Research Article

Computation of a Reference Model for Robust Fault Detection and Isolation Residual Generation

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This paper considers matrix inequality procedures to address the robust fault detection and isolation (FDI) problem for linear time-invariant systems subject to disturbances, faults, and polytopic or norm-bounded uncertainties. We propose a design procedure for an FDI filter that aims to minimize a weighted combination of the sensitivity of the residual signal to disturbances and modeling errors, and the deviation of the faults to residual dynamics from a fault to residual reference model, using the $\mathcal{H}_\infty$-norm as a measure. A key step in our procedure is the design of an optimal fault reference model. We show that the optimal design requires the solution of a quadratic matrix inequality (QMI) optimization problem. Since the solution of the optimal problem is intractable, we propose a linearization technique to derive a numerically tractable suboptimal design procedure that requires the solution of a linear matrix inequality (LMI) optimization. A jet engine example is employed to demonstrate the effectiveness of the proposed approach.

1. INTRODUCTION

In the past decade, great attention has been devoted to the design of model-based fault detection systems and their robustness [1, 2]. With the rapid development of robust control theory and $\mathcal{H}_\infty$ optimization control techniques, more and more methods have been presented to solve the robust FDI problem. The $\mathcal{H}_\infty$-filter is designed such that the $\mathcal{H}_\infty$-norm of the estimation error is minimized (see [3–5] and the references therein). Some of the approaches used for this problem include frequency domain approaches [6], left and right eigenvector assignment [7], structure parity equation [8], and an unknown input observer with disturbances decoupled in the state estimation error [9]. Recently developed LMI approaches offer numerically attractive techniques [10–12].

A reference residual model is an ideal solution for robust FDI under the assumption that there are no disturbances or model uncertainty. The idea is to design a filter for the uncertain system that approximates the solution given by the reference model [13]. In [14], a new performance index is proposed using such a reference residual model. Frisk and Nielsen [15] give an algorithm to design a reference model and a robust FDI filter that fits into the framework of standard robust $\mathcal{H}_\infty$-filtering relying on established and efficient methods. However, their framework consists in solving two optimization problems successively, which results in a suboptimal solution.

In this paper, we propose a performance index that captures the requirements of fault isolation and disturbance rejection as well as the design of the optimal reference model. The fault isolation performance is measured by the size of the deviation of the fault to residual dynamics from the reference dynamics model, while the disturbance rejection performance is measured by the size of the input to residual and disturbance to residual dynamics. In all cases, the $\mathcal{H}_\infty$ norm is used as a measure. The design of the optimal reference model is incorporated in the robust FDI framework. We consider systems subject to norm-bounded or polytopic uncertainties. For systems described by polytopic and unstructured norm-bounded uncertainties, we derive an optimal FDI filter obtained as the solution of a QMI optimization. For systems described by structured uncertainties, we derive a suboptimal QMI-based solution. Since the solution

of a QMI optimization is, in general, intractable, we propose a linearization technique to derive a suboptimal design procedure that requires the solution of a numerically tractable LMI optimization. This note extends our work in [16] by proposing algorithms for the design of suboptimal reference dynamics.

The structure of the work is as follows. After defining the notation, we review filter-based FDI techniques for residual signal generation and give the problem formulation in Section 2. Section 3 presents a matrix inequality formulation for the FDI problem, and gives the solution and the design of the optimal reference model for both norm-bounded and polytopic uncertainties in a form of QMI's. Section 4 gives a suboptimal solution in both cases through the use of an algorithm that necessitates solving LMI's. Finally, a numerical example is presented in Section 5, and Section 6 summarizes our results.

The notation we use is fairly standard. The set of real $n \times m$ matrices is denoted by $\mathcal{R}^{n \times m}$. For $A \in \mathcal{R}^{n \times m}$, we use the notation A^T to denote transpose. For $A = A^T \in \mathcal{R}^{n \times n}$, $A \succ 0$ ($A \prec 0$) denotes that A is positive (negative) definite, that is, all the eigenvalues of A are greater (less) than zero. The $n \times n$ identity matrix is denoted as I_n and the $n \times m$ null matrix is denoted as $0_{n,m}$ with the subscripts occasionally dropped if they can be inferred from context.

Let $\mathcal{L}_2$ be the set of square integrable functions. The $\mathcal{L}_2$-norm of $u \in \mathcal{L}_2$ is defined as $\|u\|_2 = \sqrt{\int_0^\infty u(t)^T u(t) dt}$. A transfer matrix $G(s) = D + C(sI - A)^{-1}B$ will be denoted as $G(s) \stackrel{s}{=} (A, B, C, D)$ or

$$G(s) \stackrel{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right], \tag{1}$$

and dependence on the variable s will be suppressed. For a stable transfer matrix G, we define

$$\begin{aligned} \|G\|_\infty &= \sup \{ \|Gu\|_2 / \|u\|_2 : 0 \neq u \in \mathcal{L}_2 \}, \\ \|G\|_- &= \inf \{ \|Gu\|_2 / \|u\|_2 : 0 \neq u \in \mathcal{L}_2 \}. \end{aligned} \tag{2}$$

In section 3, we use the following result.

Lemma 1 (see [17]). *Let $\phi(s) = R + B^T(-sI - A^T)^{-1}C + C^T(sI - A)^{-1}B + B^T(-sI - A^T)^{-1}Q(sI - A)^{-1}B$ with (A, B) sign controllable, $R^T = R$, and $Q^T = Q$. Then $\phi(s)$ has a spectral factor $G(s) \in \mathcal{RL}_\infty^{m \times p}$ (i.e., $\phi(s) = G^T(-s)G(s)$) if and only if there exists symmetric P that satisfies the following linear matrix inequality:*

$$\left[\begin{array}{cc} PA + A^T P + Q & PB + C \\ (PB + C)^T & R \end{array} \right] \succeq 0. \tag{3}$$

2. PROBLEM FORMULATION

Consider a linear time-invariant dynamic system subject to disturbances, modeling errors and process, sensor and actu-ator faults modeled as

$$\left[\begin{array}{c} \dot{x}(t) \\ y(t) \end{array} \right] = \overbrace{\left[\begin{array}{cccc} A & B & B_d & B_f \\ C & D & D_d & D_f \end{array} \right]}^{\mathcal{M}} \left[\begin{array}{c} x(t) \\ u(t) \\ d(t) \\ f(t) \end{array} \right], \tag{4}$$

where $x(t) \in \mathcal{R}^n$, $u(t) \in \mathcal{R}^{n_u}$, and $y(t) \in \mathcal{R}^{n_y}$ are the process state and input and output vectors, respectively, and where $d(t) \in \mathcal{R}^{n_d}$ and $f(t) \in \mathcal{R}^{n_f}$ are the disturbance and fault vectors, respectively. Here, $B_f \in \mathcal{R}^{n \times n_f}$ and $D_f \in \mathcal{R}^{n_y \times n_f}$ are the component and instrument fault distribution matrices, respectively, while $B_d \in \mathcal{R}^{n \times n_d}$ and $D_d \in \mathcal{R}^{n_y \times n_d}$ are the corresponding disturbance distribution matrices [18].

We consider two types of uncertainties: norm-bounded and polytopic uncertainties. In the case of norm-bounded uncertainties,

$$\mathcal{M} \in \left\{ \overbrace{\left[\begin{array}{cccc} A^o & B^o & B_d^o & B_f^o \\ C^o & D^o & D_d^o & D_f^o \end{array} \right]}^{\mathcal{M}^o} \right. \\ \left. + \left[\begin{array}{c} F_A \\ F_C \end{array} \right] \Delta_H \left[\begin{array}{cccc} E_A & E_B & E_d & E_f \end{array} \right] : \Delta \in \mathbf{\Delta} \right\} =: \mathcal{M}_\Delta, \tag{5}$$

where $\mathcal{M}^o$ represents the nominal model, $\Delta_H = \Delta(I - H\Delta)^{-1}$, where

$$\begin{aligned} \mathbf{\Delta} := \{ \Delta = \text{diag} \left(\delta_1 I_{q1}, \ldots, \delta_l I_{ql}, \Delta_{l+1}, \ldots, \Delta_{l+f} \right) : \|\Delta\| \leq 1, \\ \delta_i \in \mathcal{R}, \Delta_i \in \mathcal{R}^{q_i \times q_i} \} \subset \mathcal{R}^{n_\Delta \times n_\Delta}, \end{aligned} \tag{6}$$

and where F_A, F_C, E_A, E_B, E_d, E_f, and H are known and constant matrices with appropriate dimensions. This linear fractional representation of uncertainty, which is assumed to be well-posed over $\mathbf{\Delta}$ (i.e., $\det(I - H\Delta) \neq 0$ for all $\Delta \in \mathbf{\Delta}$), has great generality and is widely used in control theories.

In the case of polytopic uncertainties,

$$\mathcal{M} \in \left\{ \sum_{i=1}^p \xi_i \overbrace{\left[\begin{array}{cccc} A^i & B^i & B_d^i & B_f^i \\ C^i & D^i & D_d^i & D_f^i \end{array} \right]}^{\mathcal{M}^i} : \sum_{i=1}^p \xi_i = 1, \ \xi_i \geq 0 \right\} =: \mathcal{M}_\xi, \tag{7}$$

where $\mathcal{M}^i$, $i = 1, \ldots, p$, are known constant matrices with appropriate dimensions.

A residual signal in an FDI system should represent the inconsistency between the system variables and the mathematical model. The objective is to design an FDI filter of the form

$$\left[\begin{array}{c} \dot{\hat{x}}(t) \\ r(t) \end{array} \right] = \left[\begin{array}{ccc} A_k & B_{ku} & B_{ky} \\ C_k & D_{ku} & D_{ky} \end{array} \right] \left[\begin{array}{c} \hat{x}(t) \\ u(t) \\ y(t) \end{array} \right], \tag{8}$$
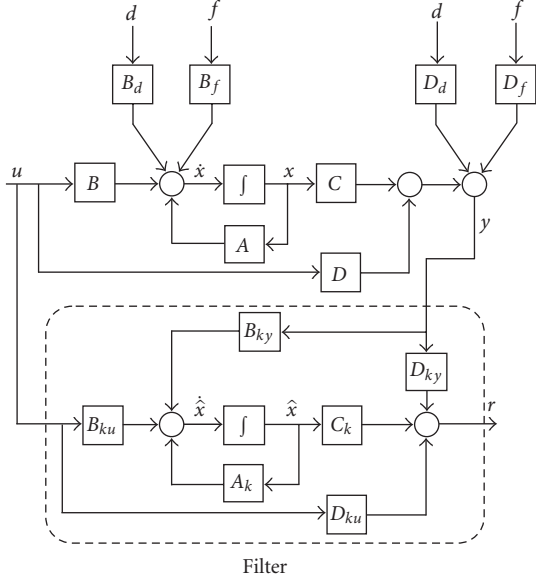
FIGURE 1: Filter-based robust FDI scheme.

where $\hat{x}(t) \in \mathcal{R}^{n_k}$ is the filter state and $r(t) \in \mathcal{R}^{n_f}$ is the residual signal. Figure 1 illustrates this filter in the robust residual generation scheme.

