One Kind of Fully Coupled Linear Quadratic Stochastic Control Problem with Random Jumps

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Abstract One kind of fully coupled linear quadratic stochastic control problem with random jumps is studied. The explicit form of the optimal control is obtained. The optimal control can be proved to be unique. One kind of generalized Riccati equation system is introduced and its solvability is discussed. The linear feedback regulator for the optimal control problem with random jumps is given by the solution of the generalized Riccati equation system.

Key words Backward stochastic differential equation, poisson process, linear quadratic stochastic control, Riccati equation

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a stochastic basis satisfying the usual conditions. We suppose that the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is generated by the following two mutually independent processes:

1) A standard Brownian motion $\{\boldsymbol{B}(t)\}_{t\geq 0}$;

2) A poisson random measure N on $\mathbf{E} \times \mathbf{R}_+$, where $\mathbf{E} \subset \mathbf{R}$ is a nonempty open set equipped with its Borel field $\mathcal{B}(\mathbf{E})$ and compensator $\hat{N}(\mathrm{d}e\mathrm{d}t) = \pi(\mathrm{d}e)\mathrm{d}t$, such that $\tilde{N}(A\times [0,t]) = (N - \hat{N})(A \times [0,t])_{t\geq 0}$ is a martingale for all $A \in \mathcal{B}(\mathbf{E})$ satisfying $\pi(A) < \infty$. π is assumed to be a σ-finite measure on $(\mathbf{E}, \mathcal{B}(\mathbf{E}))$ and is called the characteristic measure.

Let $\mathcal{H}$ be a finite-dimensional vector space, and $T > 0$ be a fixed real number called time duration. And denote by $\mathbf{L}^2(\Omega, \mathcal{F}_T; \mathcal{H})$ the space of $\mathcal{H}$-valued square-integrable $\mathcal{F}_T$-measurable random variables, by $\mathbf{L}^2_{\mathcal{F}}([0,T];\mathcal{H})$ the space of $\mathcal{H}$-valued square-integrable $\mathcal{F}_t$-adapted processes, by $\mathbf{L}^2_{\mathcal{F},p}([0,T];\mathcal{H})$ the space of $\mathcal{H}$-valued square-integrable $\mathcal{F}_t$-predictable processes, and by $\mathbf{F}^2_p([0,T];\mathcal{H})$ the space of $\mathcal{H}$-valued $\mathcal{F}_t$-predictable processes $f(\cdot,\cdot,\cdot)$ defined on $\Omega \times [0,T] \times \mathbf{E}$ such that $\mathrm{E}\int_{\mathbf{E}}\int_0^T |f(\cdot,t,e)|^2\pi(\mathrm{d}e)\mathrm{d}t < \infty$. For simplicity, we sometimes omit the value space $\mathcal{H}$ when it need not clarify.

The linear quadratic stochastic control problems have been studied in [1−6]. The optimal control problem with random jumps was first considered by Boel[7]. In this case, the control system is disturbed by random jumps and the optimal solution is a discontinuous stochastic process. This kind of optimal control problem has also been practically discussed in engineering and financial market.

In [8], the following linear quadratic stochastic control problem with random jumps was considered. The state variable is described by the following linear stochastic differential equation with random jumps (SDEP)

$$\begin{cases} \mathrm{d}\boldsymbol{x}(t) = [A\boldsymbol{x}(t) + B\boldsymbol{v}(t)]\mathrm{d}t + [C\boldsymbol{x}(t) + D\boldsymbol{v}(t)]\mathrm{d}B(t) + \\ \qquad \int_{\mathbf{E}}[E\boldsymbol{x}(t-) + F\boldsymbol{v}(t)]\tilde{N}(\mathrm{d}e\mathrm{d}t) \\ \boldsymbol{x}(0) = \boldsymbol{x} \end{cases} \tag{1}$$

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and the cost functional is

$$J(\boldsymbol{v}(\cdot)) = \frac{1}{2}\mathrm{E}[\int_0^T (\langle R\boldsymbol{x}(t), \boldsymbol{x}(t)\rangle + \langle N\boldsymbol{v}(t), \boldsymbol{v}(t)\rangle)\mathrm{d}t + \langle Q\boldsymbol{x}(T), \boldsymbol{x}(T)\rangle] \tag{2}$$

where $\boldsymbol{v}(\cdot)$ is an admissible control process, i.e., an $\mathcal{F}_t$-adapted square-integrable process taking values in a given subset $\mathbf{U}$ of $\mathbf{R}^k$. A, C, and E are bounded $n \times n$ matrices, B, D, and F are bounded $n \times k$ matrices, Q and R are nonnegative symmetric bounded $n \times n$ matrices, N is a positive bounded $k \times k$ matrix, and the inverse N^{-1} is also bounded.

The linear quadratic stochastic optimal control problem is to minimize the cost functional (2) over the set of admissible controls. That is, to find an admissible control $\boldsymbol{u}(\cdot)$, such that

$$J(\boldsymbol{u}(\cdot)) = \inf_{\boldsymbol{v}(\cdot)} J(\boldsymbol{v}(\cdot)) \tag{3}$$

In [8], the optimal control is obtained as

$$\boldsymbol{u}(t) = -N^{-1}[B^\tau \boldsymbol{y}(t) + D^\tau \boldsymbol{z}(t) + F^\tau \boldsymbol{c}(t,\cdot)] \tag{4}$$

where $(\boldsymbol{y}(t), \boldsymbol{z}(t), \boldsymbol{c}(t,\cdot))$ is the solution of the following linear backward stochastic differential equation with random jumps (BSDEP)

$$\begin{cases} -\mathrm{d}\boldsymbol{y}(t) = [A^\tau \boldsymbol{y}(t) + C^\tau \boldsymbol{z}(t) + E^\tau \boldsymbol{c}(t,\cdot) + R\boldsymbol{x}(t)]\mathrm{d}t - \\ \qquad z(t)\mathrm{d}B(t) - \int_{\mathbf{E}} c(t-,e)\tilde{N}(\mathrm{d}e\mathrm{d}t) \\ \boldsymbol{y}(T) = Q\boldsymbol{x}(T) \end{cases} \tag{5}$$

In this paper, we consider a more general problem. Let the state variables be described by the following linear SDEP coupled with a linear BSDEP

$$\begin{cases} \mathrm{d}\boldsymbol{x}(t) = [A(\omega,t)\boldsymbol{x}(t) + B(\omega,t)\boldsymbol{v}(t) - L^\tau(\omega,t)\boldsymbol{y}(t)]\mathrm{d}t + \\ \qquad [C(\omega,t)\boldsymbol{x}(t) + D(\omega,t)\boldsymbol{v}(t)]\mathrm{d}B(t) + \\ \qquad \int_{\mathbf{E}}[E(\omega,t)\boldsymbol{x}(t-) + F(\omega,t)\boldsymbol{v}(t)]\tilde{N}(\mathrm{d}e\mathrm{d}t) \\ -\mathrm{d}\boldsymbol{y}(t) = [A^\tau(\omega,t)\boldsymbol{y}(t) + C^\tau(\omega,t)\boldsymbol{z}(t) + E^\tau(\omega,t)\boldsymbol{c}(t,\cdot) + \\ \qquad R(\omega,t)\boldsymbol{x}(t)]\mathrm{d}t - \boldsymbol{z}(t)\mathrm{d}B(t) - \\ \qquad \int_{\mathbf{E}} \boldsymbol{c}(t-,e)\tilde{N}(\mathrm{d}e\mathrm{d}t) \\ \boldsymbol{x}(0) = \boldsymbol{x} \\ \boldsymbol{y}(T) = Q(\omega)\boldsymbol{x}(T) \end{cases} \tag{6}$$

where $\boldsymbol{v}(\cdot)$ is an admissible control process and $t-$ is the left limitation of t. We also assume that there is no constraint imposed on the control process $\mathbf{U} = \mathbf{R}^k$.

