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Hybrid Estimation of State and Input for Linear Discrete Time-varying Systems: A Game Theory Approach

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Abstract The H_∞ hybrid estimation problem for linear discrete time-varying systems is investigated in this paper, where estimated signals are linear combination of state and input. Design objective requires the worst-case energy gain from disturbance to estimation error to be less than a prescribed level. Optimal solution of the hybrid estimation problem is the saddle point of a two-player zero sum differential game. On the basis of the differential game approach, necessary and sufficient solvable conditions for the hybrid estimation problem are provided in terms of solutions to a Riccati differential equation. Moreover, one possible estimator is proposed if the solvable conditions are satisfied. The estimator is characterized by a gain matrix and an output mapping matrix, where the latter reflects the internal relations between unknown input and output estimation error. At last, a numerical example is provided to illustrate the proposed approach.

Key words Time-varying system, input estimation, game theory, Riccati equation

When estimated signal includes both state and unknown input of the system, the estimation problem is referred to as state and input hybrid estimation (in the following, only hybrid estimation will be used for brevity). Hybrid estimation is originated from practical application and theory[1]. One practical example is load current estimation of uninterruptible power supply (UPS), where load current is a linear function of capacitor voltage (state) and back electromotive force (unknown input)[2]. From a theoretical view point, either filtering (state estimation) or deconvolution (input estimation) is just a special case of the hybrid estimation. Both of the former two can be treated in the framework of hybrid estimation. Therefore, research on hybrid estimation is more general. Fault diagnosis is another important related area of hybrid estimation. Scheme of fault diagnosis can be designed on the basis of hybrid estimation approach because fault signal can be treated as unknown input.

In the past decade, H_∞ optimization-based estimation has been an active research area[3−5]. Differential game-theory approach is one of main time-domain approaches, because H_∞ estimation is a min-max problem in essential. Differential game-theory approach can directly deduce estimator′s design method from the performance specification and therefore, is a constructive approach. Moreover, the existence conditions of the proposed estimator are necessary and sufficient so that the least conservativeness might be achieved. Differential game-theory approach is also capable of dealing with time-varying problems, which makes it a powerful math tool. Banavar and Speyer[6] first investigated H_∞ filtering and smoothing for continuous linear time-varying (LTV) systems using differential game-theory approach. Later, discrete differential game-theory approach was applied to H_∞ filtering for discrete LTV systems[7]. In contrast, a new H_∞ deconvolution filter was derived by using game-theory approach[8]. It should be noted that existing conditions for the deconvolution filter are not provided explicitly. Moreover, the performing specification is defined in an indirect manner which makes their results unnecessarily complicated.

Other related research of H_∞ hybrid estimation is introduced in the following. Optimal performance was first presented for continuous LTV system by differential game-theory approach. However, construction of the estimator was not discussed there. Khargonekar et al. gave results on H_2/H_∞ hybrid estimation for continuous linear time-invariant (LTI) systems[9]. In [10], H_∞ filtering was explored, where uncertain initial state is deemed as a fictitious external input and the H_∞ filtering was converted to an equivalent hybrid estimation problem. At last, Cuzzola and Ferrante proposed LMI conditions for H_2 estimation for discrete LTI systems[1]. They also illustrated explicitly the theoretical and practical sense of hybrid estimation.

Above research on hybrid estimation mostly focused on LTI systems. There is however still a lack of results for H_∞ hybrid estimation problem for discrete LTV systems, which will be the subject of this paper. We will use a game theory approach that incorporates maximum principle arguments to study such a problem over a finite horizon. The connection is first established between H_∞ hybrid estimation problem and a two-player zero sum differential game. On the basis of the differential game approach, necessary and sufficient solvable conditions for the hybrid estimation problem are then provided in terms of solutions to a Riccati differential equation. Moreover, one possible estimator is proposed if the solvable conditions are satisfied. The estimator is characterized by a gain matrix and an output mapping matrix, where the latter reflects the internal relations between unknown input and output estimation error. At last, effectiveness of the proposed approach is shown through a numerical example.

Notation. $\mathbf{R}^n$ and $\mathbf{R}^{m\times n}$ denote n-dimensional and $(m\times n)$-dimensional Euclidean space, respectively. L_2 denotes square summable real sequences and $\|\|$ stands for inner product in Euclidean space. For any $h\in L_2$, $\|h\|=(h^{\mathrm T}h)^{1/2}$.