By defining an augmented state $z(t) = \begin{bmatrix} x(t)^T & \hat{x}(t)^T \end{bmatrix}^T$ the residual dynamics are given by

$$\begin{bmatrix} \dot{z}(t) \\ r(t) \end{bmatrix} = \begin{bmatrix} \widetilde{A} & \widetilde{B}_f & \widetilde{B}_d & \widetilde{B}_u \\ \widetilde{C} & \widetilde{D}_f & \widetilde{D}_d & \widetilde{D}_u \end{bmatrix} \begin{bmatrix} z(t) \\ f(t) \\ d(t) \\ u(t) \end{bmatrix} \qquad (9)$$

or

$$\begin{bmatrix} T_{rf}^{\mathcal{M}} & T_{rd}^{\mathcal{M}} & T_{ru}^{\mathcal{M}} \end{bmatrix}$$
$$\stackrel{s}{=} \left[\begin{array}{c|ccc} \widetilde{A} & \widetilde{B}_f & \widetilde{B}_d & \widetilde{B}_u \\ \hline \widetilde{C} & \widetilde{D}_f & \widetilde{D}_d & \widetilde{D}_u \end{array} \right]$$
$$= \left[\begin{array}{cc|ccc} A & 0 & B_f & B_d & B \\ B_{ky}C & A_k & B_{ky}D_f & B_{ky}D_d & B_{ku} + B_{ky}D \\ \hline D_{ky}C & C_k & D_{ky}D_f & D_{ky}D_d & D_{ku} + D_{ky}D \end{array} \right],$$
$$(10)$$

where $T_{rf}^{\mathcal{M}}$, $T_{rd}^{\mathcal{M}}$, and $T_{ru}^{\mathcal{M}}$ denote the dynamics from faults, disturbances, and inputs to the residual, respectively. Note that dependence on the uncertain data is indicated by a superscript $\mathcal{M}$.

Ideally, the residual signal is required to be sensitive only to faults. This corresponds to $T_{rd}^{\mathcal{M}} = 0$, $T_{ru}^{\mathcal{M}} = 0$, and $T_{rf}^{\mathcal{M}} \neq 0$. For fault isolation, it is further required that the fault signature can be deduced from the residual. This corresponds to $T_{rf}^{\mathcal{M}} \in \mathcal{S}$, where $\mathcal{S}$ is a set of transfer matrices with a special structure that depends on the nature of the faults, for example, if the faults can occur simultaneously, $\mathcal{S}$ is the set of stable diagonal transfer matrices with nonzero diagonal en-

tries. Unfortunately, characterizing a general structured set $\mathcal{S}$ is intractable, and we will assume that we can define a subset

$$\mathcal{S} = \Big\{ T_{\mathrm{ref}} \stackrel{s}{=} (A_{\mathrm{ref}}, B_{\mathrm{ref}}, C_{\mathrm{ref}}, D_{\mathrm{ref}}) : A_{\mathrm{ref}} \in \mathcal{S}_A \subseteq \mathcal{R}^{n_{\mathrm{ref}} \times n_{\mathrm{ref}}},$$
$$B_{\mathrm{ref}} \in \mathcal{S}_B \subseteq \mathcal{R}^{n_{\mathrm{ref}} \times n_f}, \ C_{\mathrm{ref}} \in \mathcal{S}_C \subseteq \mathcal{R}^{n_f \times n_{\mathrm{ref}}},$$
$$D_{\mathrm{ref}} \in \mathcal{S}_D \subseteq \mathcal{R}^{n_f \times n_f} \Big\} \subseteq \mathcal{S}$$
$$(11)$$

such that subsequent optimizations over the structured state-space data sets $\mathcal{S}_A$, $\mathcal{S}_B$, $\mathcal{S}_C$, and $\mathcal{S}_D$ are tractable. For example, if $\mathcal{S}$ denotes the set of all stable $n_f \times n_f$ diagonal transfer matrices with nonzero diagonal entries, we may define $\mathcal{S}_A$ as the set of all $n_f \times n_f$ diagonal matrices with negative entries, and $\mathcal{S}_B$, $\mathcal{S}_C$, and $\mathcal{S}_D$ as the sets of all $n_f \times n_f$ diagonal matrices. An example of this simplification procedure is given in Section 5 below. We also replace the requirement of nonzero diagonal entries for $\mathcal{S}$ by a condition of the form $\|T_{\mathrm{ref}}\|_- \geq \beta > 0$ for all $T_{\mathrm{ref}} \in \mathcal{S}$.

Due to the presence of disturbances and modelling uncertainties, exact FDI is not possible. For robust FDI, we propose the following, more realistic, problem formulation.

Problem 1. Assume that the system dynamics in (4) are quadratically stable. For any $\gamma > 0$, find a stable fault reference dynamics $T_{\mathrm{ref}} \stackrel{s}{=} (A_{\mathrm{ref}}, B_{\mathrm{ref}}, C_{\mathrm{ref}}, D_{\mathrm{ref}}) \in \mathcal{S}$ such that $\|T_{\mathrm{ref}}\|_- \geq 1$ (to ensure the requirement of nonzero diagonal entries for $\mathcal{S}$), and find a stable filter of the form given in (8), if it exists, such that the residual dynamics in (10) are quadratically stable and

$$\sup_{\mathcal{M} \in \mathcal{M}} \left\| \begin{bmatrix} T_{rf}^{\mathcal{M}} - T_{\mathrm{ref}} & T_{rd}^{\mathcal{M}} & T_{ru}^{\mathcal{M}} \end{bmatrix} \right\|_\infty < \gamma, \qquad (12)$$

where $\mathcal{M} = \mathcal{M}_\Delta$ for norm-bounded uncertainties and $\mathcal{M} = \mathcal{M}_\xi$ for polytopic uncertainties.

Recall that quadratic stability for the dynamics in (4) is equivalent to the existence of $P = P^T \succ 0$ such that $A^T P + PA \prec 0$ for all A (see [19] for more details).

A modified version of Problem 1 uses a weighted cost function, say, of the form

$$\left\| \begin{bmatrix} (T_{rf}^{\mathcal{M}} - T_{\mathrm{ref}})W_f & T_{rd}^{\mathcal{M}}W_d & T_{ru}^{\mathcal{M}}W_u \end{bmatrix} \right\|_\infty, \qquad (13)$$

where W_f, W_d, and W_u are constant or frequency-dependent weighting functions that can vary the emphasis between fault detection (small $\|T_{rd}^{\mathcal{M}}\|_\infty$ and $\|T_{ru}^{\mathcal{M}}\|_\infty$) and fault isolation (small $\|T_{rf}^{\mathcal{M}} - T_{\mathrm{ref}}\|_\infty$). In the sequel, we assume that any such weighting functions are absorbed in the system data.

Remark 1. The objective is to find the smallest γ for which (12) is satisfied. Indeed, by minimizing γ, we will ensure $\|T_{rd}^{\mathcal{M}}\|_\infty$ (which measures the disturbance rejection level), $\|T_{ru}^{\mathcal{M}}\|_\infty$ (which measures the effect of uncertainty), and $\|T_{rf}^{\mathcal{M}} - T_{\mathrm{ref}}\|_\infty$ (which measures the deviation of the fault to residual dynamics from the reference dynamics, and hence is a measure of fault isolation capability) to be small.

A poorly chosen reference model can result in a residual generator with poor robustness. Here, we incorporate its design into our scheme so as to improve the robustness of the FDI filter.

In some approaches, a common assumption is that $D_f = 0$ and/or CB_f has full rank [20, 21]; in others, the assumption D_f has full rank and is widely used [22, 23]. Here, we do not impose any of these assumptions. Note, however, that if D_f does not have full column rank, for example, if $D_f = 0$, then this will have an adverse effect on the minimum values of γ since $\gamma \geq \|[\ T_{rf}^{\mathcal{M}} - T_{\text{ref}}\ \ T_{rd}^{\mathcal{M}}\ \ T_{ru}^{\mathcal{M}}\ ]\|_{\infty} \geq \|T_{rf}^{\mathcal{M}}(\infty) - T_{\text{ref}}(\infty)\| = \|D_{ky}D_f - D_{\text{ref}}\| = \|D_{\text{ref}}\| \geq 1$. This would therefore limit the overall performance of the filter, which is measured by the value of γ.

3. MATRIX INEQUALITY FORMULATION

In this section, we derive a matrix inequality formulation of Problem 1. The main idea is to express (12) in terms of QMIs using the bounded real lemma and change of variables techniques, and then to derive necessary and sufficient conditions for solvability.

The dynamics in (12) can be written as

$$
\left[\ T_{rf}^{\mathcal{M}} - T_{\text{ref}}\ \ T_{rd}^{\mathcal{M}}\ \ T_{ru}^{\mathcal{M}}\ \right] \overset{s}{=} \left[\begin{array}{c|c} A_c^{\mathcal{M}} & B_c^{\mathcal{M}} \\ \hline C_c^{\mathcal{M}} & D_c^{\mathcal{M}} \end{array}\right]
$$

$$
:= \left[\begin{array}{cc|ccc} A_{\text{ref}} & 0 & B_{\text{ref}} & 0 & 0 \\ 0 & \widetilde{A} & \widetilde{B}_f & \widetilde{B}_d & \widetilde{B}_u \\ \hline -C_{\text{ref}} & \widetilde{C} & \widetilde{D}_f - D_{\text{ref}} & \widetilde{D}_d & \widetilde{D}_u \end{array}\right].
$$

$$\tag{14}$$

It follows from the bounded real lemma that there exists a stable filter of the form given in (8) such that (12) is satisfied if and only if there exists $P_c = P_c^T$ such that $P_c \succ 0$ and

$$
\left[\begin{array}{ccc} (A_c^{\mathcal{M}})^T P_c + P_c A_c^{\mathcal{M}} & P_c B_c^{\mathcal{M}} & (C_c^{\mathcal{M}})^T \\ (B_c^{\mathcal{M}})^T P_c & -\gamma I & (D_c^{\mathcal{M}})^T \\ C_c^{\mathcal{M}} & D_c^{\mathcal{M}} & -\gamma I \end{array}\right] \prec 0 \tag{15}
$$

for all $\mathcal{M}$ (see [24, Theorem 3]). We deal separately with norm-bounded and polytopic uncertainties.

