq d_\tau \quad (5)$$

where h_1, h_2, and d_τ are constants. Here, h_1 may not be equal to 0, and when $d_\tau = 0$ it is clear that $h_2 = h_1$.

Throughout this paper, we use the following definition for the system Σ.

Definition 1. The uncertain stochastic time-delay system Σ is said to be robustly stochastically stable if there exists a scalar $\alpha > 0$ such that for all admissible uncertainties,

$$\mathrm{E}\left\{\int_0^\infty \boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{x}(t)\mathrm{d}t\right\} \leq \alpha\mathrm{E}\left\{\sup_{s\in[-\tau,0]}\|\boldsymbol{\phi}(s)\|^2\right\} \quad (6)$$

where $\boldsymbol{x}(t)$ denotes the solution of system (1) at time t under initial condition in (2).

Lemma 1[3]**.** For any vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{R}^n$, matrices A, D, E, P, and F are real matrices of appropriate dimensions with $P > 0$ and $F^{\mathrm{T}}F \leq I$, the following inequalities hold:

1) $2\boldsymbol{x}^{\mathrm{T}}DFE\boldsymbol{y} \leq \varepsilon^{-1}\boldsymbol{x}^{\mathrm{T}}DD^{\mathrm{T}}\boldsymbol{x} + \varepsilon\boldsymbol{y}^{\mathrm{T}}E^{\mathrm{T}}E\boldsymbol{y}$;

2) For any scalar $\varepsilon > 0$ such that $P - \varepsilon DD^{\mathrm{T}} > 0$, then

$$(A+DFE)^{\mathrm{T}}P^{-1}(A+DFE) \leq \varepsilon^{-1}E^{\mathrm{T}}E + A^{\mathrm{T}}(P-\varepsilon DD^{\mathrm{T}})^{-1}A$$

3) $2\boldsymbol{x}^{\mathrm{T}}\boldsymbol{y} \leq \boldsymbol{x}^{\mathrm{T}}P^{-1}\boldsymbol{x} + \boldsymbol{y}^{\mathrm{T}}P\boldsymbol{y}$.

2 Robust stability analysis

In this section, a robustly stochastically stable criterion for the uncertain linear time-delay stochastic system Σ will be established by applying the Lyapunov-Krasovskii theory.

For convenience, define a new state variable

$$\boldsymbol{y}(t) = (A+\triangle A(t))\boldsymbol{x}(t) + (A_d+\triangle A_d(t))\boldsymbol{x}(t-\tau(t)) \quad (7)$$

and a new perturbation variable

$$\boldsymbol{g}(t) = (E+\triangle E(t))\boldsymbol{x}(t) + (E_d+\triangle E_d(t))\boldsymbol{x}(t-\tau(t)) \quad (8)$$

Then, system (1) becomes

$$\mathrm{d}\boldsymbol{x}(t) = \boldsymbol{y}(t)\mathrm{d}t + \boldsymbol{g}(t)\mathrm{d}w(t) \quad (9)$$

First, we introduce the following zero equations which will be used in our main result

$$2\boldsymbol{\xi}^{\mathrm{T}}(t)N\left[\boldsymbol{x}(t)-\boldsymbol{x}(t-\tau(t))-\int_{t-\tau(t)}^{t}\mathrm{d}\boldsymbol{x}(s)\right] = 0 \quad (10)$$

$$2\boldsymbol{\xi}^{\mathrm{T}}(t)H\left[\boldsymbol{x}(t-h_1)-\boldsymbol{x}(t-\tau(t))-\int_{t-\tau(t)}^{t-h_1}\mathrm{d}\boldsymbol{x}(s)\right] = 0 \quad (11)$$

$$2\boldsymbol{\xi}^{\mathrm{T}}(t)M\left[\boldsymbol{x}(t-\tau(t))-\boldsymbol{x}(t-h_2)-\int_{t-h_2}^{t-\tau(t)}\mathrm{d}\boldsymbol{x}(s)\right] = 0 \quad (12)$$

where N, H, and M are any matrices with appropriate dimensions, and

$$\boldsymbol{\xi}(t) = [\boldsymbol{x}^{\mathrm{T}}(t) \quad \boldsymbol{x}^{\mathrm{T}}(t-\tau(t)) \quad \boldsymbol{x}^{\mathrm{T}}(t-h_1) \quad \boldsymbol{x}^{\mathrm{T}}(t-h_2)]^{\mathrm{T}}$$