1 Problem formulation

Consider the following discrete-time LTV system Σ_1

$$\left\{\begin{array}{ll}\boldsymbol{x}_{k+1}=A_k\boldsymbol{x}_k+B_k\boldsymbol{u}_k, & \boldsymbol{x}_0=\bar{\boldsymbol{x}}_0\\ \boldsymbol{y}_k=C_k\boldsymbol{x}_k+D_k\boldsymbol{u}_k+\boldsymbol{v}_k & \end{array}\right. \tag{1}$$

where $\boldsymbol{x}_k\in\mathbf{R}^n$ is state vector, $\boldsymbol{y}_k\in\mathbf{R}^m$ is measured output, $\boldsymbol{u}_k\in\mathbf{R}^p$ is unknown input, and $\boldsymbol{v}_k\in\mathbf{R}^q$ is measurement noise. Matrices $A_k\in\mathbf{R}^{n\times n}$, $B_k\in\mathbf{R}^{n\times p}$, $C_k\in\mathbf{R}^{m\times n}$, and $D_k\in\mathbf{R}^{m\times p}$ are known time-varying parameters. State initial value $\bar{\boldsymbol{x}}_0$ is unknown. The control input does not influence the results and is not included in

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system Σ_1. Estimated signal is given by

$$\boldsymbol{z}_k = L_{xk}\boldsymbol{x}_k + L_{uk}\boldsymbol{u}_k \tag{2}$$

where $\boldsymbol{z}_k \in \mathbf{R}^l$, $L_{xk} \in \mathbf{R}^{l\times n}$, and $L_{uk} \in \mathbf{R}^{l\times p}$ are predefined parameter matrices to describe estimated objectives.

With different forms of $\boldsymbol{z}_k$ in (2), three kinds of estimation problem can be defined as follows: 1) $L_{xk} \neq 0$, $L_{uk} = 0$, filtering; 2) $L_{xk} = 0$, $L_{uk} \neq 0$, input estimation; 3) $L_{xk} \neq 0$ and $L_{uk} \neq 0$, SISE (State and input simultaneous estimation). Obviously, both filtering and input estimation are just special cases of SISE.

Define a scalar γ to denote disturbance attenuation level and a worst-case estimation performance index

$$J_{wrt}(\hat{\boldsymbol{z}}_k, \boldsymbol{u}_k, \boldsymbol{v}_k, \boldsymbol{x}_0) = \sup_{(\boldsymbol{u},\boldsymbol{v})\in L_2, \boldsymbol{x}_0\in\mathbf{R}^n} \left[\frac{\sum\limits_{k=0}^{N-1} \|\boldsymbol{z}_k - \hat{\boldsymbol{z}}_k\|^2}{\boldsymbol{x}_0^{\mathrm{T}} R \boldsymbol{x}_0 + \sum\limits_{k=0}^{N-1}\left(\|\boldsymbol{u}_k\|^2 + \|\boldsymbol{v}_k\|^2\right)}\right]^{1/2} \tag{3}$$

where the subscript "wrt" of J_{wrt} denotes "worst", $\hat{\boldsymbol{z}}_k \in \mathbf{R}^l$ is estimate of $\boldsymbol{z}_k$ and $R \in \mathbf{R}^{n\times n}$ is a positive definite weighting matrix on uncertainty of initial state. $N \to \infty$ corresponds to infinite horizon case and in this paper we will focus on the finite horizon case ($N < \infty$).

H_∞ SISE estimator for discrete-time LTV systems:

Consider system (1) and (2). Given a positive scalar γ, an estimator $\hat{\boldsymbol{z}}_k = \Im(\boldsymbol{y}_k)$ satisfying the performance specification $J_{wrt} < \gamma$ for all possible $\boldsymbol{u}, \boldsymbol{v}$, and $\boldsymbol{x}_0$ is referred to as H_∞ SISE estimator for discrete-time LTV systems.

2 Main results

From the output equation in (1) and the performance specification $J_{wrt} < \gamma$, we formulate a new performance measure

$$J(\hat{\boldsymbol{z}}_k, \boldsymbol{u}_k, \boldsymbol{y}_k, \boldsymbol{x}_0) = \sum_{k=0}^{N-1} \|\boldsymbol{z}_k - \hat{\boldsymbol{z}}_k\|^2 - \gamma^2\left[\boldsymbol{x}_0^{\mathrm{T}} R\boldsymbol{x}_0 + \sum_{k=0}^{N-1}\left(\|\boldsymbol{u}_k\|^2 + \|\boldsymbol{y}_k - C_k\boldsymbol{x}_k - D_k\boldsymbol{u}_k\|^2\right)\right] \tag{4}$$

To seek H_∞ SISE estimator, we are involved with solving a two-player zero sum differential game

$$\inf_{\hat{\boldsymbol{z}}_k} \sup_{\boldsymbol{y}} \sup_{\boldsymbol{x}_0} \sup_{\boldsymbol{u}} J \tag{5}$$