3.1. Solution with norm-bounded uncertainties

For norm-bounded uncertainties, we separate the terms involving modeling uncertainties from the other terms as

$$
(A_c^{\mathcal{M}}, B_c^{\mathcal{M}}, C_c^{\mathcal{M}}, D_c^{\mathcal{M}}) = (A_c^o + A_c^{\Delta}, B_c^o + B_c^{\Delta}, C_c^o + C_c^{\Delta}, D_c^o + D_c^{\Delta}),
$$

$$\tag{16}$$

where

$$
\left[\begin{array}{c|c} A_c^o & B_c^o \\ \hline C_c^o & D_c^o \end{array}\right]
$$

$$
= \left[\begin{array}{ccc|ccc} A_{\text{ref}} & 0 & 0 & B_{\text{ref}} & 0 & 0 \\ 0 & A^o & 0 & B_f^o & B_d^o & B^o \\ 0 & B_{ky}C^o & A_k & B_{ky}D_f^o & B_{ky}D_d^o & B_{ku}+B_{ky}D^o \\ \hline -C_{\text{ref}} & D_{ky}C^o & C_k & D_{ky}D_f^o - D_{\text{ref}} & D_{ky}D_d^o & D_{ku}+D_{ky}D^o \end{array}\right],
$$

$$\tag{17}$$

$$
\left[\begin{array}{c|c} A_c^{\Delta} & B_c^{\Delta} \\ \hline C_c^{\Delta} & D_c^{\Delta} \end{array}\right] = \left[\begin{array}{c} F_{r1} \\ F_{r2} \end{array}\right] \Delta_H \left[E_{r1}\ \ E_{r2} \right]
$$

$$
:= \left[\begin{array}{c} 0 \\ F_A \\ B_{ky}F_C \\ \hline D_{ky}F_C \end{array}\right] \Delta_H \left[\ 0\ \ E_A\ \ 0\ |\ E_f\ \ E_d\ \ E_B\ \right].
$$

$$\tag{18}$$

Using this separation, the inequality (15) can be rewritten as

$$
\overbrace{\left[\begin{array}{ccc} (A_c^o)^T P_c + P_c A_c^o & P_c B_c^o & (C_c^o)^T \\ (B_c^o)^T P_c & -\gamma I & (D_c^o)^T \\ C_c^o & D_c^o & -\gamma I \end{array}\right]}^{T_c^o}
$$

$$
+ \overbrace{\left[\begin{array}{ccc} (A_c^{\Delta})^T P_c + P_c A_c^{\Delta} & P_c B_c^{\Delta} & (C_c^{\Delta})^T \\ (B_c^{\Delta})^T P_c & 0 & (D_c^{\Delta})^T \\ C_c^{\Delta} & D_c^{\Delta} & 0 \end{array}\right]}^{T_c^{\Delta}} \prec 0. \tag{19}
$$

A calculation shows that $T_c^{\Delta} = \widetilde{F}\Delta_H \widetilde{E} + \widetilde{E}^T \Delta_H \widetilde{F}^T$, where

$$
\widetilde{F} = \left[(P_c F_{r1})^T\ \ 0\ \ F_{r2}^T \right]^T, \qquad \widetilde{E} = \left[E_{r1}\ \ E_{r2}\ \ 0 \right]. \tag{20}
$$

The next result uses the fact that $\Delta \in \mathbf{\Delta}$ to remove explicit dependence on Δ.

Lemma 2 (see [25]). *Let $\mathbf{\Delta}$ be as described in (6) and define the subspaces*

$$
\mathbf{\Sigma} = \{\text{diag}\,(S_1, \ldots, S_l, \lambda_1 I_{ql+1}, \ldots, \lambda_s I_{ql+f}) :
$$

$$
S_i = S_i^T \in \mathcal{R}^{q_i \times q_i}, \lambda_j \in \mathcal{R}\},
$$

$$
\mathbf{\Gamma} = \{\text{diag}\,(G_1, \ldots, G_l, 0_{ql+1}, \ldots, 0_{ql+f}) :
$$

$$
G_i = -G_i^T \in \mathcal{R}^{q_i \times q_i}\}. \tag{21}
$$

Let $R = R^T$, F, E, and H be matrices with appropriate dimensions. We have $\det(I - H\Delta) \neq 0$ and $R + F\Delta(I - H\Delta)^{-1}E + E^T(I - \Delta^T H^T)^{-1}\Delta^T F^T \prec 0$ for every $\Delta \in \mathbf{\Delta}$ if there exist $S \in \mathbf{\Sigma}$ and $G \in \mathbf{\Gamma}$ such that $S \succ 0$ and

$$
\left[\begin{array}{cc} R + E^T S E & F + E^T S H + E^T G \\ F^T + H^T S E + G E & H^T S H + H^T G + G H - S \end{array}\right] \prec 0. \tag{22}
$$

If Δ *is unstructured (i.e., if* $\Delta = \{\Delta \in \mathcal{R}^{n_\Delta \times n_\Delta} : \|\Delta\| \leq 1\})$, *then* (22) *becomes*

$$\begin{bmatrix} R + \lambda E^T E & F + \lambda E^T H \\ F^T + \lambda H^T E & \lambda(H^T H - I) \end{bmatrix} \prec 0 \qquad (23)$$

for some scalar $\lambda \geq 0$. *In this case, condition* (23) *is both necessary and sufficient.*

When the uncertainty set is unstructured, then **Lemma 2** implies that

$$(19) \Longleftrightarrow T_c^o + \widetilde{F}\Delta_H \widetilde{E} + \widetilde{E}^T \Delta_H^T \widetilde{F}^T \prec 0$$

$$\Longleftrightarrow \begin{bmatrix} T_c^o + \lambda \widetilde{E}^T \widetilde{E} & \widetilde{F} + \lambda \widetilde{E}^T H \\ \widetilde{F}^T + \lambda H^T \widetilde{E} & \lambda(H^T H - I) \end{bmatrix} \prec 0 \qquad (24)$$

for some $\lambda \geq 0$. Using a Schur complement argument shows that (19) is equivalent to

$$\overline{T}_c := \begin{bmatrix} (A_c^o)^T P_c + \star & P_c B_c^o & (C_c^o)^T & P_c F_{r1} & \lambda E_{r1}^T \\ \star & -\gamma I & (D_c^o)^T & 0 & \lambda E_{r2}^T \\ \star & \star & -\gamma I & F_{r2} & 0 \\ \star & \star & \star & -\lambda I & \lambda H^T \\ \star & \star & \star & \star & -\lambda I \end{bmatrix} \prec 0, \qquad (25)$$

where $\star$ denotes terms readily inferred from symmetry. Next, we introduce a change of variables to linearize the above matrix inequality [26]. Assume that $n_k = n_{\text{ref}} + n$, that is, the filter order is equal to the order of the system plus the order of the reference model. Let

$$P_c = \begin{bmatrix} Y & N \\ N^T & \widehat{Y} \end{bmatrix}, \qquad P_c^{-1} = \begin{bmatrix} X & M \\ M^T & \widehat{X} \end{bmatrix},$$

$$\Pi_1 = \begin{bmatrix} I & I \\ M^T Z & 0 \end{bmatrix}, \qquad (26)$$

where $X, Y, \widehat{X}, \widehat{Y} \in \mathcal{R}^{n_k \times n_k}$ are symmetric matrices, and $Z = X^{-1}$. Define $T = \text{diag}(\Pi_1, I)$. Then $\overline{T}_c \prec 0$ if and only if $T^T \overline{T}_c T \prec 0$. From $P_c P_c^{-1} = I$, we have the following calculations, where boldface letters are used to indicate optimization variables:

$$\mathcal{L}_{11} := \Pi_1^T P_c A_c^o \Pi_1$$

$$= \begin{bmatrix} \mathbf{Z}\overline{A} + \mathbf{Z}E_1 \mathbf{A}_{\text{ref}} E_1^T & \mathbf{Z}\overline{A} + \mathbf{Z}E_1 \mathbf{A}_{\text{ref}} E_1^T \\ \mathbf{Y}\overline{A} + \mathbf{Y}E_1 \mathbf{A}_{\text{ref}} E_1^T + \widehat{\mathbf{A}} + \widehat{\mathbf{B}}_y \overline{C} & \mathbf{Y}\overline{A} + \mathbf{Y}E_1 \mathbf{A}_{\text{ref}} E_1^T + \widehat{\mathbf{B}}_y \overline{C} \end{bmatrix}, \qquad (27)$$

$$\mathcal{L}_{12} := \Pi_1^T P_c B_c^o$$

$$= \begin{bmatrix} \mathbf{Z}\overline{B}_f + \mathbf{Z}E_1 \mathbf{B}_{\text{ref}} & \mathbf{Z}\overline{B}_d & \mathbf{Z}\overline{B} \\ \mathbf{Y}\overline{B}_f + \mathbf{Y}E_1 \mathbf{B}_{\text{ref}} + \widehat{\mathbf{B}}_y D_f^o & \mathbf{Y}\overline{B}_d + \widehat{\mathbf{B}}_y D_d^o & \mathbf{Y}\overline{B} + \widehat{\mathbf{B}}_y D^o + \widehat{\mathbf{B}}_u \end{bmatrix}, \qquad (28)$$

$$\mathcal{L}_{13} := (C_c^o \Pi_1)^T = \begin{bmatrix} (\widehat{\mathbf{D}}_y \overline{C} - \mathbf{C}_{\text{ref}} E_2 + \widehat{\mathbf{C}})^T \\ (\widehat{\mathbf{D}}_y \overline{C} - \mathbf{C}_{\text{ref}} E_2)^T \end{bmatrix}, \qquad (29)$$

$$\mathcal{L}_{14} := \Pi_1^T P_c F_{r1} = \begin{bmatrix} \mathbf{Z}\overline{F}_A \\ \mathbf{Y}\overline{F}_A + \widehat{\mathbf{B}}_y F_C \end{bmatrix}, \qquad (30)$$

$$\mathcal{L}_{15} := \Pi_1^T E_{r1}^T = \begin{bmatrix} \overline{E}_A^T \\ \overline{E}_A^T \end{bmatrix}, \qquad (31)$$

$$\mathcal{L}_{23}^T = \begin{bmatrix} \widehat{\mathbf{D}}_y D_f^o - \mathbf{D}_{\text{ref}} & \widehat{\mathbf{D}}_y D_d^o & \widehat{\mathbf{D}}_y D^o + \widehat{\mathbf{D}}_u \end{bmatrix}, \qquad (32)$$

$$\mathcal{L}_{25}^T = \begin{bmatrix} E_f & E_d & E_B \end{bmatrix}, \qquad (33)$$

where we have defined

$$\begin{bmatrix} \overline{A} & E_1 & \overline{B}_f & \overline{B}_d & \overline{B} & \overline{F}_A \end{bmatrix} := \begin{bmatrix} 0 & 0 & I_{n_{\text{ref}}} & 0 & 0 & 0 & 0 \\ 0 & A^o & 0 & B_f^o & B_d^o & B^o & F_A \end{bmatrix}, \qquad (34)$$

$$\begin{bmatrix} \overline{C} \\ E_2 \\ \overline{E}_A \end{bmatrix} := \begin{bmatrix} 0 & C^o \\ I_{n_{\text{ref}}} & 0 \\ 0 & E_A \end{bmatrix}, \qquad (35)$$

$$\begin{bmatrix} \widehat{\mathbf{A}} & \widehat{\mathbf{B}}_y & \widehat{\mathbf{B}}_u \\ \widehat{\mathbf{C}} & \widehat{\mathbf{D}}_y & \widehat{\mathbf{D}}_u \end{bmatrix} = \begin{bmatrix} NA_k M^T Z & NB_{ky} & NB_{ku} \\ C_k M^T Z & D_{ky} & D_{ku} \end{bmatrix}. \qquad (36)$$