And the cost functional be

$$J(v(\cdot)) = \frac{1}{2}\mathrm{E}[\int_0^T [\langle R(\omega,t)\boldsymbol{x}(t), \boldsymbol{x}(t)\rangle + \langle N(\omega,t)\boldsymbol{v}(t), \boldsymbol{v}(t)\rangle + \langle L(\omega,t)\boldsymbol{y}(t), \boldsymbol{y}(t)\rangle]\mathrm{d}t + \langle Q(\omega)\boldsymbol{x}(T), \boldsymbol{x}(T)\rangle] \tag{7}$$

In the above, $A(\omega,t)$, $C(\omega,t)$, and $E(\omega,t)$ are bounded $n \times n$ progressively measurable matrix-valued processes, $B(\omega,t)$, $D(\omega,t)$, and $F(\omega,t)$ are bounded $n \times k$ progressively measurable matrix-valued processes, $Q(\omega)$ is a $\mathcal{F}_T$-adapted nonnegative symmetric bounded matrix-valued random variable, $R(\omega,t)$ and $L(\omega,t)$ are nonnegative symmetric bounded $n \times n$ progressively measurable matrix-valued processes, $N(\omega,t)$ is a positive bounded $k \times k$ progressively measurable matrix-valued process and the inverse $N^{-1}(\omega,t)$ is also bounded.

Note that (6) itself is a fully coupled forward-backward stochastic differential equation with random jumps (FBSDEP). Under some monotonic assumptions, Wu[9] first obtained the existence and uniqueness result of the solutions to the general FBSDEP in an arbitrary fixed time duration. Wu[8] also obtained another existence and uniqueness result of FBSDEP under some monotonic assumptions suitable for the optimal control problem. In fact, fully coupled forward-backward stochastic differential equation with Brownian motion can be encountered in the optimization problem when applying stochastic maximum principle[4] and in mathematical finance when considering large investor in security market[10].

This paper is organized as follows. In Section 1, we give some preliminaries for FBSDEP including a existence and uniqueness result. In Section 2, we prove that there exists a unique optimal control and give the explicit linear state feedback form. When all the coefficient matrices are deterministic, we can give the linear feedback regulator for the optimal control using the solution of one kind of generalized matrix-valued Riccati equation system. In Section 3, the solvability of this kind of generalized matrix-valued Riccati equation is discussed.

1 The preliminary results of FBSDEP

We consider the following coupled FBSDEP

$$\left\{\begin{array}{rcl} \mathrm{d}\boldsymbol{x}(t) &=& \boldsymbol{b}(t,\boldsymbol{x}(t),\boldsymbol{y}(t))\mathrm{d}t+\boldsymbol{\sigma}(t,\boldsymbol{x}(t),\boldsymbol{y}(t))\mathrm{d}B(t)+ \\ && \int_{\mathbf{E}}\boldsymbol{g}(t,\boldsymbol{x}(t-),\boldsymbol{y}(t-),e)\tilde{N}(\mathrm{d}e\mathrm{d}t) \\ -\mathrm{d}\boldsymbol{y}(t) &=& \boldsymbol{f}(t,x(t),\boldsymbol{y}(t),\boldsymbol{z}(t),\boldsymbol{c}(t,\cdot))\mathrm{d}t- \\ && \boldsymbol{z}(t)\mathrm{d}B(t)-\int_{\mathbf{E}}\boldsymbol{c}(t-,e)\tilde{N}(\mathrm{d}e\mathrm{d}t) \\ \boldsymbol{x}(0) &=& \boldsymbol{x} \\ \boldsymbol{y}(T) &=& \boldsymbol{\phi}(\boldsymbol{x}(T)) \end{array}\right. \tag{8}$$

where $(\boldsymbol{x}(\cdot),\boldsymbol{y}(\cdot),\boldsymbol{z}(\cdot),\boldsymbol{c}(\cdot,\cdot))$ takes values in $\mathbf{R}^n\times\mathbf{R}^n\times\mathbf{R}^n\times\mathbf{R}^n$. $x\in\mathbf{R}^n$ is given and

$$\begin{array}{l} \boldsymbol{b}:\Omega\times[0,T]\times\mathbf{R}^n\times\mathbf{R}^n\to\mathbf{R}^n \\ \boldsymbol{\sigma}:\Omega\times[0,T]\times\mathbf{R}^n\times\mathbf{R}^n\to\mathbf{R}^n \\ \boldsymbol{g}:\Omega\times[0,T]\times\mathbf{R}^n\times\mathbf{R}^n\times\mathbf{E}\to\mathbf{R}^n \\ \boldsymbol{f}:\Omega\times[0,T]\times\mathbf{R}^n\times\mathbf{R}^n\times\mathbf{R}^n\times\mathbf{R}^n\to\mathbf{R}^n \\ \boldsymbol{\phi}:\Omega\times\mathbf{R}^n\to\mathbf{R}^n \end{array}$$

Assumption 1. We assume that

1) $\boldsymbol{b},\boldsymbol{\sigma},\boldsymbol{g}$ are uniformly Lipschitz continuous with respect to $(\boldsymbol{x},\boldsymbol{y})$ and $\boldsymbol{f}$ is uniformly Lipschitz continuous with respect to $(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\boldsymbol{c}(\cdot))$;

2) $\boldsymbol{l}(\omega,t,0,0)\in\mathbf{L}^2_{\mathcal{F}}([0,T])$, where $\boldsymbol{l}=\boldsymbol{b},\boldsymbol{\sigma},\boldsymbol{f}$ respectively and $\boldsymbol{g}(\omega,t,0,0,0,0)\in\mathbf{F}^2_p([0,T])$, for all $(\omega,t)\in\Omega\times[0,T]$;

3) $\boldsymbol{\phi}(\boldsymbol{x})$ is uniformly Lipschitz continuous with respect to $\boldsymbol{x}\in\mathbf{R}^n$;

4) For each $\boldsymbol{x}$, $\boldsymbol{\phi}(\boldsymbol{x})$ is in $\mathbf{L}^2(\Omega,\mathcal{F}_T;\mathbf{R}^n)$, we set

$$\boldsymbol{\lambda}=\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\\ \boldsymbol{z}\\ \boldsymbol{c}(\cdot)\end{pmatrix},\ \boldsymbol{A}(t,\boldsymbol{\lambda})=\begin{pmatrix}-\boldsymbol{f}(t,\boldsymbol{\lambda})\\ \boldsymbol{b}(t,\boldsymbol{x},\boldsymbol{y})\\ \boldsymbol{\sigma}(t,\boldsymbol{x},\boldsymbol{y})\\ \boldsymbol{g}(t,\boldsymbol{x},\boldsymbol{y},\cdot)\end{pmatrix}$$