As the two opponents in the game, unknown input $\boldsymbol{u}$, output $\boldsymbol{y}$, and unknown initial state $\boldsymbol{x}_0$ try to make the performance measure J maximize, the estimate $\hat{\boldsymbol{z}}_k$ actes counter. Optimal solution to H_∞ SISE problem is exactly the game′s saddle point $(\hat{\boldsymbol{z}}_k^*, \boldsymbol{u}^*, \boldsymbol{y}^*, \boldsymbol{x}_0^*)$ satisfying

$$J(\hat{\boldsymbol{z}}_k^*, \boldsymbol{u}_k, \boldsymbol{y}_k, \boldsymbol{x}_0) \le J(\hat{\boldsymbol{z}}_k^*, \boldsymbol{u}_k^*, \boldsymbol{y}_k^*, \boldsymbol{x}_0^*) \le J(\hat{\boldsymbol{z}}_k, \boldsymbol{u}_k^*, \boldsymbol{y}_k^*, \boldsymbol{x}_0^*) \tag{6}$$

For notation brevity, define matrices

$$F_k = L_{xk}^{\mathrm{T}} L_{uk} - \gamma^2 C_k^{\mathrm{T}} D_k, \quad E_k = \gamma^2(I + D_k^{\mathrm{T}} D_k) - L_{uk}^{\mathrm{T}} L_{uk}$$

$$S_k = L_{xk}^{\mathrm{T}} L_{xk} - \gamma^2 C_k^{\mathrm{T}} C_k + F_k E_k^{-1} F_k^{\mathrm{T}}, \ A_{ck} = A_k + B_k E_k^{-1} F_k^{\mathrm{T}}$$

Theorem 1. Given a positive scalar γ, for discrete-time LTV system (1) and (2), an estimator satisfying performance specification $J_{wrt} < \gamma$ exists if and only if there exists bounded positive matrix function Q_k for $\forall k \in [0, N-1]$ such that $(I - S_k Q_k)$ is invertible and the following conditions hold

$$\gamma^2(I + D_k^{\mathrm{T}} D_k) - L_{uk}^{\mathrm{T}} L_{uk} > 0 \tag{7}$$

$$I + L_{uk}E_k^{-1}L_{uk}^{\mathrm{T}} + (L_{xk}^{\mathrm{T}} + F_k E_k^{-1} L_{uk}^{\mathrm{T}})^{\mathrm{T}} Q_k (I - S_k Q_k)^{-1} (L_{xk}^{\mathrm{T}} + F_k E_k^{-1} L_{uk}^{\mathrm{T}}) > 0 \tag{8}$$

$$Q_{k+1} = (A_k + B_k E_k^{-1} F_k^{\mathrm{T}}) Q_k (I - S_k Q_k)^{-1} (A_k + B_k E_k^{-1} F_k^{\mathrm{T}})^{\mathrm{T}} + B_k\left[\gamma^2(I + D_k^{\mathrm{T}} D_k) - L_{uk}^{\mathrm{T}} L_{uk}\right]^{-1} B_k^{\mathrm{T}} \tag{9}$$

$$Q_0 = \gamma^{-2} R^{-1} \tag{10}$$

Furthermore, if such an estimator exists, one H_∞ SISE estimator is given by

$$\begin{cases} \hat{\boldsymbol{x}}_{k+1} = A\hat{\boldsymbol{x}}_k + (\gamma^2 A_k Q_k C_k^{\mathrm{T}} + B_k D_k^{\mathrm{T}})(D_k D_k^{\mathrm{T}} + I\gamma^2 C_k Q_k C_k^{\mathrm{T}})^{-1}(\boldsymbol{y}_k - C_k\hat{\boldsymbol{x}}_k) \\ \hat{\boldsymbol{z}}_k = L_{xk}\hat{\boldsymbol{x}}_k + (L_{uk} D_k^{\mathrm{T}} + \gamma^2 L_{xk} Q_k C_k^{\mathrm{T}})(D_k D_k^{\mathrm{T}} + I + \gamma^2 C_k Q_k C_k^{\mathrm{T}})^{-1}(\boldsymbol{y}_k - C_k\hat{\boldsymbol{x}}_k), \end{cases}$$
$$\hat{\boldsymbol{x}}_0 = 0 \tag{11}$$

Theorem 1 for existing conditions of H_∞ SISE estimator for discrete-time LTV systems can be checked by inequalities (7) and (8) and by solving a differential Riccati recursion, which is propagated by (9) from initial condition (10). In case that Q_k exists, one H_∞ SISE estimator can be constructed as (11) with a special innovation structure. Innovation information $(\boldsymbol{y}_k - C_k\hat{\boldsymbol{x}}_k)$ is used to update estimator′s state equation and provide input estimation through gain matrix and projector matrix, respectively. Let $L_{uk} = 0$, conditions (7)$\sim$(10) recover the results on H_∞ filtering for discrete-time LTV systems[11]. Let $L_{xk} = 0$, and we can obtain an H_∞ deconvolution filter for discrete-time LTV systems.