If M and N are invertible, the variables $A_k, C_k, B_{ky}, B_{ku}, D_{ky}, D_{ku}$ can be replaced by the new variables $\widehat{\mathbf{A}}, \widehat{\mathbf{B}}_y, \widehat{\mathbf{B}}_u, \widehat{\mathbf{C}}, \widehat{\mathbf{D}}_y, \widehat{\mathbf{D}}_u$ without loss of generality. We can now rewrite (19) as

$$\begin{bmatrix} \mathcal{L}_{11} + \mathcal{L}_{11}^T & \mathcal{L}_{12} & \mathcal{L}_{13} & \mathcal{L}_{14} & \lambda \mathcal{L}_{15} \\ \star & -\gamma I & \mathcal{L}_{23} & 0 & \lambda \mathcal{L}_{25} \\ \star & \star & -\gamma I & \widehat{\mathbf{D}}_y F_C & 0 \\ \star & \star & \star & -\lambda I & \lambda H^T \\ \star & \star & \star & \star & -\lambda I \end{bmatrix} \prec 0, \qquad (37)$$

which is nonlinear in the variables. Note that the nonlinearities involve the terms $\mathbf{A}_{\text{ref}}$ and $\mathbf{B}_{\text{ref}}$ only. The constraint $P_c \succ 0$ can be expressed as an LMI as follows:

$$P_c \succ 0 \Longleftrightarrow \Pi_1^T P_c \Pi_1 \succ 0 \Longleftrightarrow \begin{bmatrix} \mathbf{Z} & \mathbf{Z} \\ \mathbf{Z} & \mathbf{Y} \end{bmatrix} \succ 0. \qquad (38)$$

The constraint $\|T_{\text{ref}}\|_- \geq 1$ can be expressed as a quadratic matrix inequality using the next lemma.

Lemma 3. *Let* T_{ref} *be as defined above. Then* $\|T_{\text{ref}}\|_- \geq 1$ *if and only if there exists* $\mathbf{P}_{\text{ref}} = \mathbf{P}_{\text{ref}}^T \in \mathcal{R}^{n_{\text{ref}} \times n_{\text{ref}}}$ *such that*

$$T_{\text{ro}}$$

$$:= \begin{bmatrix} \mathbf{P}_{\text{ref}} \mathbf{A}_{\text{ref}} + \mathbf{A}_{\text{ref}}^T \mathbf{P}_{\text{ref}} + \mathbf{C}_{\text{ref}}^T \mathbf{C}_{\text{ref}} & \mathbf{P}_{\text{ref}} \mathbf{B}_{\text{ref}} + \mathbf{C}_{\text{ref}}^T \mathbf{D}_{\text{ref}} \\ \mathbf{B}_{\text{ref}}^T \mathbf{P}_{\text{ref}} + \mathbf{D}_{\text{ref}}^T \mathbf{C}_{\text{ref}} & \mathbf{D}_{\text{ref}}^T \mathbf{D}_{\text{ref}} - I_{n_f} \end{bmatrix}$$

$$\succeq 0. \qquad (39)$$

Proof. Let $\phi(s) = T^{\sim}_{\text{ref}}(s)T_{\text{ref}}(s) - I_{n_f}$. Then $\phi(s) \succeq 0$ if and only if $\|T_{\text{ref}}\|_{-} \geq 1$. It is easy to show that $\phi(s)$ can be written as follows:

$$
\begin{aligned}
\phi(s) = {}& \mathbf{D}_{\text{ref}}{}^T\mathbf{D}_{\text{ref}} - I_{nf} + \mathbf{B}_{\text{ref}}{}^T\left(-sI_{nf} - \mathbf{A}_{\text{ref}}{}^T\right)^{-1}\mathbf{C}_{\text{ref}}{}^T\mathbf{D}_{\text{ref}} \\
& + \left(\mathbf{C}_{\text{ref}}{}^T\mathbf{D}_{\text{ref}}\right)^T\left(sI_{nf} - \mathbf{A}_{\text{ref}}\right)^{-1}\mathbf{B}_{\text{ref}} \\
& + \mathbf{B}_{\text{ref}}{}^T\left(-sI_{nf} - \mathbf{A}_{\text{ref}}{}^T\right)^{-1}\mathbf{C}_{\text{ref}}{}^T\mathbf{C}_{\text{ref}}\left(sI_{nf} - \mathbf{A}_{\text{ref}}\right)^{-1}\mathbf{B}_{\text{ref}}.
\end{aligned}
\tag{40}
$$

The result then follows from Lemma 1. □

The next lemma summarizes our result so far by giving necessary and sufficient conditions for the solution of Problem 1, in the case that the uncertainty set is unstructured, in the form of a QMI feasibility problem.

Lemma 4. *Assume that Δ is unstructured. Then there exist a stable filter of the form given in* (8) *and a stable fault reference dynamics model $T_{\text{ref}} \overset{s}{=} (\mathbf{A}_{\text{ref}}, \mathbf{B}_{\text{ref}}, \mathbf{C}_{\text{ref}}, \mathbf{D}_{\text{ref}}) \in \mathcal{S}$, where $\mathcal{S}$ is defined in* (11), *such that $\|T_{\text{ref}}\|_{-} \geq 1$, residual dynamics in* (10) *are quadratically stable, and* (12) *is satisfied for $\mathcal{M} = \mathcal{M}_{\Delta}$ if and only if there exist $\mathbf{A}_{\text{ref}} \in \mathcal{S}_A$, $\mathbf{B}_{\text{ref}} \in \mathcal{S}_B$, $\mathbf{C}_{\text{ref}} \in \mathcal{S}_C$, $\mathbf{D}_{\text{ref}} \in \mathcal{S}_D$, $\hat{\mathbf{A}}$, $\hat{\mathbf{B}}_y$, $\hat{\mathbf{B}}_u$, $\hat{\mathbf{C}}$, $\hat{\mathbf{D}}_y$, $\hat{\mathbf{D}}_u, \lambda$, and symmetric matrices $\mathbf{P}_{\text{ref}}$, $\mathbf{Y}$, and $\mathbf{Z}$ such that* (37), (38), *and* (39) *are satisfied. If such variables exist, the filter dynamics are obtained by solving* (36) *where M and N are chosen such that $NM^T = I - \mathbf{YZ}^{-1}$.*

Approximate solutions to these QMIs can be obtained by using algorithms with guaranteed global convergence [27, 28], as well as local numerical search algorithms that converge (without a guarantee) much faster [29, 30]. A related discussion of the solution algorithms for QMIs can also be found in [31]. In Section 4 below, we develop an alternative procedure for the approximate solution of these QMIs.

Remark 2. In the case that T_{ref} is preassigned to a known value, (37) becomes linear and (39) becomes irrelevant, therefore, the optimal solution is given in a form of an LMI optimization. This case has been considered in [16].

When Δ is structured, we proceed as follows. By using (22) from Lemma 2, we get

$$
\begin{aligned}
(19) &\iff T_c^o + \widetilde{F}\Delta_H\widetilde{E} + \widetilde{E}^T\Delta_H{}^T\widetilde{F}^T \prec 0 \\
&\iff \exists \mathbf{S} \in \Sigma \ \& \ \mathbf{G} \in \Gamma \quad \text{s.t. } \mathbf{S} \succ 0 \ \& \ T_{SG} \prec 0,
\end{aligned}
\tag{41}
$$

where

$$
T_{SG} = \begin{bmatrix} T_c^o & \widetilde{F} + \widetilde{E}^T\mathbf{G} \\ \widetilde{F}^T + \mathbf{G}^T\widetilde{E} & H^T\mathbf{G} + \mathbf{G}^TH - \mathbf{S} \end{bmatrix} + \begin{bmatrix} \widetilde{E}^T \\ H^T \end{bmatrix} \mathbf{S} \begin{bmatrix} \widetilde{E} & H \end{bmatrix}.
\tag{42}
$$

Using a Schur complement argument and the expression of $\widetilde{E}$ and $\widetilde{F}$ in (20), we get

$$
T_{SG} \prec 0 \iff \overline{T}_{SG}
$$
$$
:= \begin{bmatrix} (A_c^o)^T P_c + \star & P_c B_c^o & (C_c^o)^T & P_c F_{r1} + E_{r1}^T\mathbf{G} & E_{r1}^T\mathbf{S} \\ \star & -\gamma I & (D_c^o)^T & E_{r2}^T\mathbf{G} & E_{r2}^T\mathbf{S} \\ \star & \star & -\gamma I & F_{r2} & 0 \\ \star & \star & \star & H^T\mathbf{G} + \mathbf{G}^TH - \mathbf{S} & H^T\mathbf{S} \\ \star & \star & \star & \star & -\mathbf{S} \end{bmatrix}
$$
$$
\prec 0.
\tag{43}
$$

As we did for unstructured uncertainties, we use the same matrix $T = \text{diag}(\Pi_1, I)$ to allow to change variables, it follows that $\overline{T}_{SG} \prec 0$ if and only if $\widetilde{T}_{SG} := T\overline{T}_{SG}T^T \prec 0$. We multiply $\widetilde{T}_{SG}$ by $K = \text{diag}(I, \mathbf{S})$ and K^T from left and right, respectively, to get $T_{SG} \prec 0$ if and only if

$$
\begin{bmatrix} \mathcal{L}_{11} + \mathcal{L}_{11}^T & \mathcal{L}_{12} & \mathcal{L}_{13} & \mathcal{L}_{14} + \mathcal{L}_{15}\mathbf{G} & \mathcal{L}_{15}\mathbf{S} \\ \star & -\gamma I & \mathcal{L}_{23} & 0 & \mathcal{L}_{25}\mathbf{S} \\ \star & \star & -\gamma I & \hat{\mathbf{D}}_y F_C & 0 \\ \star & \star & \star & H^T\mathbf{G} + \mathbf{G}^TH - \mathbf{S} & H^T\mathbf{S} \\ \star & \star & \star & \star & -\mathbf{S} \end{bmatrix} \prec 0.
\tag{44}
$$

Therefore, when Δ is structured, we have the following sufficient condition for solvability.