Assumption 2. We also assume that

$$\left\{\begin{array}{l} \langle A(t,\boldsymbol{\lambda})-A(t,\bar{\boldsymbol{\lambda}}),\boldsymbol{\lambda}-\bar{\boldsymbol{\lambda}}\rangle\le-\beta_1|\boldsymbol{x}-\bar{\boldsymbol{x}}|^2-\beta_2|\boldsymbol{y}-\bar{\boldsymbol{y}}|^2 \\ \langle\boldsymbol{\phi}(\boldsymbol{x})-\boldsymbol{\phi}(\bar{\boldsymbol{x}}),\boldsymbol{x}-\bar{\boldsymbol{x}}\rangle\ge\mu_1|\boldsymbol{x}-\bar{\boldsymbol{x}}|^2 \end{array}\right.$$

where $\boldsymbol{\lambda}=(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\boldsymbol{c}(\cdot)),\bar{\boldsymbol{\lambda}}=(\bar{\boldsymbol{x}},\bar{\boldsymbol{y}},\bar{\boldsymbol{z}},\bar{\boldsymbol{c}}(\cdot)),\beta_1,\beta_2,\mu_1$ are nonnegative constants with $\beta_1+\beta_2>0$ and $\beta_2+\mu_1>0$.

Theorem 1. We assume Assumptions 1 and 2 hold. Then, for all $t\in[0,T]$, there exists a unique quartet $(\boldsymbol{x}(t),\boldsymbol{y}(t),\boldsymbol{z}(t),\boldsymbol{c}(t,\cdot))\in\mathbf{L}^2_{\mathcal{F}}([0,T];\mathbf{R}^n)\times\mathbf{L}^2_{\mathcal{F}}([0,T];\mathbf{R}^n)\times\mathbf{L}^2_{\mathcal{F}}([0,T];\mathbf{R}^{n\times d})\times\mathbf{F}^2_p([0,T];\mathbf{R}^n)$ satisfying (8).

This theorem is a special case of FBSDEP (1) in [9].

2 Linear quadratic stochastic control problem with random jumps

In this section, we consider the linear quadratic stochastic control problem with random jumps (6) and (7). We prove that there exists a unique optimal control and give the explicit linear state feedback form. When all the coefficient matrices are deterministic, we can give the linear feedback regulator for the optimal control using the solution of one kind of generalized matrix-valued Riccati equation system. We first have the following result.

Theorem 2. There exists a unique optimal control for (6) and (7)

$$\begin{aligned}\boldsymbol{u}(t)\ =\ &-N^{-1}(\omega,t)[B^\tau(\omega,t)\boldsymbol{y}(t)+D^\tau(\omega,t)\boldsymbol{z}(t)+\\ &F^\tau(\omega,t)\boldsymbol{c}(t,\cdot)]\end{aligned}\tag{9}$$

Proof. It is easy to check that for given admissible control $\boldsymbol{v}(\cdot)$, (6) satisfies Assumptions 1 and 2. So, by Theorem 1, FBSDEP (6) has a unique solution $(\boldsymbol{x}(t),\boldsymbol{y}(t),\boldsymbol{z}(t),\boldsymbol{c}(t,\cdot))\in\mathbf{L}^2_{\mathcal{F}}([0,T];\mathbf{R}^n)\times\mathbf{L}^2_{\mathcal{F}}([0,T];\mathbf{R}^n)\times\mathbf{L}^2_{\mathcal{F}}([0,T];\mathbf{R}^{n\times d})\times\mathbf{F}^2_p([0,T];\mathbf{R}^n)$.

Existence. For any admissible control $\boldsymbol{v}(\cdot)$, we denote by $\boldsymbol{x}^v(t)$ the corresponding trajectory and by $(\boldsymbol{y}^v(t),\boldsymbol{z}^v(t))$ the corresponding solution of the BSDEP in the system (6). Then,

$$\begin{aligned} &J(\boldsymbol{v}(\cdot))-J(\boldsymbol{u}(\cdot))=\\ &\quad\frac{1}{2}\mathrm{E}[\int_0^T(\langle R(\omega,t)(\boldsymbol{x}^v(t)-\boldsymbol{x}(t)),\boldsymbol{x}^v(t)-\boldsymbol{x}(t)\rangle+\\ &\quad\langle N(\omega,t)(\boldsymbol{v}(t)-\boldsymbol{u}(t)),\boldsymbol{v}(t)-\boldsymbol{u}(t)\rangle+\\ &\quad\langle L(\omega,t)(\boldsymbol{y}^v(t)-\boldsymbol{y}(t)),\boldsymbol{y}^v(t)-\boldsymbol{y}(t)\rangle+\\ &\quad\langle 2R(\omega,t)\boldsymbol{x}(t),\boldsymbol{x}^v(t)-\boldsymbol{x}(t)\rangle+\langle 2N(\omega,t)\boldsymbol{u}(t),\boldsymbol{v}(t)-\\ &\quad\boldsymbol{u}(t)\rangle+\langle 2L(\omega,t)\boldsymbol{y}(t),\boldsymbol{y}^v(t)-\boldsymbol{y}(t)\rangle)\mathrm{d}t+\\ &\quad\langle Q(\omega)(\boldsymbol{x}^v(T)-\boldsymbol{x}(T)),\boldsymbol{x}^v(T)-\boldsymbol{x}(T)\rangle+\\ &\quad\langle 2Q(\omega)\boldsymbol{x}(T),\boldsymbol{x}^v(T)-\boldsymbol{x}(T)\rangle]\end{aligned}$$

Applying Ito's formula to $\langle\boldsymbol{x}^v(t)-\boldsymbol{x}(t),\boldsymbol{y}(t)\rangle$ and noting that $\boldsymbol{y}(T)=Q(\omega)\boldsymbol{x}(T)$, we have

$$\begin{aligned} &\mathrm{E}[\langle\boldsymbol{x}^v(T)-\boldsymbol{x}(T),\boldsymbol{y}(T)\rangle]=\\ &\mathrm{E}[\int_0^T(\langle-R(\omega,t)\boldsymbol{x}(t),\boldsymbol{x}^v(t)-\boldsymbol{x}(t)\rangle+\\ &\langle B^\tau(\omega,t)\boldsymbol{y}(t),\boldsymbol{v}(t)-\boldsymbol{u}(t)\rangle-\langle L(\omega,t)\boldsymbol{y}(t),\boldsymbol{y}^v(t)-\boldsymbol{y}(t)\rangle+\\ &\langle D^\tau(\omega,t)\boldsymbol{z}(t),\boldsymbol{v}(t)-\boldsymbol{u}(t)\rangle+\langle F^\tau(\omega,t)\boldsymbol{c}(t,\cdot),\boldsymbol{v}(t)-\boldsymbol{u}(t)\rangle)\mathrm{d}t]\end{aligned}$$