Proof of Theorem 1.

We will propose the necessary proof of Theorem 1 and it is easy to finish its sufficient proof by reversing the process of the necessary proof.

Step 1. Seek optimal $\boldsymbol{u}_k^*$ and $\boldsymbol{x}_0^*$ of unknown input $\boldsymbol{u}_k$ and initial state $\boldsymbol{x}_0$.

Having known the performing specification $J_{wrt} < \gamma$ holds for discrete-time LTV systems (1) and (2), following (4), and constraint condition (1), define Hamiltonian function as

$$H_{ak}(\hat{\boldsymbol{z}}_k, \boldsymbol{u}_k, \boldsymbol{y}_k, \boldsymbol{x}_0, \lambda_{k+1}) = \frac{1}{2}\|\boldsymbol{z}_k - \hat{\boldsymbol{z}}_k\|^2 - \frac{1}{2}\gamma^2(\|\boldsymbol{u}_k\|^2 + \|\boldsymbol{y}_k - C_k\boldsymbol{x}_k - D_k\boldsymbol{u}_k\|^2) + \lambda_{k+1}^{\mathrm{T}}(A_k\boldsymbol{x}_k + B_k\boldsymbol{u}_k) \tag{12}$$

where λ_{k+1} is a Lagrange multiplier.

First order necessary conditions are given by

$$NC_1\text{: } \lambda_k = -\frac{\partial H_{ak}}{\partial \boldsymbol{x}_k}, \ NC_2\text{: } \frac{\partial H_{ak}}{\partial \boldsymbol{u}_k} = 0$$
$$NC_3\text{: } \lambda_0 = \gamma^2 R\boldsymbol{x}_0, \ NC_4\text{: } \lambda_N = 0$$

NC_2 gives the optimal unknown input value as

$$\boldsymbol{u}_k^* = E_k^{-1}(F_k^{\mathrm{T}}\boldsymbol{x}_k - L_{uk}^{\mathrm{T}}\hat{\boldsymbol{z}}_k + \gamma^2 D_k^{\mathrm{T}}\boldsymbol{y}_k + B_k^{\mathrm{T}}\lambda_{k+1}) \tag{13}$$

To make sure $\boldsymbol{u}_k^*$ maximizes performance measure J requires that

$$E_k > 0 \tag{14}$$

A new Hamiltonian accompanying system is obtained from $NC_1 \sim NC_4$.

$$\begin{bmatrix} \boldsymbol{x}_{k+1} \\ \lambda_k \end{bmatrix} = \begin{bmatrix} A_{ck} & B_k E_k^{-1} B_k^{\mathrm{T}} \\ S_k & A_{ck}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_k \\ \lambda_{k+1} \end{bmatrix} - \begin{bmatrix} B_k E_k^{-1} L_{uk}^{\mathrm{T}} \\ F_k E_k^{-1} L_{uk}^{\mathrm{T}} + L_{xk}^{\mathrm{T}} \end{bmatrix} \hat{\boldsymbol{z}}_k + \gamma^2 \begin{bmatrix} B_k E_k^{-1} D_k^{\mathrm{T}} \\ F_k E_k^{-1} D_k^{\mathrm{T}} + C_k^{\mathrm{T}} \end{bmatrix} \boldsymbol{y}_k$$
$$\begin{bmatrix} \boldsymbol{x}_0 \\ \lambda_N \end{bmatrix} = \begin{bmatrix} \bar{\boldsymbol{x}}_0 \\ 0 \end{bmatrix} \tag{15}$$

(15) is a two point boundary value problem and its solution is given by

$$\boldsymbol{x}_k^* = \hat{\boldsymbol{x}}_k^* + Q_k \lambda_k^* \tag{16}$$

Substituting (16) into (15), we have

$$Q_{k+1} = A_{ck} Q_k (I - S_k Q_k)^{-1} A_{ck}^{\mathrm{T}} + B_k E_k^{-1} B_k^{\mathrm{T}} \tag{17}$$

$$Q_0 = \gamma^{-2} R^{-1} \tag{18}$$

$$\boldsymbol{x}_0^* = \gamma^{-2} R^{-1} \lambda_0^* \tag{19}$$

$$\hat{\boldsymbol{x}}_{k+1} = A\hat{\boldsymbol{x}}_k - [B_k E_k^{-1} L_{uk}^{\mathrm{T}} + A_{ck} Q_k (I - SQ_k)^{-1} (F_k \times E_k^{-1} L_{uk}^{\mathrm{T}} + L_{xk}^{\mathrm{T}})](L_{xk}\hat{\boldsymbol{x}}_k - \hat{\boldsymbol{z}}_k) + \gamma^2 [B_k E_k^{-1} D_k^{\mathrm{T}} + A_{ck} Q_k \times (I - SQ_k)^{-1} (F_k E_k^{-1} D_k^{\mathrm{T}} + C_k^{\mathrm{T}})](\boldsymbol{y}_k - C_k \hat{\boldsymbol{x}}_k), \hat{\boldsymbol{x}}_0 = 0 \tag{20}$$