Lemma 5. *Suppose that Δ has the structure defined in* (6). *Then there exist a stable filter of the form given in* (8) *and a stable fault reference dynamics model $T_{\text{ref}} \overset{s}{=} (\mathbf{A}_{\text{ref}}, \mathbf{B}_{\text{ref}}, \mathbf{C}_{\text{ref}}, \mathbf{D}_{\text{ref}}) \in \mathcal{S}$, where $\mathcal{S}$ is defined in* (11), *such that $\|T_{\text{ref}}\|_{-} \geq 1$, the residual dynamics in* (10) *are quadratically stable, and* (12) *is satisfied for $\mathcal{M} = \mathcal{M}_{\Delta}$ if there exist $\mathbf{A}_{\text{ref}} \in \mathcal{S}_A$, $\mathbf{B}_{\text{ref}} \in \mathcal{S}_B$, $\mathbf{C}_{\text{ref}} \in \mathcal{S}_C$, $\mathbf{D}_{\text{ref}} \in \mathcal{S}_D$, $\hat{\mathbf{A}}$, $\hat{\mathbf{B}}_y$, $\hat{\mathbf{B}}_u$, $\hat{\mathbf{C}}$, $\hat{\mathbf{D}}_y$, $\hat{\mathbf{D}}_u$, $\mathbf{S} \in \Sigma$, $\mathbf{G} \in \Gamma$, and symmetric matrices $\mathbf{P}_{\text{ref}}$, $\mathbf{Y}$, and $\mathbf{Z}$ such that* (38), (39), *and* (44) *are satisfied.*

3.2. Solution with polytopic uncertainties

In this section, we derive necessary and sufficient conditions for solvability of the robust FDI problem for a system subject to polytopic uncertainties, in the form of LMIs. Now,

$$
\left[\begin{array}{c|c} A_c^{\mathcal{M}} & B_c^{\mathcal{M}} \\ \hline C_c^{\mathcal{M}} & D_c^{\mathcal{M}} \end{array} \right] \in \left\{ \sum_{i=1}^p \xi_i \left[\begin{array}{c|c} A_c^i & B_c^i \\ \hline C_c^i & D_c^i \end{array} \right], \sum_{i=1}^p \xi_i = 1, \xi_i \geq 0 \right\},
\tag{45}
$$

where $(A_c^i, B_c^i, C_c^i, D_c^i)$ are as defined in (17), but with superscript (o) replaced by (i). We assume that the polytopic system is quadratically stable. Recall that (12) is satisfied if and only if (15) is satisfied for all $\mathcal{M}$. Now,

$$(15) \Longleftrightarrow \sum_{i=1}^{p} \xi_i \overbrace{\begin{bmatrix} (A_c^i)^T P_c + P_c A_c^i & P_c B_c^i & (C_c^i)^T \\ (B_c^i)^T P_c & -\gamma I & (D_c^i)^T \\ C_c^i & D_c^i & -\gamma I \end{bmatrix}}^{T_{\text{pol}}^i} \prec 0,$$

$$\forall \xi_i \text{ s.t. } \sum_{i=1}^{p} \xi_i = 1, \quad \xi_i \succeq 0,$$

$$\Longleftrightarrow T_{\text{pol}}^i \prec 0, \quad i = 1, \dots, p. \tag{46}$$

Noting that the change of variable defined in (36) is independent of $\mathcal{M}^i$, we can therefore use it in this scheme as well. Letting $T = \text{diag}(\Pi_1, I)$, it follows that

$$T_{\text{pol}}^i \prec 0 \Longleftrightarrow \overline{T}_{\text{pol}}^i := T T_{\text{pol}}^i T^T$$

$$= \begin{bmatrix} \mathcal{L}_{11}^i + (\mathcal{L}_{11}^i)^T & \mathcal{L}_{12}^i & \mathcal{L}_{13}^i \\ \star & -\gamma I & \mathcal{L}_{23}^i \\ \star & \star & -\gamma I \end{bmatrix} \prec 0, \tag{47}$$

where the $(\mathcal{L}_{jk}^i)^T s$ are as defined in (27)–(33) and (34)-(35), but with the nominal model data $\mathcal{M}^o$ replaced by $\mathcal{M}^i$. Therefore, for polytopic uncertainties, we can derive necessary and sufficient conditions for solvability of Problem 1 in the form of a QMI feasibility problem as follows.

Lemma 6. *Let $\mathcal{M} = \mathcal{M}_\xi$. Then there exist a stable filter of the form given in (8) and a stable fault reference dynamics model $T_{\text{ref}} \stackrel{s}{=} (\mathbf{A}_{\text{ref}}, \mathbf{B}_{\text{ref}}, \mathbf{C}_{\text{ref}}, \mathbf{D}_{\text{ref}}) \in \mathcal{S}$, where $\mathcal{S}$ is defined in (11), such that $\|T_{\text{ref}}\|_- \geq 1$, the residual dynamics in (10) are quadratically stable, and (12) is satisfied for $\mathcal{M} = \mathcal{M}_\Delta$ if and only there exist $\mathbf{A}_{\text{ref}} \in \mathcal{S}_A, \mathbf{B}_{\text{ref}} \in \mathcal{S}_B, \mathbf{C}_{\text{ref}} \in \mathcal{S}_C, \mathbf{D}_{\text{ref}} \in \mathcal{S}_D, \hat{\mathbf{A}}, \hat{\mathbf{B}}_y, \hat{\mathbf{B}}_u, \hat{\mathbf{C}}, \hat{\mathbf{D}}_y, \hat{\mathbf{D}}_u, \mathbf{S} \in \Sigma, \mathbf{G} \in \Gamma$, and symmetric matrices $\mathbf{P}_{\text{ref}}, \mathbf{Y}$, and $\mathbf{Z}$ such that (38), (39), and $\overline{T}_{\text{pol}}^i \prec 0$ are satisfied for $i = 1, \dots, p$.*

4. ROBUST FDI FILTER DESIGN USING LMIS

In this section, we give a suboptimal solution to Problem 1. An optimal solution would necessitate the solution of a quadratic matrix inequality and is in general intractable. Here, we propose a linearization procedure to derive an upper bound on the optimal solution that requires solving an LMI optimization problem.

The following general result demonstrates that if we have one feasible solution to a QMI optimization, then we can construct an LMI optimization problem whose solution gives an upper bound on the QMI problem.

Lemma 7. *Let $Q_o, R_o \in \mathcal{R}^{m \times n}$ and $S = S^T \in \mathcal{R}^{m \times m}$ be given and let $E_o = Q_o - R_o$ and $F_o = Q_o + R_o$. For $Q, R \in \mathcal{R}^{m \times n}$, define*

$$\mathcal{M}(Q, R) = S + QR^T + RQ^T,$$

$$\mathcal{N}(Q, R)$$

$$= \begin{bmatrix} S + E_o E_o^T - (Q - R)E_o^T - E_o(Q - R)^T & R & Q \\ R^T & -I & 0 \\ Q^T & 0 & -I \end{bmatrix},$$

$$\mathcal{O}(Q, R)$$

$$= \begin{bmatrix} S - F_o F_o^T + (Q + R)F_o^T + F_o(Q + R)^T & R & Q \\ R^T & I & 0 \\ Q^T & 0 & I \end{bmatrix}. \tag{48}$$

Then

$$\mathcal{N}(Q, R) \prec 0 \Longrightarrow \mathcal{M}(Q, R) \prec 0, \tag{49}$$

$$\mathcal{M}(Q_o, R_o) \prec 0 \Longrightarrow \mathcal{N}(Q_o, R_o) \prec 0, \tag{50}$$

$$\mathcal{O}(Q, R) \succ 0 \Longrightarrow \mathcal{M}(Q, R) \succ 0, \tag{51}$$

$$\mathcal{M}(Q_o, R_o) \succ 0 \Longrightarrow \mathcal{O}(Q_o, R_o) \succ 0. \tag{52}$$

Proof. Let $\mathcal{J}(Q, R) = S + QQ^T + RR^T + E_o E_o^T - (Q - R)E_o^T - E_o(Q - R)^T$. Then by using a Schur complement argument, we get

$$\mathcal{N}(Q, R) \prec 0 \Longleftrightarrow \mathcal{J}(Q, R) \prec 0. \tag{53}$$

Let $\xi(Q, R) = (Q - R - E_o)(Q - R - E_o)^T$. Now,

$$S + \xi(Q, R) = S + (Q - R)(Q - R)^T - (Q - R)E_o^T$$
$$- E_o(Q - R)^T + E_o E_o^T, \tag{54}$$

and it follows that

$$S + QR^T + RQ^T = S + QQ^T + RR^T + E_o E_o^T - (Q - R)E_o^T$$
$$- E_o(Q - R)^T - \xi(Q, R). \tag{55}$$

That is,

$$\mathcal{M}(Q, R) = \mathcal{J}(Q, R) - \xi(Q, R). \tag{56}$$

Using (53) and noting that $\xi(Q, R) \succeq 0$ and $\xi(Q_o, R_o) = 0$, we have

$$\mathcal{N}(Q, R) \prec 0 \Longrightarrow \mathcal{J}(Q, R) \prec 0 \Longrightarrow \mathcal{M}(Q, R) \prec 0,$$
$$\mathcal{M}(Q_o, R_o) \prec 0 \Longrightarrow \mathcal{J}(Q_o, R_o) \prec 0 \Longrightarrow \mathcal{N}(Q_o, R_o) \prec 0. \tag{57}$$

A similar proof can be used to derive (51) and (52). $\square$

In order to simplify the subsequent analysis, we adopt the convention that variables appended with a subscript "x" denote feasible values of the variables for the QMIs (37) and (39).

In (37), the only nonlinear terms are $\mathbf{Z}E_1 \mathbf{A}_{\text{ref}} E_1^T$, $\mathbf{Y}IE_1 \mathbf{A}_{\text{ref}} E_1^T$, $\mathbf{Z}E_1 \mathbf{B}_{\text{ref}}$, and $\mathbf{Y}E_1 \mathbf{B}_{\text{ref}}$. We denote the matrix in (37) by T_{rfw}, and set S_{rfw} to be the matrix T_{rfw}, with the nonlinear terms removed.