As $R(\omega,t)$, $L(\omega,t)$, and $Q(\omega)$ are nonnegative and $N(\omega,t)$ is positive, we have

$$\begin{aligned} &J(\boldsymbol{v}(\cdot))-J(\boldsymbol{u}(\cdot))\ge\mathrm{E}[\int_0^T(\langle B^\tau(\omega,t)\boldsymbol{y}(t)+\\ &\quad D^\tau(\omega,t)\boldsymbol{z}(t)+F^\tau(\omega,t)\boldsymbol{c}(t,\cdot),\boldsymbol{v}(t)-\boldsymbol{u}(t)\rangle+\\ &\quad\langle N(\omega,t),\boldsymbol{v}(t)-\boldsymbol{u}(t)\rangle)\mathrm{d}t=\mathrm{E}[\int_0^T(\langle B^\tau(\omega,t)\boldsymbol{y}(t)+\\ &\quad D^\tau(\omega,t)\boldsymbol{z}(t)+F^\tau(\omega,t)\boldsymbol{c}(t,\cdot),\boldsymbol{v}(t)-\boldsymbol{u}(t)\rangle+\\ &\quad\langle-N(\omega,t)N^{-1}(\omega,t)[B^\tau(\omega,t)\boldsymbol{y}(t)+D^\tau(\omega,t)\boldsymbol{z}(t)+\\ &\quad F^\tau(\omega,t)\boldsymbol{c}(t,\cdot)],\boldsymbol{v}(t)-\boldsymbol{u}(t)\rangle)\mathrm{d}t=0\end{aligned}$$

So, $\boldsymbol{u}(t)=-N^{-1}(\omega,t)[B^\tau(\omega,t)\boldsymbol{y}(t)+D^\tau(\omega,t)\boldsymbol{z}(t)+F^\tau(\omega,t)\boldsymbol{c}(t,\cdot)]$ is the optimal control.

Uniqueness. To prove the uniqueness of the optimal control, the method is classical. We assume $\boldsymbol{u}^1(\cdot)$ and $\boldsymbol{u}^2(\cdot)$ are both optimal controls, and the corresponding trajectories are $\boldsymbol{x}^1(\cdot)$ and $\boldsymbol{x}^2(\cdot)$, respectively. It is easy to know that the trajectories corresponding to $\dfrac{\boldsymbol{u}^1(\cdot)+\boldsymbol{u}^2(\cdot)}{2}$ and $\dfrac{\boldsymbol{u}^1(\cdot)-\boldsymbol{u}^2(\cdot)}{2}$ are $\dfrac{\boldsymbol{x}^1(\cdot)+\boldsymbol{x}^2(\cdot)}{2}$ and $\dfrac{\boldsymbol{x}^1(\cdot)-\boldsymbol{x}^2(\cdot)}{2}$, respectively.

Since $R(\omega,t)$, $L(\omega,t)$, and $Q(\omega)$ are nonnegative, $N(\omega,t)$ is positive, we have $J(\boldsymbol{u}^1(\cdot))=J(\boldsymbol{u}^2(\cdot))=\alpha\geq 0$ and

$$\begin{aligned}
2\alpha = {} & J(\boldsymbol{u}^1(\cdot))+J(\boldsymbol{u}^2(\cdot)) = 2J\left(\frac{\boldsymbol{u}^1(\cdot)+\boldsymbol{u}^2(\cdot)}{2}\right)+\\
& \mathrm{E}\Bigg[\int_0^T\Bigg(\left\langle R(\omega,t)\frac{\boldsymbol{x}^1(t)-\boldsymbol{x}^2(t)}{2},\frac{\boldsymbol{x}^1(t)-\boldsymbol{x}^2(t)}{2}\right\rangle+\\
& \left\langle N(\omega,t)\frac{\boldsymbol{u}^1(t)-\boldsymbol{u}^2(t)}{2},\frac{\boldsymbol{u}^1(t)-\boldsymbol{u}^2(t)}{2}\right\rangle+\\
& \left\langle L(\omega,t)\frac{\boldsymbol{y}^1(t)-\boldsymbol{y}^2(t)}{2},\frac{\boldsymbol{y}^1(t)-\boldsymbol{y}^2(t)}{2}\right\rangle\Bigg)\mathrm{d}t+\\
& \left\langle Q(\omega,T)\frac{\boldsymbol{x}^1(T)-\boldsymbol{x}^2(T)}{2},\frac{\boldsymbol{x}^1(T)-\boldsymbol{x}^2(T)}{2}\right\rangle\Bigg]\geq\\
& 2J\left(\frac{\boldsymbol{u}^1(\cdot)+\boldsymbol{u}^2(\cdot)}{2}\right)+\\
& \mathrm{E}\int_0^T\left\langle N(\omega,t)\frac{\boldsymbol{u}^1(t)-\boldsymbol{u}^2(t)}{2},\frac{\boldsymbol{u}^1(t)-\boldsymbol{u}^2(t)}{2}\right\rangle\mathrm{d}t\geq\\
& 2\alpha+\frac{\delta}{4}\mathrm{E}\int_0^T|\boldsymbol{u}^1(t)-\boldsymbol{u}^2(t)|^2\mathrm{d}t
\end{aligned}$$

Here, $\delta>0$. So $\mathrm{E}\int_0^T|\boldsymbol{u}^1(t)-\boldsymbol{u}^2(t)|^2\mathrm{d}t\leq 0$

Hence, $\boldsymbol{u}^1(\cdot)=\boldsymbol{u}^2(\cdot)$, in the sense of $\mathbf{L}^2_{\mathcal{F}}([0,T])$. □

Now, we assume that the matrices $A(t)$, $B(t)$, $C(t)$, $D(t)$, $E(t)$, $F(t)$, $R(t)$, $N(t)$, $L(t)$, and Q are all deterministic. We introduce the following generalized $n\times n$ matrix-valued Riccati equation system of $(K(t),M(t),Y(t,\cdot)),t\in[0,T]$

$$\left\{\begin{aligned}
-\dot{K}(t) = {} & A^\tau(t)K(t)+K(t)A(t)+C^\tau(t)M(t)+\\
& E^\tau(t)Y(t,\cdot)-K(t)[L^\tau(t)+\\
& B(t)N^{-1}(t)B^\tau(t)]K(t)-\\
& K(t)B(t)N^{-1}(t)D^\tau(t)M(t)-\\
& K(t)B(t)N^{-1}(t)F^\tau(t)Y(t,\cdot)+R(t)\\
M(t) = {} & K(t)C(t)-K(t)D(t)N^{-1}(t)B^\tau(t)K(t)-\\
& K(t)D(t)N^{-1}(t)D^\tau(t)M(t)-\\
& K(t)D(t)N^{-1}(t)F^\tau(t)Y(t,\cdot)\\
Y(t,\cdot) = {} & K(t)E(t)-K(t)F(t)N^{-1}(t)B^\tau(t)K(t)-\\
& K(t)F(t)N^{-1}(t)D^\tau(t)M(t)-\\
& K(t)F(t)N^{-1}(t)F^\tau(t)Y(t,\cdot)\\
K(T) = {} & Q
\end{aligned}\right. \tag{10}$$

This is a kind of generalized $n\times n$ matrix-valued Riccati equation system formed by a matrix-valued ordinary differential equation and two algebraic equations. This equation system is more complicated than the one in [9], where there is only one algebraic equation. Note that this equation system is similar to the one in [8] in form, it is more complicated because our stochastic control system (9) is a coupled FBSDEP other than an SDEP in [8]. So, the solvability of (10) is more difficult. We will discuss its solvability in the next section. Now, we have Theorem 3.