If (20) is used to solve the differential game, the result will be $\hat{\boldsymbol{z}}_k = L_{xk}\hat{\boldsymbol{x}}_k$, i.e., only state observation term occurs in $\hat{\boldsymbol{z}}_k$. By inspection, unknown input part in $\boldsymbol{z}_k$ is estimated by $H_k(\boldsymbol{y}_k - C_k\hat{\boldsymbol{x}}_k)$, where projector matrix H_k can be determined in a similar way to that in [11] using the bounded real lemma approach.

$$H_k = (L_{uk} D_k^{\mathrm{T}} + \gamma^2 L_{xk} Q_k C_k^{\mathrm{T}})(D_k D_k^{\mathrm{T}} + I + \gamma^2 C_k Q_k C_k^{\mathrm{T}}) \tag{21}$$

Introduce a zero term by adding and subtracting term

$$\gamma^2 \begin{bmatrix} B_k E_k^{-1} D_k^{\mathrm{T}} + A_{ck} Q_k (I - SQ_k)^{-1} \times \\ (F_k E_k^{-1} D_k^{\mathrm{T}} + C_k^{\mathrm{T}}) \end{bmatrix} H_k (\boldsymbol{y}_k - C_k \hat{\boldsymbol{x}}_k)$$

in the right side of (20). After rearrangement, (20) is transformed to

$$\hat{\boldsymbol{x}}_{k+1} = A\hat{\boldsymbol{x}}_k - [B_k E_k^{-1} L_{uk}^{\mathrm{T}} + A_{ck} Q_k (I - SQ_k)^{-1} (F_k E_k^{-1} \times L_{uk}^{\mathrm{T}} + L_{xk}^{\mathrm{T}})][H_k(\boldsymbol{y}_k - C_k\hat{\boldsymbol{x}}_k) + L_{xk}\hat{\boldsymbol{x}}_k - \hat{\boldsymbol{z}}_k] + (\gamma^2 A_k Q_k C_k^{\mathrm{T}} + B_k D_k^{\mathrm{T}})(D_k D_k^{\mathrm{T}} + I + \gamma^2 C_k Q_k C_k^{\mathrm{T}})^{-1} (\boldsymbol{y}_k - C_k \hat{\boldsymbol{x}}_k) \tag{22}$$

Step 2. Seek optimal $\boldsymbol{y}_k^*$ and $\boldsymbol{z}_k^*$ of $\boldsymbol{y}_k$ and $\boldsymbol{z}_k$, define function

$$L_k(\hat{\boldsymbol{z}}_k, \boldsymbol{u}_k, \boldsymbol{y}_k, \boldsymbol{x}_0) = \|\boldsymbol{z}_k - \hat{\boldsymbol{z}}_k\|^2 - \gamma^2(\|\boldsymbol{u}_k\|^2 + \|\boldsymbol{y}_k - C_k\boldsymbol{x}_k - D_k\boldsymbol{u}_k\|^2) \tag{23}$$

Then

$$J(\hat{\boldsymbol{z}}_k, \boldsymbol{u}_k, \boldsymbol{y}_k, \boldsymbol{x}_0) = -\gamma^2 \boldsymbol{x}_0^{\mathrm{T}} R \boldsymbol{x}_0 + \lambda_0^{\mathrm{T}} Q_0 \lambda_0 - \lambda_N^{\mathrm{T}} Q_N \lambda_N + \sum_{k=0}^{N-1} \Big[L_k(\hat{\boldsymbol{z}}_k, \boldsymbol{u}_k, \boldsymbol{y}_k, \boldsymbol{x}_0) + \lambda_{k+1}^{\mathrm{T}} Q_{k+1} \lambda_{k+1} - \lambda_k^{\mathrm{T}} Q_k \lambda_k\Big] \tag{24}$$

and

$$J(\hat{\boldsymbol{z}}_k, \boldsymbol{u}_k, \boldsymbol{y}_k, \boldsymbol{x}_0^*) = \sum_{k=0}^{N-1} \Big[L_k(\hat{\boldsymbol{z}}_k, \boldsymbol{u}_k, \boldsymbol{y}_k, \boldsymbol{x}_0) + \lambda_{k+1}^{\mathrm{T}} Q_{k+1} \lambda_{k+1} - \lambda_k^{\mathrm{T}} Q_k \lambda_k\Big] \tag{25}$$