On the other hand, for any semi-positive matrices $X \geq 0$ and $Y \geq 0$, the following equations hold

$$h_2\boldsymbol{\xi}^{\mathrm{T}}(t)X\boldsymbol{\xi}(t) - \int_{t-\tau(t)}^{t}\boldsymbol{\xi}^{\mathrm{T}}(t)X\boldsymbol{\xi}(t)\mathrm{d}s - \int_{t-h_2}^{t-\tau(t)}\boldsymbol{\xi}^{\mathrm{T}}(t)X\boldsymbol{\xi}(t)\mathrm{d}s = 0 \quad (13)$$

$$(h_2-h_1)\boldsymbol{\xi}^{\mathrm{T}}(t)Y\boldsymbol{\xi}(t) - \int_{t-\tau(t)}^{t-h_1}\boldsymbol{\xi}^{\mathrm{T}}(t)Y\boldsymbol{\xi}(t)\mathrm{d}s - \int_{t-h_2}^{t-\tau(t)}\boldsymbol{\xi}^{\mathrm{T}}(t)Y\boldsymbol{\xi}(t)\mathrm{d}s = 0 \quad (14)$$

It is easy to see that (10) $\sim$ (14) are always satisfied. By using them, we have the following theorem.

Theorem 1. For given scalars $0 \leq h_1 \leq h_2$ and d_τ, system Σ is robustly stochastically stable for all time-varying delays satisfying (5) and for all admissible uncertainties satisfying (3) and (4) if there exist symmetric positive definite matrices $P > 0$, $Q_i > 0$, $i = 1,2,3$, $Z_j > 0$, $S_j > 0$, $j = 1,2$, semi-positive definite matrices $X \geq 0$, $Y \geq 0$, scalars $\varepsilon_l > 0$, $l = 1,2,3$ and any appropriately dimensioned matrices N, H, M, such that the LMIs (15) $\sim$ (18) hold

$$\begin{bmatrix} X & N \\ * & Z_1 \end{bmatrix} \geq 0 \quad (16)$$

$$\begin{bmatrix} Y & H \\ * & Z_2 \end{bmatrix} \geq 0 \quad (17)$$

$$\begin{bmatrix} X+Y & M \\ * & Z_1+Z_2 \end{bmatrix} \geq 0 \quad (18)$$

where

$$\begin{aligned}\Theta &= \Phi + \Psi + \Psi^{\mathrm{T}} + h_2X + (h_2-h_1)Y\\ \hat{P} &= [P \quad 0 \quad 0 \quad 0]^{\mathrm{T}}\\ W_1 &= [A \quad A_d \quad 0 \quad 0]\\ W_2 &= [E \quad E_d \quad 0 \quad 0]\\ \hat{Z} &= h_2Z_1 + (h_2-h_1)Z_2\\ \hat{S} &= P + h_2S_1 + (h_2-h_1)S_2\end{aligned}$$

$$\begin{bmatrix} \Theta & \hat{P}D & W_1^{\mathrm{T}}\hat{Z} & 0 & W_2^{\mathrm{T}}\hat{S} & 0 & N & H & M \\ * & -\varepsilon_1 I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & -\hat{Z} & \hat{Z}D & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\varepsilon_2 I & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -\hat{S} & \hat{S}D & 0 & 0 & 0 \\ * & * & * & * & * & -\varepsilon_3 I & 0 & 0 & 0 \\ * & * & * & * & * & * & -S_1 & 0 & 0 \\ * & * & * & * & * & * & * & -S_2 & 0 \\ * & * & * & * & * & * & * & * & -S_1-S_2 \end{bmatrix} < 0 \tag{15}$$

with

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} & 0 & 0 \\ * & \Phi_{22} & 0 & 0 \\ * & * & -Q_1 & 0 \\ * & * & * & -Q_2 \end{bmatrix}$$

$$\Psi = [N \quad M-N-H \quad H \quad -M]$$

and

$$\Phi_{11} = \sum_{i=1}^{3} Q_i + PA + A^{\mathrm{T}}P + (\varepsilon_1+\varepsilon_2)G_1^{\mathrm{T}}G_1 + \varepsilon_3 G_3^{\mathrm{T}}G_3$$

$$\Phi_{12} = PA_d + (\varepsilon_1+\varepsilon_2)G_1^{\mathrm{T}}G_2 + \varepsilon_3 G_3^{\mathrm{T}}G_4$$

$$\Phi_{22} = -(1-d_\tau)Q_3 + (\varepsilon_1+\varepsilon_2)G_2^{\mathrm{T}}G_2 + \varepsilon_3 G_4^{\mathrm{T}}G_4$$