Theorem 3. Suppose for all $t\in[0,T]$, there exist matrices $(K(t),M(t),Y(t,\cdot))$ satisfying the generalized matrix-valued Riccati equation system (10). Then, the optimal linear feedback regulator for our linear quadratic optimal control problem (6) and (7) is

$$\boldsymbol{u}(t)=-N^{-1}(t)[B^\tau(t)K(t)+D^\tau(t)M(t)+F^\tau(t)Y(t,\cdot)]\boldsymbol{x}(t) \tag{11}$$

and the optimal value function is

$$J(\boldsymbol{u}(\cdot))=\frac{1}{2}\langle K(0)\boldsymbol{x},\boldsymbol{x}\rangle \tag{12}$$

Proof. It is easy to check that if $(K(t),M(t),Y(t,\cdot))$ is the solution of (10), then the solution $(\boldsymbol{x}(t),\boldsymbol{y}(t),\boldsymbol{z}(t),\boldsymbol{c}(t,\cdot)$ of FBSDEP (6) satisfies

$$\boldsymbol{y}(t)=K(t)\boldsymbol{x}(t),\quad \boldsymbol{z}(t)=M(t)\boldsymbol{x}(t),\quad \boldsymbol{c}(t,\cdot)=Y(t,\cdot)\boldsymbol{x}(t)$$

So, the optimal control $\boldsymbol{u}(\cdot)$ satisfies the equation (11). Applying Ito′s formula to $\langle\boldsymbol{x}(t),\boldsymbol{y}(t)\rangle$, we can easily get (12). □

We discuss a special case: $L(\omega,t)\equiv 0$. So, the state equation (6) is reduced to

$$\left\{\begin{aligned}
\mathrm{d}\boldsymbol{x}(t) = {} & [A(\omega,t)\boldsymbol{x}(t)+B(\omega,t)\boldsymbol{v}(t)]\mathrm{d}t+\\
& [C(\omega,t)\boldsymbol{x}(t)+D(\omega,t)\boldsymbol{v}(t)]\mathrm{d}\boldsymbol{B}(t)+\\
& \textstyle\int_{\mathbf{E}}[E(\omega,t)\boldsymbol{x}(t-)+F(\omega,t)\boldsymbol{v}(t)]\tilde{N}(\mathrm{d}e\mathrm{d}t)\\
\boldsymbol{x}(0) = {} & \boldsymbol{x}
\end{aligned}\right. \tag{13}$$

And the cost functional (7) is reduced to

$$\begin{aligned}
J(\boldsymbol{v}(\cdot)) = {} & \mathrm{E}\Bigg[\frac{1}{2}\int_0^T[\langle R(\omega,t)\boldsymbol{x}(t),\boldsymbol{x}(t)\rangle+\\
& \langle N(\omega,t)\boldsymbol{v}(t),\boldsymbol{v}(t)\rangle]\mathrm{d}t+\langle Q(\omega)\boldsymbol{x}(T),\boldsymbol{x}(T)\rangle\Bigg]
\end{aligned} \tag{14}$$

From Theorem 3, we can easily get Corollaries 1 and 2.

Corollary 1. For the linear quadratic stochastic control problem with random jumps (13) and (14), there exists a unique optimal control

$$\begin{aligned}
\boldsymbol{u}(t) = {} & -N^{-1}(\omega,t)[B^\tau(\omega,t)\boldsymbol{y}(t)+\\
& D^\tau(\omega,t)\boldsymbol{z}(t)+F^\tau(\omega,t)\boldsymbol{c}(t,\cdot)]
\end{aligned} \tag{15}$$

where $(\boldsymbol{y}(t),\boldsymbol{z}(t),\boldsymbol{c}(t,\cdot))$ is the solution of BSDEP:

$$\left\{\begin{aligned}
-\mathrm{d}\boldsymbol{y}(t) = {} & [A^\tau(\omega,t)\boldsymbol{y}(t)+C^\tau(\omega,t)\boldsymbol{z}(t)+\\
& E^\tau(\omega,t)\boldsymbol{c}(t,\cdot)+R(\omega,t)\boldsymbol{x}(t)]\mathrm{d}t-\\
& \boldsymbol{z}(t)\mathrm{d}B(t)-\textstyle\int_{\mathbf{E}}\boldsymbol{c}(t-,e)\tilde{N}(\mathrm{d}e\mathrm{d}t)\\
\boldsymbol{y}(T) = {} & Q(\omega)\boldsymbol{x}(T)
\end{aligned}\right. \tag{16}$$

Similarly, if we assume that the matrices $A(t)$, $B(t)$, $C(t)$, $D(t)$, $E(t)$, $F(t)$, $R(t)$, $N(t)$, and Q are all deterministic, then the generalized $n\times n$ matrix-valued Riccati equation system (10) of $(K(t),M(t),Y(t,\cdot)),t\in[0,T]$ is reduced to

$$\left\{\begin{aligned}
-\dot{K}(t) = {} & K(t)A(t)+A^\tau(t)K(t)+C^\tau(t)M(t)+\\
& E^\tau(t)Y(t)-K(t)B(t)N^{-1}(t)B^\tau(t)K(t)-\\
& K(t)B(t)N^{-1}(t)D^\tau(t)M(t)-\\
& K(t)B(t)N^{-1}(t)F^\tau(t)Y(t,\cdot)+R(t)\\
M(t) = {} & K(t)C(t)-K(t)D(t)N^{-1}(t)B^\tau(t)K(t)-\\
& K(t)D(t)N^{-1}(t)D^\tau(t)M(t)-\\
& K(t)D(t)N^{-1}(t)F^\tau(t)Y(t,\cdot)\\
Y(t,\cdot) = {} & K(t)E(t)-K(t)F(t)N^{-1}(t)B^\tau(t)K(t)-\\
& K(t)F(t)N^{-1}(t)D^\tau(t)M(t)-\\
& K(t)F(t)N^{-1}(t)F^\tau(t)Y(t,\cdot)\\
K(T) = {} & Q
\end{aligned}\right. \tag{17}$$

Corollary 2. Suppose for all $t \in [0,T]$, there exist matrices $(K(t), M(t), Y(t,\cdot))$ satisfying the generalized matrix-valued Riccati equation system (17). Then, the optimal linear feedback regulator for our linear quadratic optimal control problem (13) and (14) is

$$\begin{aligned}\boldsymbol{u}(t) = & -N^{-1}(t)[B^\tau(t)K(t) + D^\tau(t)M(t)+\\ & F^\tau(t)Y(t,\cdot)]\boldsymbol{x}(t)\end{aligned} \tag{18}$$

and the optimal value function is

$$J(\boldsymbol{u}(\cdot)) = \frac{1}{2}\langle K(0)\boldsymbol{x}, \boldsymbol{x}\rangle \tag{19}$$

In this case, using the above optimal control as the linear feedback, we can also get the optimal state trajectory $\boldsymbol{x}(\cdot)$ from (13).