Substituting $\boldsymbol{u}_k^*$ described by (13) into (25), and through a rather involved algebraic operation, we can obtain

$$J(\hat{\boldsymbol{z}}_k, \boldsymbol{u}_k^*, \boldsymbol{y}_k, \boldsymbol{x}_0^*) = \sum_{k=0}^{N-1} \{\tilde{\boldsymbol{z}}_k^{\mathrm{T}} [(L_{xk}^{\mathrm{T}} + F_k E_k^{-1} L_{uk}^{\mathrm{T}})^{\mathrm{T}} Q_k (I - S_k Q_k)^{-1} (L_{xk}^{\mathrm{T}} + F_k E_k^{-1} L_{uk}^{\mathrm{T}})] \tilde{\boldsymbol{z}}_k - \gamma^2 \tilde{\boldsymbol{y}}_k^{\mathrm{T}} (D_k D_k^{\mathrm{T}} + I + \gamma^2 C_k Q_k C_k^{\mathrm{T}}) \tilde{\boldsymbol{y}}_k\} \tag{26}$$

where $\tilde{\boldsymbol{z}}_k = L_{xk}\hat{\boldsymbol{x}}_k + H_k(\boldsymbol{y}_k - C_k\hat{\boldsymbol{x}}_k) - \hat{\boldsymbol{z}}_k$ and $\tilde{\boldsymbol{y}}_k = \boldsymbol{y}_k - C_k\hat{\boldsymbol{x}}_k$.

For positive matrix $D_k D_k^{\mathrm{T}} + I + \gamma^2 C_k Q_k C_k^{\mathrm{T}}$, to maximize $J(\hat{\boldsymbol{z}}_k, \boldsymbol{u}_k^*, \boldsymbol{y}_k, \boldsymbol{x}_0^*)$, the optimal solution of $\boldsymbol{y}_k$ should be given by

$$\boldsymbol{y}_k^* = C_k \hat{\boldsymbol{x}}_k \tag{27}$$

In contrast, matrix inequality relation

$$(L_{xk}^{\mathrm{T}} + F_k E_k^{-1} L_{uk}^{\mathrm{T}})^{\mathrm{T}} Q_k (I - S_k Q_k)^{-1} (L_{xk}^{\mathrm{T}} + F_k E_k^{-1} L_{uk}^{\mathrm{T}}) > 0 \tag{28}$$

must be held to guarantee that the optimal solution of $\boldsymbol{z}_k$ exists. Otherwise, $\boldsymbol{z}_k$, that is bounded, does not minimize $J(\hat{\boldsymbol{z}}_k, \boldsymbol{u}_k^*, \boldsymbol{y}_k, \boldsymbol{x}_0^*)$. In case that (28) holds, optimal $\boldsymbol{z}_k^*$ is expressed by

$$\hat{\boldsymbol{z}}_k^* = L_{xk}\hat{\boldsymbol{x}}_k + H(\boldsymbol{y}_k - C_k\hat{\boldsymbol{x}}_k) \tag{29}$$

Step 3. Verify $(\hat{\boldsymbol{z}}_k^*, \boldsymbol{u}_k^*, \boldsymbol{y}_k^*, \boldsymbol{x}_0^*)$ to guarantee saddle point condition (6).

It follows from (26), (27), and (29) that

$$J(\hat{\boldsymbol{z}}_k^*, \boldsymbol{u}_k^*, \boldsymbol{y}_k^*, \boldsymbol{x}_0^*) = 0 \tag{30}$$

On the one hand

$$J(\hat{\boldsymbol{z}}_k^*, \boldsymbol{u}_k, \boldsymbol{y}_k, \boldsymbol{x}_0) \le J(\hat{\boldsymbol{z}}_k^*, \boldsymbol{u}_k^*, \boldsymbol{y}_k, \boldsymbol{x}_0^*) = \sum_{k=0}^{N-1} -\gamma^2 \tilde{\boldsymbol{y}}_k^{\mathrm{T}} (D_k D_k^{\mathrm{T}} + I + \gamma^2 C_k Q_k C_k^{\mathrm{T}}) \tilde{\boldsymbol{y}}_k \le 0 \tag{31}$$

On the other hand, with (28), we have

$$J(\hat{\boldsymbol{z}}_k, \boldsymbol{u}_k^*, \boldsymbol{y}_k^*, \boldsymbol{x}_0^*) = \sum_{k=0}^{N-1} \{\tilde{\boldsymbol{z}}_k^{\mathrm{T}} [(L_{xk}^{\mathrm{T}} + F_k E_k^{-1} L_{uk}^{\mathrm{T}})^{\mathrm{T}} Q_k (I - S_k Q_k)^{-1} (L_{xk}^{\mathrm{T}} + F_k E_k^{-1} L_{uk}^{\mathrm{T}})] \tilde{\boldsymbol{z}}_k\} \ge 0 \tag{32}$$