Proof. Define $\boldsymbol{x}_t$ by $\boldsymbol{x}_t(s) = \boldsymbol{x}(t+s)$, $-h_2 \le s \le 0$, and a stochastic Lyapunov-Krasovskii functional candidate as

$$V(\boldsymbol{x}_t,t) = \sum_{i=1}^{6} V_i(\boldsymbol{x}_t,t) \tag{19}$$

where

$$V_1(\boldsymbol{x}_t,t) = \boldsymbol{x}^{\mathrm{T}}(t)P\boldsymbol{x}(t)$$

$$V_2(\boldsymbol{x}_t,t) = \sum_{i=1}^{2}\int_{t-h_i}^{t} \boldsymbol{x}^{\mathrm{T}}(s)Q_i\boldsymbol{x}(s)\mathrm{d}s + \int_{t-\tau(t)}^{t} \boldsymbol{x}^{\mathrm{T}}(s)Q_3\boldsymbol{x}(s)\mathrm{d}s$$

$$V_3(\boldsymbol{x}_t,t) = \int_{-h_2}^{0}\int_{t+\theta}^{t} \boldsymbol{y}^{\mathrm{T}}(s)Z_1\boldsymbol{y}(s)\mathrm{d}s\mathrm{d}\theta$$

$$V_4(\boldsymbol{x}_t,t) = \int_{-h_2}^{-h_1}\int_{t+\theta}^{t} \boldsymbol{y}^{\mathrm{T}}(s)Z_2\boldsymbol{y}(s)\mathrm{d}s\mathrm{d}\theta$$

$$V_5(\boldsymbol{x}_t,t) = \int_{-h_2}^{0}\int_{t+\theta}^{t} \mathrm{tr}\left(\boldsymbol{g}^{\mathrm{T}}(s)S_1\boldsymbol{g}(s)\right)\mathrm{d}s\mathrm{d}\theta$$

$$V_6(\boldsymbol{x}_t,t) = \int_{-h_2}^{-h_1}\int_{t+\theta}^{t} \mathrm{tr}\left(\boldsymbol{g}^{\mathrm{T}}(s)S_2\boldsymbol{g}(s)\right)\mathrm{d}s\mathrm{d}\theta$$

in which P, Q_i, $i=1,2,3$, Z_j, S_j, $j=1,2$, are all symmetric positive definite matrices with appropriate dimensions and to be determined.

Then, the weak infinitesimal operator $\mathcal{L}$ of the stochastic process $\{\boldsymbol{x}_t, t \ge h_2\}$ along the evolution of $V(\boldsymbol{x}_t,t)$ is given as

$$\begin{aligned}\mathcal{L}V(\boldsymbol{x}_t,t) = {} & 2\boldsymbol{x}^{\mathrm{T}}(t)P\boldsymbol{y}(t) + \mathrm{tr}\left(\boldsymbol{g}^{\mathrm{T}}(t)P\boldsymbol{g}(t)\right) + \\ & \sum_{i=1}^{3}\boldsymbol{x}^{\mathrm{T}}(t)Q_i\boldsymbol{x}(t) - \boldsymbol{x}^{\mathrm{T}}(t-h_1)Q_1\boldsymbol{x}(t-h_1) - \\ & \boldsymbol{x}^{\mathrm{T}}(t-h_2)Q_2\boldsymbol{x}(t-h_2) - \\ & (1-\dot{\tau}(t))\boldsymbol{x}^{\mathrm{T}}(t-\tau(t))Q_3\boldsymbol{x}(t-\tau(t)) + \\ & h_2\boldsymbol{y}^{\mathrm{T}}(t)Z_1\boldsymbol{y}(t) - \int_{t-h_2}^{t}\boldsymbol{y}^{\mathrm{T}}(s)Z_1\boldsymbol{y}(s)\mathrm{d}s + \\ & (h_2-h_1)\boldsymbol{y}^{\mathrm{T}}(t)Z_2\boldsymbol{y}(t) - \int_{t-h_2}^{t-h_1}\boldsymbol{y}^{\mathrm{T}}(s)Z_2\boldsymbol{y}(s)\mathrm{d}s + \\ & h_2\mathrm{tr}(\boldsymbol{g}^{\mathrm{T}}(s)S_1\boldsymbol{g}(s)) - \\ & \int_{t-h_2}^{t}\mathrm{tr}(\boldsymbol{g}^{\mathrm{T}}(s)S_1\boldsymbol{g}(s))\mathrm{d}s + \\ & (h_2-h_1)\mathrm{tr}(\boldsymbol{g}^{\mathrm{T}}(s)S_2\boldsymbol{g}(s)) - \\ & \int_{t-h_2}^{t-h_1}\mathrm{tr}(\boldsymbol{g}^{\mathrm{T}}(s)S_2\boldsymbol{g}(s))\mathrm{d}s \end{aligned} \tag{20}$$