3 Solvability of the generalized Riccati equation

In Section 2, we obtained the optimal linear feedback regulator for the linear quadratic stochastic control problem (6) and (7) by the solution of the generalized matrix-valued Riccati equation (10). However, (10) is so complicated that we cannot prove the existence and uniqueness of the general case at this moment. In this section, we will discuss the solvability for a special case of (10) when $D(t) \equiv 0$ using technique introduced in [3] (In fact, similar technique can give the solvability of (10) when $F(t) \equiv 0$). Now, (10) is reduced to

$$\left\{\begin{aligned}-\dot{K}(t) = & A^\tau(t)K(t) + K(t)A(t) + C^\tau(t)M(t)+\\ & E^\tau(t)Y(t) - K(t)[L^\tau(t)+\\ & B(t)N^{-1}(t)B^\tau(t)]K(t)-\\ & K(t)B(t)N^{-1}(t)F^\tau(t)Y(t,\cdot) + R(t)\\ M(t) = & K(t)C(t)\\ Y(t,\cdot) = & K(t)E(t) - K(t)F(t)N^{-1}(t)B^\tau(t)K(t)-\\ & K(t)F(t)N^{-1}(t)F^\tau(t)Y(t,\cdot)\\ K(T) = & Q\end{aligned}\right. \tag{20}$$

Now, we can study the following equation

$$\left\{\begin{aligned}-\dot{K}(t) = & A^\tau(t)K(t) + K(t)A(t)+\\ & E^\tau(t)[I_n + K(t)F(t)N^{-1}(t)F^\tau(t)]^{-1}\\ & [K(t)E(t) - K(t)F(t)N^{-1}(t)B^\tau(t)K(t)]-\\ & K(t)[L^\tau(t) + B(t)N^{-1}(t)B^\tau(t)]K(t)+\\ & C^\tau(t)K(t)C(t) + R(t)-\\ & K(t)B(t)N^{-1}(t)F^\tau(t)[I_n+\\ & K(t)F(t)N^{-1}(t)F^\tau(t)]^{-1}K(t)E(t)+\\ & K(t)B(t)N^{-1}(t)F^\tau(t)[I_n+\\ & K(t)F(t)N^{-1}(t)F^\tau(t)]^{-1}\\ & K(t)F(t)N^{-1}(t)B^\tau(t)K(t)\\ K(T) = & Q,\ I_n + K(t)F(t)N^{-1}(t)F^\tau(t) > 0\end{aligned}\right. \tag{21}$$

If we can get the solution $K(t)$ of (21), then we can let

$$\begin{aligned}M(t) = & K(t)C(t)\\ Y(t,\cdot) = & [I_n + K(t)F(t)N^{-1}(t)F^\tau(t)]^{-1}[K(t)E(t)-\\ & K(t)F(t)N^{-1}(t)B^\tau(t)K(t)]\end{aligned} \tag{22}$$

to get the solution of (20).

At first we have the following uniqueness result.

Theorem 4. Riccati equation (21) admits at most one solution $K(\cdot) \in C([0,T]; \mathbf{S}_+^n)$, here $C([0,T]; \mathbf{S}_+^n)$ denotes the Banach space of $\mathbf{S}_+^n$-valued continuous functions on $[0,T]$ and $\mathbf{S}_+^n$ denotes the space of all $n \times n$ nonnegative symmetric matrices.

Proof. That $K(\cdot) \in C([0,T]; \mathbf{S}_+^n)$ is clear from the conventional Riccati equation theory. Now, suppose $\tilde{K}(t)$ is another solution of (21). Set $\hat{K}(t) = K(t) - \tilde{K}(t)$. Then, $\hat{K}(t)$ satisfies

$$\left\{\begin{aligned}-\dot{\hat{K}}(t) = & A^\tau(t)\hat{K}(t) + \hat{K}(t)A(t) + C^\tau(t)\hat{K}(t)C(t)-\\ & \hat{K}(t)[L^\tau(t) + B(t)N^{-1}(t)B^\tau(t)]K(t)-\\ & \tilde{K}(t)[L^\tau(t) + B(t)N^{-1}(t)B^\tau(t)]\hat{K}(t)+\\ & I + II + III + IV\\ \hat{K}(T) = & 0,\quad I_n + K(t)F(t)N^{-1}(t)F^\tau(t) > 0\end{aligned}\right.$$

where $I_n + K(t)F(t)N^{-1}(t)F^\tau(t) > 0$, $I_n + \tilde{K}(t) \times F(t)N^{-1}(t)F^\tau(t) > 0$, and

$$\begin{aligned}I = & \hat{K}(t)B(t)N^{-1}(t)F^\tau(t)[I_n + K(t)F(t)N^{-1}(t)\times\\ & F^\tau(t)]^{-1}K(t)F(t)N^{-1}(t)B^\tau(t)K(t) + \tilde{K}(t)\times\\ & B(t)N^{-1}(t)F^\tau(t)[I_n + K(t)F(t)N^{-1}(t)\times\\ & F^\tau(t)]^{-1}\hat{K}(t)F(t)N^{-1}(t)B^\tau(t)K(t) + \tilde{K}(t)\times\\ & B(t)N^{-1}(t)F^\tau(t)[I_n + K(t)F(t)N^{-1}(t)\times\\ & F^\tau(t)]^{-1}\tilde{K}(t)F(t)N^{-1}(t)B^\tau(t)\hat{K}(t) - \tilde{K}(t)\times\\ & B(t)N^{-1}(t)F^\tau(t)[I_n + K(t)F(t)N^{-1}(t)\times\\ & F^\tau(t)]^{-1}\hat{K}(t)F(t)N^{-1}(t)F^\tau(t)[I_n + \tilde{K}(t)F(t)\times\\ & N^{-1}(t)F^\tau(t)]^{-1}\tilde{K}(t)F(t)N^{-1}(t)B^\tau(t)\tilde{K}(t)\end{aligned}$$

$$\begin{aligned}II = & E^\tau(t)[I_n + K(t)F(t)N^{-1}(t)F^\tau(t)]^{-1}\hat{K}(t)E(t)-\\ & E^\tau(t)[I_n + K(t)F(t)N^{-1}(t)F^\tau(t)]^{-1}\hat{K}(t)\times\\ & F(t)N^{-1}(t)F^\tau(t)[I_n + \tilde{K}(t)F(t)N^{-1}(t)\times\\ & F^\tau(t)]^{-1}\tilde{K}(t)E(t)\end{aligned}$$

$$\begin{aligned}III = & -\hat{K}(t)B(t)N^{-1}(t)F^\tau(t)[I_n + K(t)F(t)\times\\ & N^{-1}(t)F^\tau(t)]^{-1}K(t)E(t) - \tilde{K}(t)B(t)N^{-1}(t)\times\\ & F^\tau(t)[I_n + K(t)F(t)N^{-1}(t)F^\tau(t)]^{-1}\hat{K}(t)E(t)+\\ & \tilde{K}(t)B(t)N^{-1}(t)F^\tau(t)[I_n + K(t)F(t)\times\\ & N^{-1}(t)F^\tau(t)]^{-1}\hat{K}(t)F(t)N^{-1}(t)F^\tau(t)\times\\ & [I_n + \tilde{K}(t)F(t)N^{-1}(t)F^\tau(t)]^{-1}\tilde{K}(t)E(t)\end{aligned}$$