By (30) $\sim$ (32), we can conclude that the optimal solution $(\hat{\boldsymbol{z}}_k^*, \boldsymbol{u}_k^*, \boldsymbol{y}_k^*, \boldsymbol{x}_0^*)$ satisfies the saddle point condition

$$J(\hat{\boldsymbol{z}}_k^*, \boldsymbol{u}_k, \boldsymbol{y}_k, \boldsymbol{x}_0) \le J(\hat{\boldsymbol{z}}_k^*, \boldsymbol{u}_k^*, \boldsymbol{y}_k^*, \boldsymbol{x}_0^*) \le J(\hat{\boldsymbol{z}}_k, \boldsymbol{u}_k^*, \boldsymbol{y}_k^*, \boldsymbol{x}_0^*) \tag{33}$$

Finally, necessary and sufficient existing conditions for H_∞ SISE estimator in Theorem 1 are followed by (14), (17), (18), and (28). If we choose the optimal strategy (29) for estimated signal $\boldsymbol{z}_k$, from (20) estimator described by (11) is an H_∞ SISE estimator that satisfies the performance specification $J_{wrt} < \gamma$.

3 Simulation example

In this paper, we exploit a modified example from [7]. Considering a damped harmonic oscillator with velocity measurements described by (1) and (2), the system′s parameters are given by

$$A_k = \begin{bmatrix} 0.5079 & 0.7594 \\ -0.7594 & 0.2801 \end{bmatrix}, B_k = \begin{bmatrix} 0.4921 \\ 0.7594 \end{bmatrix}$$

$$C_k = \begin{bmatrix} 0 & 1 \end{bmatrix}, D_k = 5, L_{xk} = \begin{bmatrix} 1 & 0 \end{bmatrix}, L_{uk} = 1$$

The state vector $\boldsymbol{x}_k$ stands for position and velocity, and output $\boldsymbol{y}_k$ is the measured velocity. Unknown input $\boldsymbol{u}_k$ here stands for a step fault signal and its waveform is shown in Fig. 1. Measurement noise $\boldsymbol{v}_k$ is a band-limited white noise with power 0.02, and its waveform is shown in Fig. 2.

For clarity, in (2), let $\boldsymbol{z}_{xk} = L_{xk}\boldsymbol{x}_k$ and $\boldsymbol{z}_{uk} = L_{uk}\boldsymbol{u}_k$ denote state observation and input estimation objective, respectively. Choose disturbance attenuation level $\gamma = 0.8$. By Theorem 1, the solution to differential Ricatti recursion (9) is obtained as

$$Q = \begin{bmatrix} 0.0644 & 0.0039 \\ 0.0039 & 0.0685 \end{bmatrix}$$

It is easy to verify that inequalities (7) and (8) are guaranteed. From (11), an estimator satisfying performance specification $J_{wrt} < \gamma$ is constructed as follows

$$\hat{\boldsymbol{x}}_{k+1} = \begin{bmatrix} 0.5079 & 0.7594 \\ -0.7594 & 0.2801 \end{bmatrix} \hat{\boldsymbol{x}}_k + \begin{bmatrix} 0.0958 \\ 0.1462 \end{bmatrix} (\boldsymbol{y}_k - C_k\hat{\boldsymbol{x}}_k)$$

$$\hat{\boldsymbol{z}}_k = \begin{bmatrix} 1 & 0 \end{bmatrix} \hat{\boldsymbol{x}}_k + 0.1921(\boldsymbol{y}_k - C_k\hat{\boldsymbol{x}}_k), \qquad \hat{\boldsymbol{x}}_0 = 0$$

where $\hat{\boldsymbol{z}}_{xk} = \begin{bmatrix} 1 & 0 \end{bmatrix} \hat{\boldsymbol{x}}_k$ and $\hat{\boldsymbol{z}}_{uk} = 0.1921(\boldsymbol{y}_k - C_k\hat{\boldsymbol{x}}_k)$ are estimate of $\boldsymbol{z}_{xk}$ and $\boldsymbol{z}_{uk}$, respectively. Observation of the state and input estimation results are shown in Figs. 3 and 4. Fig. 5 shows state and input simultaneous estimation results. Define estimation error vector $\tilde{\boldsymbol{z}}_k = \boldsymbol{z}_k - \hat{\boldsymbol{z}}_k$. Fig. 6 gives the estimation error of signal $\boldsymbol{z}_k$. It is seen that the estimator shows superior performance in each case. Both state and fault signals are reconstructed in a high precision despite the influence of fault and measurement noise.