Adding the left sides of (10) ~ (14) to (20), we have the weak infinitesimal operator of $V(\boldsymbol{x}_t,t)$ along the trajectory of system Σ as

$$\begin{aligned}\mathcal{L}V(\boldsymbol{x}_t,t) = {} & 2\boldsymbol{x}^{\mathrm{T}}(t)P\boldsymbol{y}(t) + \mathrm{tr}(\boldsymbol{g}^{\mathrm{T}}(t)P\boldsymbol{g}(t)) + \\ & \sum_{i=1}^{3}\boldsymbol{x}^{\mathrm{T}}(t)Q_i\boldsymbol{x}(t) - \boldsymbol{x}^{\mathrm{T}}(t-h_1)Q_1\boldsymbol{x}(t-h_1) - \\ & \boldsymbol{x}^{\mathrm{T}}(t-h_2)Q_2\boldsymbol{x}(t-h_2) - \\ & (1-\dot{\tau}(t))\boldsymbol{x}^{\mathrm{T}}(t-\tau(t))Q_3\boldsymbol{x}(t-\tau(t)) + \\ & h_2\boldsymbol{y}^{\mathrm{T}}(t)Z_1\boldsymbol{y}(t) - \int_{t-h_2}^{t}\boldsymbol{y}^{\mathrm{T}}(s)Z_1\boldsymbol{y}(s)\mathrm{d}s + \\ & (h_2-h_1)\boldsymbol{y}^{\mathrm{T}}(t)Z_2\boldsymbol{y}(t) - \int_{t-h_2}^{t-h_1}\boldsymbol{y}^{\mathrm{T}}(s)Z_2\boldsymbol{y}(s)\mathrm{d}s + \\ & h_2\mathrm{tr}(\boldsymbol{g}^{\mathrm{T}}(s)S_1\boldsymbol{g}(s)) - \\ & \int_{t-h_2}^{t}\mathrm{tr}(\boldsymbol{g}^{\mathrm{T}}(s)S_1\boldsymbol{g}(s))\mathrm{d}s + \\ & (h_2-h_1)\mathrm{tr}(\boldsymbol{g}^{\mathrm{T}}(s)S_2\boldsymbol{g}(s)) - \\ & \int_{t-h_2}^{t-h_1}\mathrm{tr}(\boldsymbol{g}^{\mathrm{T}}(s)S_2\boldsymbol{g}(s))\mathrm{d}s + \\ & 2\boldsymbol{\xi}^{\mathrm{T}}(t)N\left[\boldsymbol{x}(t) - \boldsymbol{x}(t-\tau(t)) - \int_{t-\tau(t)}^{t}\mathrm{d}\boldsymbol{x}(s)\right] + \\ & 2\boldsymbol{\xi}^{\mathrm{T}}(t)H\left[\boldsymbol{x}(t-h_1) - \boldsymbol{x}(t-\tau(t)) - \int_{t-\tau(t)}^{t-h_1}\mathrm{d}\boldsymbol{x}(s)\right] + \\ & 2\boldsymbol{\xi}^{\mathrm{T}}(t)M\left[\boldsymbol{x}(t-\tau(t)) - \boldsymbol{x}(t-h_2) - \int_{t-h_2}^{t-\tau(t)}\mathrm{d}\boldsymbol{x}(s)\right] + \\ & h_2\boldsymbol{\xi}^{\mathrm{T}}(t)X\boldsymbol{\xi}(t) - \int_{t-\tau(t)}^{t}\boldsymbol{\xi}^{\mathrm{T}}(t)X\boldsymbol{\xi}(t)\mathrm{d}s - \end{aligned}$$