$$\begin{aligned}IV = & -E^\tau(t)[I_n + K(t)F(t)N^{-1}(t)F^\tau(t)]^{-1}\hat{K}(t)\times\\ & F(t)N^{-1}(t)B^\tau(t)K(t) - E^\tau(t)[I_n + K(t)\times\\ & F(t)N^{-1}(t)F^\tau(t)]^{-1}\tilde{K}(t)F(t)N^{-1}(t)B^\tau(t)\times\\ & \hat{K}(t) + E^\tau(t)[I_n + K(t)F(t)N^{-1}(t)F^\tau(t)]^{-1}\times\\ & \hat{K}(t)F(t)N^{-1}(t)F^\tau(t)[I_n + \tilde{K}(t)F(t)N^{-1}(t)\times\\ & F^\tau(t)]^{-1}\tilde{K}(t)F(t)N^{-1}(t)B^\tau(t)\tilde{K}(t) - \hat{K}(t)\times\\ & B(t)N^{-1}(t)F^\tau(t)[I_n + K(t)F(t)N^{-1}(t)\times\\ & F^\tau(t)]^{-1}K(t)E(t) - \tilde{K}(t)B(t)N^{-1}(t)F^\tau(t)\times\\ & [I_n + K(t)F(t)N^{-1}(t)F^\tau(t)]^{-1}\hat{K}(t)E(t)+\\ & \tilde{K}(t)B(t)N^{-1}(t)F^\tau(t)[I_n + K(t)F(t)\times\\ & N^{-1}(t)F^\tau(t)]^{-1}\hat{K}(t)F(t)N^{-1}(t)F^\tau(t)\times\\ & [I_n + \tilde{K}(t)F(t)N^{-1}(t)F^\tau(t)]^{-1}\tilde{K}(t)E(t)\end{aligned}$$

Since $|[I_n + K(t)F(t)N^{-1}(t)F^\tau(t)]^{-1}|, |[I_n + \tilde{K}(t)F(t)\times N^{-1}(t)F^\tau(t)]^{-1}|$ are uniformly bounded due to their continuity, we can apply Grownwall's inequality to get $\hat{K}(t) \equiv 0, t \in [0,T]$. This proves the uniqueness. □

Now, let us discuss the existence of the solution of (21) step by step.

If we let $\Phi = \Lambda(K) = [I_n + KFN^{-1}F^\tau]^{-1}K$, then for all $\Phi(\cdot) \in C([0,T]; \mathbf{S}_+^n)$, when

$$[L^\tau + BN^{-1}B^\tau - BN^{-1}F^\tau\Phi FN^{-1}B^\tau] \in C([0,T]; \mathbf{S}_+^n) \tag{23}$$

the following conventional Riccati equation

$$\begin{cases} -\dot{K}(t) = [A(t) - B(t)N^{-1}(t)F^\tau(t)\Phi(t)E(t)]^\tau K(t)+ \\ \quad K(t)[A(t) - B(t)N^{-1}(t)F^\tau(t)\Phi(t)E(t)]+ \\ \quad C^\tau(t)K(t)C(t) + E^\tau(t)\Phi(t)E(t) + R(t)- \\ \quad K(t)[L^\tau(t) + B(t)N^{-1}(t)B^\tau(t)- \\ \quad B(t)N^{-1}(t)F^\tau(t)\Phi(t)F(t)N^{-1}(t)B^\tau(t)]K(t) \\ K(T) = Q,\ I_n + K(t)F(t)N^{-1}(t)F^\tau(t) > 0 \end{cases} \tag{24}$$

admits a unique solution $K(\cdot) \in C([0,T]; \mathbf{S}_+^n)$.

We denote by $\mathbf{S}_s^n$ the subspace of $\mathbf{S}_+^n$ formed by the symmetric matrices satisfying (23). Obviously $\Phi \equiv 0 \in \mathbf{S}_s^n$, so this definition is reasonable. Thus, we can define a mapping $\Psi : C([0,T]; \mathbf{S}_s^n) \rightarrow C([0,T]; \mathbf{S}_+^n)$ as $K = \Psi(\Phi)$ and have the following lemmas.

Lemma 1. The operator $\Lambda(K)$ is monotonously increasing when $K > 0$.

Proof. When $K > 0$, we notice that

$$\begin{aligned} \Lambda(K) = & [I_n + KFN^{-1}F^\tau]^{-1}K = \\ & [K^{-1}(I_n + KFN^{-1}F^\tau)]^{-1} = \\ & [K^{-1} + FN^{-1}F^\tau]^{-1} \end{aligned}$$

So, if $K_1 \geq K_2$, then $\Lambda(K_1) \geq \Lambda(K_2)$. □

Lemma 2. The operator Ψ is monotonously increasing and continuous.

Proof. Let $K = \Psi(\Phi)$, $\bar{K} = \Psi(\bar{\Phi})$, and $\hat{K} = K - \bar{K}$. We rewrite (24) as

$$\begin{cases} -\dot{K}(t) = A^\tau(t)K(t) + K(t)A(t)- \\ \quad K(t)[L^\tau(t) + B(t)N^{-1}(t)B^\tau(t)]K(t)+ \\ \quad R(t) + C^\tau(t)K(t)C(t) + [E(t)- \\ \quad F(t)N^{-1}(t)B^\tau(t)K(t)]^\tau\Phi(t)\cdot \\ \quad [E(t) - F(t)N^{-1}(t)B^\tau(t)K(t)] \\ K(T) = Q \end{cases} \tag{25}$$

From Lemma 8.2 in [2] and Lemma 1, we know that if $\Phi \geq \bar{\Phi}$, then $K \geq \bar{K}$. This proves the monotonicity of Ψ. On the other hand, applying Grownwall′s inequality, we easily see that if $\Phi \rightarrow \bar{\Phi}$, then $K - \bar{K} = \hat{K} \rightarrow 0$. This yields the continuity of Ψ. □

Looking back at equation (24), it is easy to know the following theorem.

Theorem 5. If there exists $\Phi(\cdot) \in C([0,T]; \mathbf{S}_s^n)$ such that

$$\Phi = [I_n + \Psi(\Phi)FN^{-1}F^\tau]^{-1}\Psi(\Phi) \tag{26}$$

then Riccati equation (23) admits a unique solution.

The following task is to find the suitable $\Phi \in C([0,T]; \mathbf{S}_s^n)$ satisfying (26). We need the following result.

Lemma 3. If there exist $\Phi^+, \Phi^- \in C(0,T; \mathbf{S}_s^n)$ which satisfy

$$\begin{aligned} \Phi^+ \geq & [I_n + \Psi(\Phi^+)FN^{-1}F^\tau]^{-1}\Psi(\Phi^+) \geq \\ & [I_n + \Psi(\Phi^-)FN^{-1}F^\tau]^{-1}\Psi(\Phi^-) \geq \Phi^- \end{aligned} \tag{27}$$

then (21) admits a solution $K(\cdot) \in C([0,T]; \mathbf{S}_+^n)$.