Let $Z_x = Z_x^T \in \mathcal{R}^{n_k \times n_k}$, $Y_x = Y_x^T \in \mathcal{R}^{n_k \times n_k}$, $\mathbf{A}_{\text{ref}_x} \in \mathcal{R}^{n_{\text{ref}} \times n_{\text{ref}}}$, and $\mathbf{B}_{\text{ref}_x} \in \mathcal{R}^{n_{\text{ref}} \times n_f}$ be given. We use **Lemma 7** to derive an LMI formulation. We can write T_{rfw} as

$$T_{rfw} = S_{rfw} + Q_c R_c^T + R_c Q_c^T, \tag{58}$$

where

$$
Q_c = \begin{bmatrix} \mathbf{Z}E_1 & 0 \\ 0 & \mathbf{Y}E_1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad R_c = \begin{bmatrix} E_1\mathbf{A}_{\mathrm{ref}}{}^T & 0 \\ E_1\mathbf{A}_{\mathrm{ref}}{}^T & E_1\mathbf{A}_{\mathrm{ref}}{}^T \\ \mathbf{B}_{\mathrm{ref}}{}^T & \mathbf{B}_{\mathrm{ref}}{}^T \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{59}
$$

Let Q_{c_x} denote the value of Q_c when $\mathbf{Z}$ and $\mathbf{Y}$ are replaced by Z_x and Y_x, respectively, let R_{c_x} denote the value of R_c when A_{ref} and B_{ref} are replaced by A_{ref_x} and B_{ref_x}, respectively, and define $E_x = Q_{c_x} - R_{c_x}$. Using (49) from Lemma 7, we get

$$
T^{\mathrm{lin}}_{rfw} \prec 0 \Longrightarrow T_{rfw} \prec 0, \tag{60}
$$

where

$$
T^{\mathrm{lin}}_{rfw}
$$

$$
= \begin{bmatrix} S_{rfw} + E_x E_x^T - (Q_c - R_c)E_x^T - E_x(Q_c - R_c)^T & R_c & Q_c \\ R_c^T & -I & 0 \\ Q_c^T & 0 & -I \end{bmatrix}. \tag{61}
$$

To linearize the matrix inequality in (39), we need Lemma 7 and the following lemma, whose proof is similar to that of Lemma 7 and is therefore dropped.

Lemma 8. *Let $U_o \in \mathcal{R}^{m \times n}$ and $S = S^T \in \mathcal{R}^{m \times m}$. For $U \in \mathcal{R}^{m \times n}$, define*

$$
\begin{aligned}
\mathcal{M}(U) &= S + UU^T = \mathcal{M}(U)^T, \\
\mathcal{N}(U) &= S + UU_o^T + U_o U^T - U_o U_o^T = \mathcal{N}(U)^T.
\end{aligned} \tag{62}
$$

Then

$$
\begin{aligned}
\mathcal{N}(U) &\succ 0 \Longrightarrow \mathcal{M}(U) \succ 0, \\
\mathcal{M}(U_o) &\succ 0 \Longrightarrow \mathcal{N}(U_o) \succ 0.
\end{aligned} \tag{63}
$$

Let $P_{\mathrm{ref}_x} = P_{\mathrm{ref}_x}^T \in \mathcal{R}^{n_{\mathrm{ref}} \times n_{\mathrm{ref}}}$, $C_{\mathrm{ref}_x} \in \mathcal{R}^{n_f \times n_{\mathrm{ref}}}$, and $D_{\mathrm{ref}_x} \in \mathcal{R}^{n_f \times n_f}$ be given. Using (51) from Lemmas 7 and 8, it is easy to get

$$
T^{\mathrm{lin}}_{\mathrm{ro}} \succeq 0 \Longrightarrow T_{\mathrm{ro}} \succeq 0, \tag{64}
$$

where

$$
T^{\mathrm{lin}}_{\mathrm{ro}} = \begin{bmatrix} V & \star & \star & \star \\ V' & V'' & \star & \star \\ \mathbf{P}_{\mathrm{ref}} & 0 & I & \star \\ \mathbf{A}_{\mathrm{ref}} & \mathbf{B}_{\mathrm{ref}} & 0 & I \end{bmatrix}, \tag{65}
$$

where $V = \{(\mathbf{P}_{\mathrm{ref}} + \mathbf{A}_{\mathrm{ref}}{}^T)(P_{\mathrm{ref}_x} + A_{\mathrm{ref}_x}) + \star - C_{\mathrm{ref}_x}^T C_{\mathrm{ref}_x} + \mathbf{C}_{\mathrm{ref}}{}^T C_{\mathrm{ref}_x} + \star - (P_{\mathrm{ref}_x} + A_{\mathrm{ref}_x}^T)(P_{\mathrm{ref}_x} + A_{\mathrm{ref}_x})\}$, $V' = \{\mathbf{D}_{\mathrm{ref}}^T C_{\mathrm{ref}_x} +$

$D_{\mathrm{ref}_x}^T \mathbf{C}_{\mathrm{ref}} - B_{\mathrm{ref}_x}^T(P_{\mathrm{ref}_x} + A_{\mathrm{ref}_x}) - D_{\mathrm{ref}_x}^T C_{\mathrm{ref}_x} + \mathbf{B}_{\mathrm{ref}}^T(P_{\mathrm{ref}_x} + A_{\mathrm{ref}_x}) + B_{\mathrm{ref}_x}^T(\mathbf{P}_{\mathrm{ref}} + \mathbf{A}_{\mathrm{ref}})\}$, $V'' = \{\mathbf{D}_{\mathrm{ref}}{}^T D_{\mathrm{ref}_x} + D_{\mathrm{ref}_x}^T \mathbf{D}_{\mathrm{ref}} - B_{\mathrm{ref}_x}^T B_{\mathrm{ref}_x} - D_{\mathrm{ref}_x}^T D_{\mathrm{ref}_x} - I + \mathbf{B}_{\mathrm{ref}}^T B_{\mathrm{ref}_x} + \star\}$. The next lemma summarizes the results of this section by giving a linearized formulation of the optimization problem defined in Lemma 4 using (60) and (64).

Lemma 9. *Assume that Δ is unstructured. Let $Z_x = Z_x^T \in \mathcal{R}^{n_k \times n_k}$, $Y_x = Y_x^T \in \mathcal{R}^{n_k \times n_k}$, $P_{\mathrm{ref}_x} = P_{\mathrm{ref}_x}^T \in \mathcal{R}^{n_{\mathrm{ref}} \times n_{\mathrm{ref}}}$, $A_{\mathrm{ref}_x} \in \mathcal{S}_A$, $B_{\mathrm{ref}_x} \in \mathcal{S}_B$, $C_{\mathrm{ref}_x} \in \mathcal{S}_C$, and $D_{\mathrm{ref}_x} \in \mathcal{S}_D$ be given. Then there exist a stable filter of the form given in (8) and a stable fault reference dynamics model $T_{\mathrm{ref}} \stackrel{s}{=} (\mathbf{A}_{\mathrm{ref}}, \mathbf{B}_{\mathrm{ref}}, \mathbf{C}_{\mathrm{ref}}, \mathbf{D}_{\mathrm{ref}}) \in \mathcal{S}$, where $\mathcal{S}$ is defined in (11), such that $\|T_{\mathrm{ref}}\|_- \geq 1$, residual dynamics in (10) are quadratically stable, and (12) is satisfied for $\mathcal{M} = \mathcal{M}_\Delta$ if there exist $\mathbf{A}_{\mathrm{ref}} \in \mathcal{S}_A$, $\mathbf{B}_{\mathrm{ref}} \in \mathcal{S}_B$, $\mathbf{C}_{\mathrm{ref}} \in \mathcal{S}_C$, $\mathbf{D}_{\mathrm{ref}} \in \mathcal{S}_D$, $\hat{\mathbf{A}}$, $\hat{\mathbf{B}}_y$, $\hat{\mathbf{B}}_u$, $\hat{\mathbf{C}}$, $\hat{\mathbf{D}}_y$, $\hat{\mathbf{D}}_u$, λ, and symmetric matrices $\mathbf{P}_{\mathrm{ref}}$, $\mathbf{Y}$, and $\mathbf{Z}$ such that*

$$
T^{\mathrm{lin}}_{rfw} \prec 0, \tag{66}
$$

$$
T^{\mathrm{lin}}_{\mathrm{ro}} \succeq 0, \tag{67}
$$

$$
\begin{bmatrix} \mathbf{Z} & \mathbf{Z} \\ \mathbf{Z} & \mathbf{Y} \end{bmatrix} \succ 0. \tag{68}
$$

Remark 3. This scheme can also be applied to Lemmas 5 and 6 to obtain a suboptimal solution involving linear matrix inequalities.

Next, we need to choose the initial parameters (Z_x, Y_x, P_{ref_x}, A_{ref_x}, B_{ref_x}, C_{ref_x}, and D_{ref_x}) to reduce γ. The idea is to derive an algorithm where at each iteration, we solve the optimization problem given in Lemma 9, using the solution of this problem at the previous iteration, for initial parameters. The algorithm will use initial values Z_x^{init}, Y_x^{init}, $P_{\mathrm{ref}_x}^{\mathrm{init}}$, $A_{\mathrm{ref}_x}^{\mathrm{init}}$, $B_{\mathrm{ref}_x}^{\mathrm{init}}$, $C_{\mathrm{ref}_x}^{\mathrm{init}}$, and $D_{\mathrm{ref}_x}^{\mathrm{init}}$, which must guarantee that the LMI constraints (66) and (67) will be feasible.

From the above discussion, an algorithm for choosing the initial parameters can be listed as follows.

Algorithm 1. (1) Set initial values $A_{\mathrm{ref}_x}^{\mathrm{init}}$, $B_{\mathrm{ref}_x}^{\mathrm{init}}$, $C_{\mathrm{ref}_x}^{\mathrm{init}}$, and $D_{\mathrm{ref}_x}^{\mathrm{init}}$ such that $T_{\mathrm{ref}}^{\mathrm{init}} \stackrel{s}{=} (A_{\mathrm{ref}_x}^{\mathrm{init}}, B_{\mathrm{ref}_x}^{\mathrm{init}}, C_{\mathrm{ref}_x}^{\mathrm{init}}, D_{\mathrm{ref}_x}^{\mathrm{init}})$ satisfies $\|T_{\mathrm{ref}}^{\mathrm{init}}\|_- \geq 1$ and $T_{\mathrm{ref}}^{\mathrm{init}} \in \mathcal{S}$.

(2) Find the solutions $\mathbf{Z}$, $\mathbf{Y}$, and $\mathbf{P}_{\mathrm{ref}}$ of the optimization derived from Lemmas 3 and 4, which is linear since the matrix inequalities (37), (38), and (39) become linear when $T_{\mathrm{ref}} := T_{\mathrm{ref}}^{\mathrm{init}}$ is fixed. (The matrices $\mathbf{Z}$, $\mathbf{Y}$, and $\mathbf{P}_{\mathrm{ref}}$ always exist since the cost function in (12) is bounded because Δ is bounded).