$$\int_{t-h_2}^{t-\tau(t)} \boldsymbol{\xi}^{\mathrm{T}}(t)X\boldsymbol{\xi}(t)\mathrm{d}s+$$
$$(h_2-h_1)\boldsymbol{\xi}^{\mathrm{T}}(t)Y\boldsymbol{\xi}(t)-\int_{t-\tau(t)}^{t-h_1} \boldsymbol{\xi}^{\mathrm{T}}(t)Y\boldsymbol{\xi}(t)\mathrm{d}s-$$
$$\int_{t-h_2}^{t-\tau(t)} \boldsymbol{\xi}^{\mathrm{T}}(t)Y\boldsymbol{\xi}(t)\mathrm{d}s \tag{21}$$

By 1) of Lemma 1, for any scalar $\varepsilon_1 > 0$, we have

$$\begin{aligned}2\boldsymbol{x}^{\mathrm{T}}(t)P\boldsymbol{y}(t) = &2\boldsymbol{x}^{\mathrm{T}}(t)P[A \quad A_d \quad 0 \quad 0]\boldsymbol{\xi}(t)+\\ &2\boldsymbol{x}^{\mathrm{T}}(t)PDF(t)[G_1 \quad G_2 \quad 0 \quad 0]\boldsymbol{\xi}(t) \leq\\ &2\boldsymbol{x}^{\mathrm{T}}(t)P[A \quad A_d \quad 0 \quad 0]\boldsymbol{\xi}(t)+\\ &\boldsymbol{\xi}^{\mathrm{T}}(t)\hat{P}D\varepsilon_1^{-1}(\hat{P}D)^{\mathrm{T}}\boldsymbol{\xi}(t)+\\ &\boldsymbol{\xi}^{\mathrm{T}}(t)\varepsilon_1[G_1 \quad G_2 \quad 0 \quad 0]^{\mathrm{T}}[G_1 \quad G_2 \quad 0 \quad 0]\boldsymbol{\xi}(t)\end{aligned} \tag{22}$$

and by 2) of Lemma 1, for any scalar $\varepsilon_2 > 0$ satisfying $[h_2Z_1 + (h_2-h_1)Z_2]^{-1} - \varepsilon_2^{-1}DD^{\mathrm{T}} > 0$, we have

$$\begin{aligned}&\boldsymbol{y}^{\mathrm{T}}(t)[h_2Z_1 + (h_2-h_1)Z_2]\boldsymbol{y}(t) =\\ &\quad\boldsymbol{\xi}^{\mathrm{T}}(t)([A \quad A_d \quad 0 \quad 0] + DF(t)[G_1 \quad G_2 \quad 0 \quad 0])^{\mathrm{T}}\times\\ &\quad\hat{Z}([A \quad A_d \quad 0 \quad 0] + DF(t)[G_1 \quad G_2 \quad 0 \quad 0])\boldsymbol{\xi}(t) \leq\\ &\quad\boldsymbol{\xi}^{\mathrm{T}}(t)W_1^{\mathrm{T}}[\hat{Z}^{-1} - \varepsilon_2^{-1}DD^{\mathrm{T}}]^{-1}W_1\boldsymbol{\xi}(t)+\\ &\quad\boldsymbol{\xi}^{\mathrm{T}}(t)\varepsilon_2[G_1 \quad G_2 \quad 0 \quad 0]^{\mathrm{T}}[G_1 \quad G_2 \quad 0 \quad 0]\boldsymbol{\xi}(t)\end{aligned} \tag{23}$$