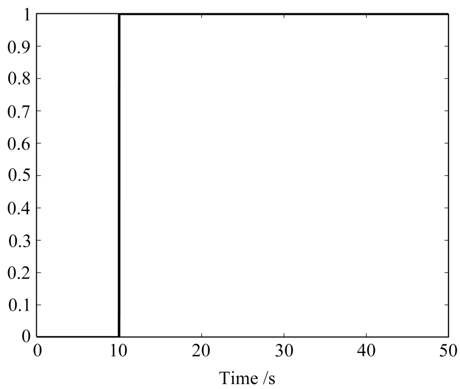

Fig. 1 Waveform of $\boldsymbol{u}_k$

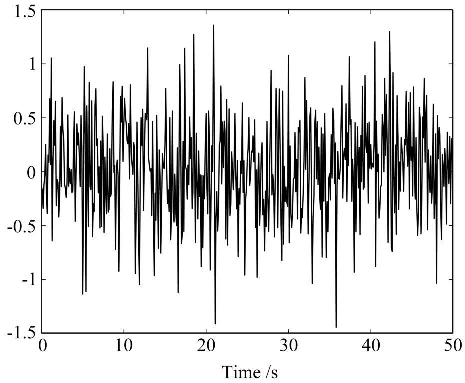

Fig. 2 Waveform of $\boldsymbol{v}_k$

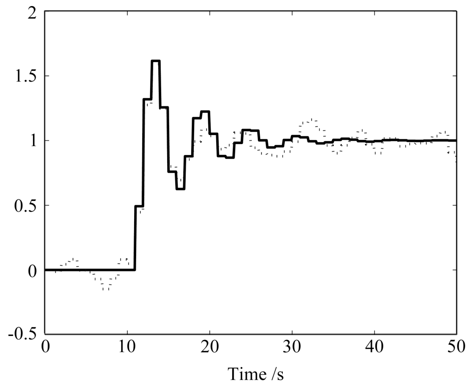

Fig. 3 $\boldsymbol{z}_{xk}$ (solid line) and $\hat{\boldsymbol{z}}_{xk}$ (dot line)

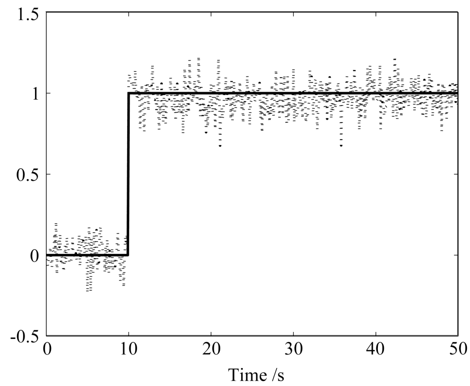

Fig. 4 $\boldsymbol{z}_{uk}$ (solid line) and $\hat{\boldsymbol{z}}_{uk}$ (dot line)

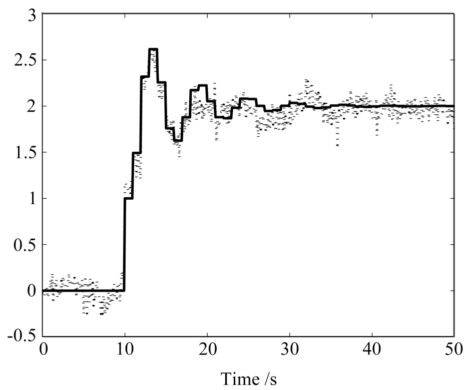

Fig. 5 $\boldsymbol{z}_k$ (solid line) and $\hat{\boldsymbol{z}}_k$ (dot line)

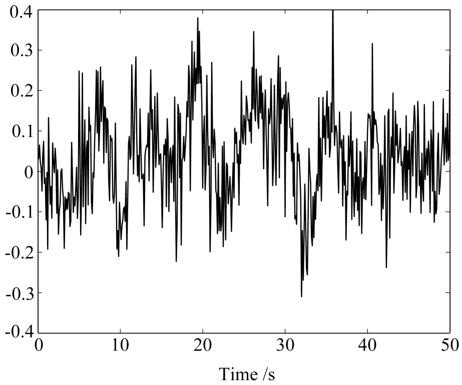

Fig. 6 Estimation error $\tilde{\boldsymbol{z}}_k$

4 Conclusion

H_∞ state and input simultaneous estimation for discrete-time LTV–hybrid estimation are investigated in this paper. On the basis of differential game-theory approach, necessary and sufficient solvable conditions for H_∞ state and input simultaneous estimation for discrete-time LTV systems are proposed, which are equivalent to solvability of a set differential Riccati recursion. An estimator is presented in case that the H_∞ SISE is solvable. The estimator is parameterized by a gain matrix and a projector matrix. The work in this paper shows that innovation information can be used to provide state observation and input estimation simultaneously. Because fault signal can be treated as exogenous input, with input estimation ability, one immediate application area of the proposed estimator is fault diagnosis.

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