(3) Set $Z_x^{\mathrm{init}} := \mathbf{Z}$, $Y_x^{\mathrm{init}} := \mathbf{Y}$, and $P_{\mathrm{ref}_x}^{\mathrm{init}} := \mathbf{P}_{\mathrm{ref}}$.

(4) Start loop.

(5) Since the initial values are feasible for (37), (38), and (39), the LMIs (66) to (68) have feasible solutions from (50) and (63) in Lemmas 7 and 8. Compute solutions ($\mathbf{Z}$, $\mathbf{Y}$, etc.) of (66) to (68) to minimize γ.

(6) Rename $Z_x^{\mathrm{init}} = \mathbf{Z}$, $Y_x^{\mathrm{init}} = \mathbf{Y}$, $P_{\mathrm{ref}_x}^{\mathrm{init}} = \mathbf{P}_{\mathrm{ref}}$, $A_{\mathrm{ref}_x}^{\mathrm{init}} = \mathbf{A}_{\mathrm{ref}}$, $B_{\mathrm{ref}_x}^{\mathrm{init}} = \mathbf{B}_{\mathrm{ref}}$, $C_{\mathrm{ref}_x}^{\mathrm{init}} = \mathbf{C}_{\mathrm{ref}}$, and $D_{\mathrm{ref}_x}^{\mathrm{init}} = \mathbf{D}_{\mathrm{ref}}$, and go to Step 5.

(7) End loop.

Algorithm 1 is convergent, possibly to a local minimum, in the sense that the quantity γ is nonincreasing after each iteration. This can be easily shown using (50) and (52) from Lemma 7, and (63) from Lemma 8.

Remark 4. Lemmas 7, 8, and Algorithm 1 can also be applied to other problems involving QMIs and bilinear matrix inequalities (BMIs). The procedure potentially gives an improvement and seems to work well in practice.

Remark 5. In the case that we choose a diagonal structure for T_{ref}, we may use

$$
\begin{aligned}
A_{\text{ref}}^{\text{init}} &= -I_{n_{\text{ref}}}, & B_{\text{ref}}^{\text{init}} &= 0_{n_{\text{ref}} \times n_f}, \\
C_{\text{ref}}^{\text{init}} &= 0_{n_f \times n_{\text{ref}}}, & D_{\text{ref}}^{\text{init}} &= I_{n_f},
\end{aligned}
\tag{69}
$$

as initial values. This will ensure that T_{ref} is stable and $\|T_{\text{ref}}\|_{-} = 1$. We can solve the LMI optimization problem given in Lemma 4. This will give Z_x^{init}, Y_x^{init}, $P_{\text{ref}_x}^{\text{init}}$, and so forth. These initial values are not unique and can be chosen using other criteria.

Remark 6. A more systematic technique for generating valid initial values is as follows: first, generate any $B_{\text{ref}}^{\text{init}}$, $C_{\text{ref}}^{\text{init}}$, $D_{\text{ref}}^{\text{init}}$, and a stable $A_{\text{ref}}^{\text{init}}$ with the structure chosen, then, compute $\alpha = \|T_{\text{ref}}^{\text{init}}\|_{-}$. If $\alpha > 0$, redefine the matrices as $A_{\text{ref}}^{\text{init}} = A_{\text{ref}}^{\text{init}}/\alpha$, $B_{\text{ref}}^{\text{init}} = B_{\text{ref}}^{\text{init}}/\alpha$, $C_{\text{ref}}^{\text{init}} = C_{\text{ref}}^{\text{init}}/\alpha$, and $D_{\text{ref}}^{\text{init}} = D_{\text{ref}}^{\text{init}}/\alpha$, which fulfill that the conditions $T_{\text{ref}}^{\text{init}}$ are stable and $\|T_{\text{ref}}^{\text{init}}\|_{-} = 1$.

Remark 7. The requirement for T_{ref} to be diagonal is to ensure fault isolability in the case that all faults may occur simultaneously. If we ignore disturbances and uncertainty, and assuming perfect fault isolation, then $r = T_{\text{ref}} f$ so that fault f_i only affects residual r_i. If none of the faults can occur simultaneously, then we need only to impose the constraint that T_{ref} is upper (or lower) triangular. While it is not difficult to modify our analysis under these conditions, we only consider the case when T_{ref} is diagonal so that all faults may occur simultaneously since our contribution is focussed on reducing the effect of disturbances and uncertainties under the most stringent fault scenarios.

5. NUMERICAL EXAMPLE

In order to illustrate the efficiency of Algorithm 1 and the importance of the choice of T_{ref}, we consider a jet engine state-space model [32] supplied by NASA Glenn Research Center given as

$$
\begin{aligned}
\dot{x}(t) &= (A^o + F_A \Delta_H E_A)x(t) + (B^o + F_A \Delta_H E_B)u(t) \\
&\quad + B_d d(t) + B_f f(t), \\
y(t) &= (C^o + F_C \Delta_H E_A)x(t) + (D^o + F_C \Delta_H E_B)u(t) \\
&\quad + D_d d(t) + D_f f(t),
\end{aligned}
\tag{70}
$$

where

$$
\begin{aligned}
A^o &= \begin{bmatrix} -0.9835 & -0.0110 & -0.0039 \\ -0.0004 & -0.9858 & -0.0026 \\ 0 & 0.0002 & -0.9891 \end{bmatrix}, \\
B^o &= \begin{bmatrix} 0.0080 & 0.2397 & -0.0383 \\ 0.0068 & 0.1565 & 0.0248 \\ 0.0003 & -0.0003 & 0.0003 \end{bmatrix}, \\
C^o &= \begin{bmatrix} 0.2383 & 0.4871 & 0.1390 \\ 0 & -.0008 & 0.0004 \\ 0.00002 & -0.00004 & 0 \end{bmatrix}, \\
D^o &= \begin{bmatrix} 0.4171 & -4.4920 & 0.4875 \\ 0.0008 & -0.0050 & 0.0003 \\ 0 & 0.0005 & -0.0021 \end{bmatrix}.
\end{aligned}
\tag{71}
$$

The system is subject to three disturbances and three potential faults. Here, the setup is given by

$$
\begin{aligned}
B_d &= \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.01 \end{bmatrix}, \\
B_f &= \begin{bmatrix} 0.0080 & 0.2397 & -0.0383 \\ 0.0068 & 0.1565 & 0.0248 \\ 0.0003 & -0.0003 & 0.0003 \end{bmatrix}, \\
D_d &= 0.1 \times I_3, \\
D_f &= \begin{bmatrix} -0.0205 & 0.6217 & 0.8115 \\ 0.2789 & -1.7506 & 0.6363 \\ 1.0583 & 0.6973 & 1.3101 \end{bmatrix}.
\end{aligned}
\tag{72}
$$

Since no uncertainty parameters were given in this example, we assume an unstructured norm-bounded uncertainty, with matrices F_A, F_C, E_A, E_B, and H randomly generated as

$$
\begin{aligned}
H &= \begin{bmatrix} -0.319 & -0.080 & 0.142 \\ -0.288 & 0.138 & 0.258 \\ 0.114 & 0.163 & 0.133 \end{bmatrix}, \\
E_A &= \begin{bmatrix} -0.060 & -0.008 & -0.135 \\ -0.015 & 0.154 & 0.047 \\ -0.044 & -0.061 & -0.090 \end{bmatrix}, \\
E_B &= \begin{bmatrix} 0.134 & 0.630 & 0.451 \\ 0.207 & 0.371 & 0.044 \\ 0.607 & 0.575 & 0.027 \end{bmatrix}, \\
F_C &= \begin{bmatrix} -0.055 & 0.066 & -0.012 \\ -0.085 & -0.085 & -0.007 \\ -0.025 & -0.120 & 0.049 \end{bmatrix}, \\
F_A &= \begin{bmatrix} 0.148 & -0.129 & -0.084 \\ 0.114 & -0.007 & 0.050 \\ -0.068 & -0.033 & 0.149 \end{bmatrix}.
\end{aligned}
\tag{73}
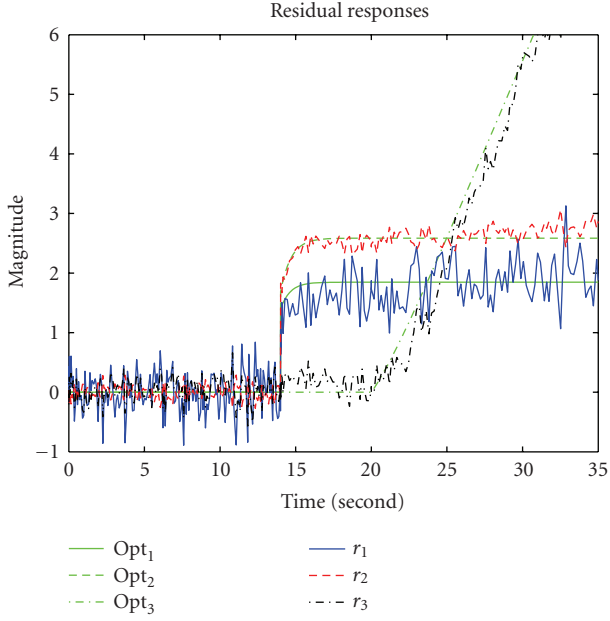$$

Residual responses

Figure 2: Time response of the residual before the optimization.

A square and diagonal structure for T_{ref} is necessary to get fault isolation in the case that the faults occur at the same time.

Remark 8. If A_{ref}, B_{ref}, and C_{ref} are chosen diagonal, then

$$T_{\text{ref}}(s) = C_{\text{ref}}(sI - A_{\text{ref}})^{-1}B_{\text{ref}} + D_{\text{ref}}$$
$$= C_{\text{ref}}B_{\text{ref}}(sI - A_{\text{ref}})^{-1} + D_{\text{ref}}. \tag{74}$$

The terms in B_{ref} can be incorporated in C_{ref}. It follows that we can set $B_{\text{ref}} = I$. Therefore, the nonlinearity in (37) comes only from bilinear terms $\mathbf{Z}E_1\mathbf{A}_{\text{ref}}E_1^T$ and $\mathbf{Y}E_1\mathbf{A}_{\text{ref}}E_1^T$.