For any scalar $\varepsilon_3 > 0$ satisfying $[h_2S_1 + (h_2-h_1)S_2]^{-1} - \varepsilon_3^{-1}DD^{\mathrm{T}} > 0$, the following inequality holds

$$\begin{aligned}&\mathrm{tr}(\boldsymbol{g}^{\mathrm{T}}(t)[P + h_2S_1 + (h_2-h_1)S_2]\boldsymbol{g}(t)) =\\ &\quad\boldsymbol{g}^{\mathrm{T}}(t)[P + h_2S_1 + (h_2-h_1)S_2]\boldsymbol{g}(t) =\\ &\quad\boldsymbol{\xi}^{\mathrm{T}}(t)([E \quad E_d \quad 0 \quad 0] + DF(t)[G_3 \quad G_4 \quad 0 \quad 0])^{\mathrm{T}}\times\\ &\quad\hat{S}([E \quad E_d \quad 0 \quad 0] + DF(t)[G_3 \quad G_4 \quad 0 \quad 0])\boldsymbol{\xi}(t) \leq\\ &\quad\boldsymbol{\xi}^{\mathrm{T}}(t)W_2^{\mathrm{T}}[\hat{S}^{-1} - \varepsilon_3^{-1}DD^{\mathrm{T}}]^{-1}W_2\boldsymbol{\xi}(t)+\\ &\quad\boldsymbol{\xi}^{\mathrm{T}}(t)\varepsilon_3[G_3 \quad G_4 \quad 0 \quad 0]^{\mathrm{T}}[G_3 \quad G_4 \quad 0 \quad 0]\boldsymbol{\xi}(t)\end{aligned} \tag{24}$$

where

$$\begin{aligned}\hat{P} &= [P \quad 0 \quad 0 \quad 0]^{\mathrm{T}}\\ W_1 &= [A \quad A_d \quad 0 \quad 0]\\ W_2 &= [E \quad E_d \quad 0 \quad 0]\\ \hat{Z} &= h_2Z_1 + (h_2-h_1)Z_2\\ \hat{S} &= P + h_2S_1 + (h_2-h_1)S_2\end{aligned}$$

In addition, by 3) of lemma 1, the following inequalities hold

$$-2\boldsymbol{\xi}^{\mathrm{T}}(t)N\int_{t-\tau(t)}^{t} \boldsymbol{g}(s)\mathrm{d}w(s) \leq \boldsymbol{\xi}^{\mathrm{T}}(t)NS_1^{-1}N^{\mathrm{T}}\boldsymbol{\xi}(t)+$$
$$\left(\int_{t-\tau(t)}^{t} \boldsymbol{g}(s)\mathrm{d}w(s)\right)^{\mathrm{T}} S_1 \left(\int_{t-\tau(t)}^{t} \boldsymbol{g}(s)\mathrm{d}w(s)\right) \tag{25}$$

$$-2\boldsymbol{\xi}^{\mathrm{T}}(t)H\int_{t-\tau(t)}^{t-h_1} \boldsymbol{g}(s)\mathrm{d}w(s) \leq \boldsymbol{\xi}^{\mathrm{T}}(t)HS_2^{-1}H^{\mathrm{T}}\boldsymbol{\xi}(t)+$$
$$\left(\int_{t-\tau(t)}^{t-h_1} \boldsymbol{g}(s)\mathrm{d}w(s)\right)^{\mathrm{T}} S_2 \left(\int_{t-\tau(t)}^{t-h_1} \boldsymbol{g}(s)\mathrm{d}w(s)\right) \tag{26}$$

$$-2\boldsymbol{\xi}^{\mathrm{T}}(t)M\int_{t-h_2}^{t-\tau(t)} g(s)\mathrm{d}w(s) \leq \boldsymbol{\xi}^{\mathrm{T}}(t)M(S_1+S_2)^{-1}M^{\mathrm{T}}\boldsymbol{\xi}(t)+$$
$$\left(\int_{t-h_2}^{t-\tau(t)} \boldsymbol{g}(s)\mathrm{d}w(s)\right)^{\mathrm{T}} (S_1+S_2)\left(\int_{t-h_2}^{t-\tau(t)} \boldsymbol{g}(s)\mathrm{d}w(s)\right) \tag{27}$$

Note that

$$\mathrm{E}\left\{\left(\int_{t-\tau(t)}^{t} \boldsymbol{g}(s)\mathrm{d}w(s)\right)^{\mathrm{T}} S_1\left(\int_{t-\tau(t)}^{t} \boldsymbol{g}(s)\mathrm{d}w(s)\right)\right\} =$$
$$\int_{t-\tau(t)}^{t} \mathrm{tr}\left(\boldsymbol{g}^{\mathrm{T}}(s)S_1\boldsymbol{g}(s)\right)\mathrm{d}s \tag{28}$$