Proof. Let Φ^+, Φ^- be given and satisfy (27). We define the sequence $\Phi_i^+, \Phi_i^-, K_i^+, K_i^-$

$$\Phi_0^+ = \Phi^+ \in \mathbf{S}_s^n, \quad K_0^+ = \Psi(\Phi_0^+)$$

$$\Phi_0^- = \Phi^- \in \mathbf{S}_s^n, \quad K_0^- = \Psi(\Phi_0^-)$$

$$\Phi_{i+1}^+ = [I_n + K_i^+FN^{-1}F^\tau]^{-1}K_i^+$$

$$\Phi_{i+1}^- = [I_n + K_i^-FN^{-1}F^\tau]^{-1}K_i^-$$

$$K_{i+1}^+ = \Psi(\Phi_{i+1}^+), \quad K_{i+1}^- = \Psi(\Phi_{i+1}^-), \quad i = 0, 1, 2 \cdots$$

From (27), we have

$$\begin{aligned} \Phi_0^+ \geq & [I_n + \Psi(\Phi_0^+)FN^{-1}F^\tau]^{-1}\Psi(\Phi_0^+) = \\ & [I_n + K_0^+FN^{-1}F^\tau]^{-1}K_0^+ = \\ & \Phi_1^+ \geq \Phi_1^- \geq \Phi_0^- \geq 0 \end{aligned}$$

By Lemma 2, we have $K_0^+ \geq K_1^+ \geq K_1^- \geq K_0^-$. By induction, we have

$$K_0^+ \geq K_i^+ \geq K_{i+1}^+ \geq K_{i+1}^- \geq K_i^- \geq K_0^- \geq 0$$

$$\Phi_0^+ \geq \Phi_i^+ \geq \Phi_{i+1}^+ \geq \Phi_{i+1}^- \geq \Phi_i^- \geq \Phi_0^- \geq 0$$

and $\Phi_i^+, \Phi_i^- \in \mathbf{S}_s^n$. So, we have

$$\lim_{i\rightarrow\infty} \Phi_i^+ = \Phi^+ \in \mathbf{S}_s^n, \quad \lim_{i\rightarrow\infty} K_i^+ = K^+ \in \mathbf{S}_+^n$$

By Lemma 2 again, we have

$$K^+ = \lim_{i\rightarrow\infty} K_i^+ = \lim_{i\rightarrow\infty} \Psi(\Phi_i^+) = \Psi(\lim_{i\rightarrow\infty} \Phi_i^+) = \Psi(\Phi^+)$$

So, K^+ is a solution of Riccati equation (24) corresponding to $\Phi = \Phi^+$. Then, $\Phi^+ = [I_n + K^+FN^{-1}F^\tau]^{-1}K^+$. By Theorem 5, K^+ is a solution of (21).

By the same step, we also can get $\lim_{i\rightarrow\infty} \Phi_i^- = \Phi^-$, $\lim_{i\rightarrow\infty} K_i^- = K^-$. So, K^- is a solution of (21). By the uniqueness result Theorem 4, we have $K^+ = K^-$. □

We only need to find Φ^+ and Φ^- satisfying (27). The existence of Φ^- is obvious. We can let $\Phi^- = 0$, which by the conventional Riccati equation theory, satisfies (27).

Now, we need a sufficient condition below to find Φ^+ satisfying (27) and ensure the existence of (21).

Assumption 3. There exists $\bar{\Phi}(\cdot) \in \mathbf{S}_s^n$, such that $F^\tau(t)\bar{\Phi}(t)F(t) = N(t)$ and $[I_n + \bar{K}(t)F(t)N^{-1}(t)F^\tau(t)]^{-1} \times \bar{K}(t) \leq \bar{\Phi}(t)$, here $\bar{K}(\cdot)$ is the unique solution of the equation

$$\begin{cases} -\dot{\bar{K}}(t) = [A(t) - B(t)N^{-1}(t)F^\tau(t)\bar{\Phi}(t)E(t)]^\tau \bar{K}(t)+ \\ \quad \bar{K}(t)[A(t) - B(t)N^{-1}(t)\times \\ \quad F^\tau(t)\bar{\Phi}(t)E(t)] - \bar{K}(t)L^\tau(t)\bar{K}(t)+ \\ \quad C^\tau(t)\bar{K}(t)C(t) + E^\tau(t)\bar{\Phi}(t)E(t) + R(t) \\ \bar{K}(T) = Q \end{cases} \tag{28}$$

Equation (28) is different from (13) in [8], where there is a linear ordinary equation and (28) is a Riccati equation. From conventional Riccati equation theory as long as we find suitable $\Phi(\cdot)$ satisfying the condition in Assumption 3, then (28) has a unique solution $\bar{K}(\cdot) \in C([0,T]; \mathbf{S}_+^n)$. Moreover, it is easy to know that when $k = n$ and the matrix $F(\cdot)$ is invertible, the Assumption 3 is satisfied. Hence, we have the main result of this section.

Theorem 6. Let Assumption 3 hold and $D \equiv 0$. Then the generalized Riccati equation (10) has a unique solution $(K(\cdot), M(\cdot), Y(\cdot,\cdot)) \in C^1([0,T]; \mathbf{S}_+^n) \times L^\infty([0,T]; \mathbf{R}^{n\times n}) \times L^\infty([0,T]; \mathbf{R}^{n\times n})$.

At last, we give a simple example of the generalized Riccati equation, which has a unique solution.

Example 1. We assume the dimensions of the state and control in the stochastic control system (6) are the same, i.e., $k = n$, and assume $D \equiv 0, F \equiv I_n$. Now, we can let $\bar{\Phi}(\cdot) = N(\cdot)$ and then check Assumption 3.

In fact, we have $\bar{\Phi}(t) = N(t) \geq 0$, so $\bar{\Phi}(t)F(t) + \bar{K}(t)F(t)N^{-1}(t)F^\tau(t)\bar{\Phi}(t)F(t) \geq \bar{K}(t)F(t)$, where $\bar{K}(t)$ is the solution of the following equation

$$\begin{cases} -\dot{\bar{K}}(t) = [A(t) - B(t)E(t)]^\tau \bar{K}(t) + K(t)[A(t) - \\ \qquad B(t)E(t)] - \bar{K}(t)L^\tau(t)\bar{K}(t) + C^\tau(t)\bar{K}(t)C(t) + \\ \qquad E^\tau(t)N(t)E(t) + R(t) \\ \bar{K}(T) = Q \end{cases}$$

and $\bar{\Phi}(t) + \bar{K}(t)F(t)N^{-1}(t)F^\tau(t)\bar{\Phi}(t) \geq \bar{K}(t)$. Then,

$$[I_n + \bar{K}(t)F(t)N^{-1}(t)F^\tau(t)]^{-1}\bar{K}(t) \leq \bar{\Phi}(t)$$

From Theorem 6, when $k = n, D \equiv 0, F \equiv I_n$ the Riccati equation (10) has a unique solution. For this case, we can get an explicit form of the linear optimal feedback value function for the linear quadratic optimal control problem (6) and (7), and also can get the optimal state trajectory from (6).

4 Conclusion

In this paper, we study one kind of fully coupled linear quadratic stochastic control problem with random jumps. The explicit form of the optimal control is obtained. The optimal control can be proved to be unique. One kind of generalized Riccati equation system is introduced and its solvability is discussed. The linear feedback regulator for the optimal control problem with random jumps is given by the solution of the generalized Riccati equation system.

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