$$\mathrm{E}\left\{\left(\int_{t-\tau(t)}^{t-h_1} \boldsymbol{g}(s)\mathrm{d}w(s)\right)^{\mathrm{T}} S_2\left(\int_{t-\tau(t)}^{t-h_1} \boldsymbol{g}(s)\mathrm{d}w(s)\right)\right\} =$$
$$\int_{t-\tau(t)}^{t-h_1} \mathrm{tr}\left(\boldsymbol{g}^{\mathrm{T}}(s)S_2\boldsymbol{g}(s)\right)\mathrm{d}s \tag{29}$$

and

$$\mathrm{E}\left\{\left(\int_{t-h_2}^{t-\tau(t)} \boldsymbol{g}(s)\mathrm{d}w(s)\right)^{\mathrm{T}} (S_1+S_2)\left(\int_{t-h_2}^{t-\tau(t)} \boldsymbol{g}(s)\mathrm{d}w(s)\right)\right\} =$$
$$\int_{t-h_2}^{t-\tau(t)} \mathrm{tr}(\boldsymbol{g}^{\mathrm{T}}(s)(S_1+S_2)\boldsymbol{g}(s))\mathrm{d}s \tag{30}$$

Then, applying inequalities (22) ∼ (27) to (21) yields

$$\begin{aligned}\mathcal{L}V(\boldsymbol{x}_t, t) \leq &\boldsymbol{\xi}^{\mathrm{T}}(t)\,\Xi\,\boldsymbol{\xi}(t)-\\ &\int_{t-d(t)}^{t} \boldsymbol{\eta}^{\mathrm{T}}(t,s)\begin{bmatrix} X & N\\ * & Z_1\end{bmatrix}\boldsymbol{\eta}(t,s)\mathrm{d}s-\\ &\int_{t-d(t)}^{t-h_1} \boldsymbol{\eta}^{\mathrm{T}}(t,s)\begin{bmatrix} Y & H\\ * & Z_2\end{bmatrix}\boldsymbol{\eta}(t,s)\mathrm{d}s-\\ &\int_{t-h_2}^{t-d(t)} \boldsymbol{\eta}^{\mathrm{T}}(t,s)\begin{bmatrix} X+Y & M\\ * & Z_1+Z_2\end{bmatrix}\boldsymbol{\eta}(t,s)\mathrm{d}s\end{aligned}$$

where $\Xi = \Theta + \hat{P}D\varepsilon_1(\hat{P}D)^{\mathrm{T}} + W_1^{\mathrm{T}}[\hat{P}^{-1} - \varepsilon_2^{-1}DD^{\mathrm{T}}]^{-1}W_1 + W_2^{\mathrm{T}}[\hat{Z}^{-1} - \varepsilon_3^{-1}DD^{\mathrm{T}}]^{-1}W_2 + NS_1^{-1}N^{\mathrm{T}} + HS_2^{-1}H^{\mathrm{T}} + M(S_1+S_2)^{-1}M^{\mathrm{T}}$ with Θ being defined in Theorem 1 and $\boldsymbol{\eta}(t,s) = [\boldsymbol{\xi}^{\mathrm{T}}(t) \quad \boldsymbol{y}^{\mathrm{T}}(s)]^{\mathrm{T}}$.

By applying the Schur complement to (15) results in $\Xi < 0$. Therefore, if (15) ∼ (18) are satisfied, then (21) implies that

$$\mathcal{L}V(\boldsymbol{x}_t, t) \leq -\lambda\|\boldsymbol{x}(t)\|^2 \tag{31}$$

where $\lambda = \lambda_{\min}(\Xi)$. Now, by Dynkin′s formula, it is true that for all $t \geq h_2$,

$$\mathrm{E}\{V(t)\} - \mathrm{E}\{V(h_2)\} \leq -\lambda\mathrm{E}\{\int_{h_2}^{t} \|\boldsymbol{x}(s)\|^2\mathrm{d}s\} \tag{32}$$

It follows that

$$\mathrm{E}\left\{\int_{h_2}^{t} \|\boldsymbol{x}(s)\|^2\mathrm{d}s\right\} \leq \frac{1}{\lambda}\mathrm{E}\left\{V(h_2)\right\} \tag{33}$$

For system (1), the proof follows a similar line to that of [16]. It is clear that there exists a positive scalar $\alpha > 0$ such that

$$\mathrm{E}\left\{\int_0^{h_2} \|\boldsymbol{x}(s)\|^2 \mathrm{d}s\right\} \le \alpha \sup_{s\in[-h_2,0]} \left\{\|\boldsymbol{\phi}(s)\|^2\right\} \tag{34}$$

Therefore, by the definitions of $V(\boldsymbol{x}_t, t)$ and $\boldsymbol{x}(t)$, there always exists a scalar $\tilde{\alpha} > 0$ such that

$$\lim_{T\to\infty} \mathrm{E}\left\{\int_0^{T} \|\boldsymbol{x}(t)\|^2 \mathrm{d}t\right\} \le \tilde{\alpha}\mathrm{E}\left\{\sup_{s\in[-\tau,0]} \|\boldsymbol{\phi}(s)\|^2\right\} \tag{35}$$

which means that system Σ is robustly stochastically stable by Definition 1. □

Remark 1. It is seen that $\tau(t)$, $h_2 - \tau(t)$, and $\tau(t) - h_1$ are not simply enlarged as h_2, $h_2 - h_1$, and $h_2 - h_1$, respectively. Instead, the relationship that $\tau(t) + h_2 - \tau(t) = h_2$ and $\tau(t) - h_1 + h_2 - \tau(t) = h_2 - h_1$ are considered in the process of the proof.

Remark 2. If the perturbation terms are not considered, then system Σ becomes uncertain linear system with interval time-varying delay. Using the similar method, the stability criterion for uncertain linear system with interval time-varying delay can be obtained. In fact, if we choose some weighting matrices that are relative with stochastic perturbation to zero matrices in Theorem 1, then a stability criterion without stochastic perturbation can be obtained.

3 Numerical example

In this section, we shall present a numerical example to demonstrate the effectiveness of the proposed method.

Consider the uncertain linear stochastic system (3) with

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad A_d = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}$$

and $E = 0$, $E_d = 0$, the uncertainties are described by (3) and with

$$\|\triangle A(t)\| \le 0.2, \quad \|\triangle A_d(t)\| \le 0.2$$
$$\|\triangle E(t)\| \le 0.2, \quad \|\triangle E_d(t)\| \le 0.2$$
$$D = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \quad G_1 = G_2 = G_3 = G_4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

According to Theorem 1, for $h_1 = 0$, the upper bounds on the time delay to guarantee the system is robustly stochastically stable are listed in Table 1. At the same time, Table 1 also lists the upper bounds obtained from the criterion in [22].

Table 1 Allowable upper bounds of h_2 for different d_τ

d_τ	0.3	0.5	0.9	1
Reference [22]	0.7288	0.5252	0.1489	—
Theorem 1	1.2950	1.1006	0.9434	0.9424

Remark 3. In [22], the maximum allowable time delay was given by 0.6822 for $d_\tau = 0.9$. However, after our computation by the method in [22], the true maximum allowable time delay should be 0.1489. In addition, "–" means that the result is not applicable to the corresponding case.

In the sequel, for given h_1, Table 2 lists the upper bounds on the time delay to guarantee the system is robustly stochastically stable.

Table 2 Allowable upper bounds of h_2 with given h_1 and d_τ

h_1	0.3	0.8	1	1.5
$d_\tau = 0.3$	1.2895	1.2824	1.2822	1.5515
$d_\tau = 0.5$	1.0935	1.1204	1.2179	1.5508
$d_\tau = 0.9$	0.9388	1.1042	1.2145	1.5508

4 Conclusions

In this paper, the delay-dependent robust stability problem has been investigated for a class of linear stochastic systems with interval time-varying delay. Sufficient conditions have been established without ignoring any terms in the weak infinitesimal operator of Lyapunov-Krasovskii functional by considering the relationship among the time-varying delay, its upper bound, and their difference. A numerical example has verified its low level of conservativeness.

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ZHANG Yan Ph. D. candidate at the School of Information Science and Engineering, Central South University. Her research interest covers stochastic control of time-varying delay system.
E-mail: yanzhang24@yahoo.com.cn

HE Yong Professor at the School of Information Science and Engineering, Central South University. His research interest covers robust control and its applications.
E-mail: heyong08@yahoo.com.cn

WU Min Professor at the School of Information Science and Engineering, Central South University. His research interest covers robust control and its applications, intelligent control, and process control. Corresponding author of this paper.
E-mail: min@csu.edu.cn