To initialize Algorithm 1, $T_{\text{ref}}^{\text{init}}$ is generated following Remark 6 as

$$A_{\text{ref}}^{\text{init}} = \begin{bmatrix} -1.60 & 0 & 0 \\ 0 & -1.44 & 0 \\ 0 & 0 & -0.57 \end{bmatrix},$$

$$B_{\text{ref}}^{\text{init}} = I_3,$$

$$C_{\text{ref}}^{\text{init}} = \begin{bmatrix} 0.71 & 0 & 0 \\ 0 & 1.29 & 0 \\ 0 & 0 & 0.67 \end{bmatrix}, \tag{75}$$

$$D_{\text{ref}}^{\text{init}} = \begin{bmatrix} 1.40 & 0 & 0 \\ 0 & 1.69 & 0 \\ 0 & 0 & 1.81 \end{bmatrix}.$$

Then, by following the first two steps of Algorithm 1, we get the optimal γ as

$$\gamma = 0.6273 \tag{76}$$

and the values given below for Z, Y, and P_{ref_x}, which will be used to initialize the main loop of the algorithm as follows.

Z_{ref}

$$= \begin{bmatrix} 0.5010 & -0.4385 & 0.0309 & 0.6192 & -0.4313 & 0.4861 \\ -0.4385 & 0.8184 & 0.0844 & -0.3629 & -0.6083 & -0.8105 \\ 0.0309 & 0.0844 & 0.0651 & 0.0986 & -0.3323 & -0.1397 \\ 0.6192 & -0.3629 & 0.0986 & 1.4722 & -1.2716 & 0.9082 \\ -0.4313 & -0.6083 & -0.3323 & -1.2716 & 5.7926 & -1.5337 \\ 0.4861 & -0.8105 & -0.1397 & 0.9082 & -1.5337 & 2.5593 \end{bmatrix},$$

Y_{ref}

$$= \begin{bmatrix} 5.0059 & 19.1311 & 2.7518 & 0.1860 & 3.0607 & 0.7016 \\ 19.1311 & 165.3233 & 12.5435 & -13.2886 & 32.4525 & -15.3595 \\ 2.7518 & 12.5435 & 7.7249 & -4.8209 & 0.6873 & -3.4439 \\ 0.1860 & -13.2886 & -4.8209 & 7.4800 & -5.3595 & 4.9329 \\ 3.0607 & 32.4525 & 0.6873 & -5.3595 & 22.7920 & -4.1575 \\ 0.7016 & -15.3595 & -3.4439 & 4.9329 & -4.1575 & 8.6769 \end{bmatrix},$$

$$P_{\text{ref}_x} = \begin{bmatrix} -0.0492 & 1.4322 & -0.1444 \\ 1.4322 & -63.1246 & 8.1628 \\ -0.1444 & 8.1628 & -1.2084 \end{bmatrix}. \tag{77}$$

Taking into account Remark 8, we implemented Algorithm 1 in Matlab to minimize γ. Table 1 shows the evolution of the optimization following Algorithm 1.

Algorithm 1 can clearly improve the result in a few iterations; γ is reduced by 33% compared to the one obtained with a fixed T_{ref}. This shows that the choice of T_{ref} is essential in our FDI scheme. We get

$$A_{\text{ref}} = \begin{bmatrix} -18.1665 & 0 & 0 \\ 0 & -9.2781 & 0 \\ 0 & 0 & -6.0914 \end{bmatrix},$$

$$B_{\text{ref}} = I_3,$$

$$C_{\text{ref}} = \begin{bmatrix} 0.0007 & 0 & 0 \\ 0 & -0.3959 & 0 \\ 0 & 0 & 0.0572 \end{bmatrix}, \tag{78}$$

$$D_{\text{ref}} = \begin{bmatrix} 1.2438 & 0 & 0 \\ 0 & 4.3768 & 0 \\ 0 & 0 & 1.8216 \end{bmatrix}.$$

Remark 9. In our numerical experimentation, other choices for $T_{\text{ref}}^{\text{init}}$ have been used; however, all converged to the same solution but with different numbers of iterations.

In order to show that our filter is robust against disturbances and model uncertainties, we introduce a randomly generated Δ given by

$$\Delta = \begin{bmatrix} 0.4058 & -0.0534 & -0.3600 \\ -0.4097 & -0.5466 & 0.4822 \\ -0.0067 & 0.0877 & -0.2743 \end{bmatrix}, \tag{79}$$

TABLE 1

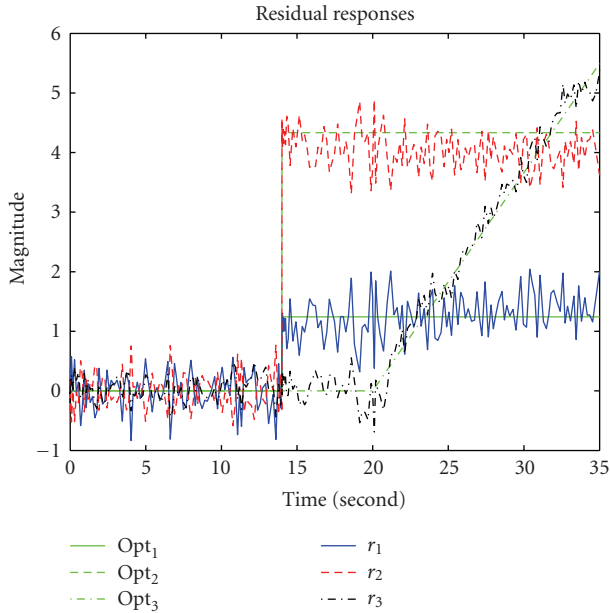| Iterations	0	1	2	5	10	20	50	100
γ	0.6273	0.6248	0.5791	0.5476	0.5253	0.4739	0.4641	0.4641

FIGURE 3: Time response of the residual after the optimization.

as well as three disturbances. Simulated through MATLAB and SIMULINK, these disturbances are three-band-limited white noises with mean 0 and standard deviation 2. Faults f_1 and f_2 are both simulated by a unit positive jump connected from the 14th second. Fault f_3, simulated by a soft bias (slope = 0.2), is connected from the 20th second. Figure 2 gives the residual responses before the algorithm, where each residual is dedicated to a particular fault, while Figure 3 gives the optimized residual response using our algorithm.

The lines opt_1, opt_2, and opt_3 represent the optimal trajectories that each residual must follow

$$\begin{bmatrix} \mathrm{opt}_1(s) \\ \mathrm{opt}_2(s) \\ \mathrm{opt}_3(s) \end{bmatrix} = T_{\mathrm{ref}} \begin{bmatrix} f_1(s) \\ f_2(s) \\ f_3(s) \end{bmatrix}. \tag{80}$$

In both figures, each fault can be distinguished from the others and the disturbances; however, in Figure 3, the faults can be more easily distinguished and each residual follows its optimal trajectory (green line) with more accuracy. Furthermore, the disturbances are more attenuated compared to Figure 2, and the jumps that indicate faults are clearer in Figure 3 since their amplitudes are bigger and therefore allow a better fault detection using thresholds [33, 34]. The isolation performance is clearly effective as each fault produces a deviation of its residual only, without modifying the trajectory of the others. This example illustrates that the designed filter satisfies the performance requirement of FDI which is sufficiently robust against disturbances and modeling errors.

Figure 3 also justifies the efficiency of Algorithm 1 to improve the design of the reference model and therefore the overall performance of our filter.

6. CONCLUSION

This paper has addressed the problem of fault detection and isolation for linear time-invariant systems subject to faults, disturbances, and model uncertainties. We proposed a performance index that captures the FDI requirements. Through QMI formulations, we gave the design of an optimal filter for polytopic or unstructured norm-bounded uncertainties, and a suboptimal filter for structured norm-bounded uncertainties. Suboptimality in the latter case is inherited from the bounded real lemma, which gives only sufficient LMI conditions for structured uncertainties (see Lemma 2). By allowing the reference model T_{ref} to be variable in our formulation, we get its optimal design, which can be used in other schemes dedicated to fault isolation. The optimal design of this reference model is initially given in a form of a QMI optimization, then a suboptimal solution is obtained by using a linearization procedure which derives an upper bound on the optimal solution of the QMI formulation that requires the solution of an LMI optimization. Note, however, that we have no indication concerning the deviation of our design from the optimal filter. We have also illustrated the effectiveness of our algorithm using a jet-engine example.

ACKNOWLEDGMENT

REFERENCES

[1] Z. Gao and H. Wang, "Descriptor observer approaches for multivariable systems with measurement noises and application in fault detection and diagnosis," *Systems and Control Letters*, vol. 55, no. 4, pp. 304–313, 2006.

[2] C. Jiang and D. H. Zhou, "Fault detection and identification for uncertain linear time-delay systems," *Computers and Chemical Engineering*, vol. 30, no. 2, pp. 228–242, 2005.

[3] M. Fu, C. E. De Souza, and Z.-Q. Luo, "Finite-horizon robust kalman filter design," *IEEE Transactions on Signal Processing*, vol. 49, no. 9, pp. 2103–2112, 2001.

[4] K. M. Nagpal and P. P. Khargonekar, "Filtering and smoothing in an $\mathcal{H}_\infty$ setting," *IEEE Transactions on Automatic Control*, vol. 36, no. 2, pp. 152–166, 1991.

[5] A. H. Sayed, "A framework for state-space estimation with uncertain models," *IEEE Transactions on Automatic Control*, vol. 46, no. 7, pp. 998–1013, 2001.

[6] S. H. Zad and M.-A. Massoumnia, "Generic solvability of the failure detection and identification problem," *Automatica*, vol. 35, no. 5, pp. 887–893, 1999.

[7] R. J. Patton and J. Chen, "On eigenstructure assignment for robust fault diagnosis," *International Journal of Robust and Nonlinear Control*, vol. 10, no. 14, pp. 1193–1208, 2000.

[8] P. Zhang, H. Ye, S. X. Ding, G. Z. Wang, and D. H. Zhou, "On the relationship between parity space and $\mathcal{H}_2$ approaches to fault detection," *Systems and Control Letters*, vol. 55, no. 2, pp. 94–100, 2006.

[9] J. Chen, R. J. Patton, and H.-Y. Zhang, "Design of unknown input observers and robust fault detection filters," *International Journal of Control*, vol. 63, no. 1, pp. 85–105, 1996.

[10] I. M. Jaimoukha, Z. Li, and E. F.M. Mazars, "Linear matrix inequality solution to the $\mathcal{H}_-/\mathcal{H}_\infty$, fault detection problem," in *Proceedings of the 7th International Conference on Control and Applications, the International Association of Science and Technology for Development (IASTED '05)*, pp. 183–188, Cancun, Mexico, May 2005.

[11] I. M. Jaimoukha, Z. Li, and E. Mazars, "Fault isolation filter with linear matrix inequality solution to optimal decoupling," in *Proceedings of the American Control Conference (ACC '06)*, vol. 2006, pp. 2339–2344, Minneapolis, Minn, USA, 2006.

[12] Z. Li, E. Mazars, and I. M. Jaimoukha, "State space solution to the $\mathcal{H}_-/\mathcal{H}_\infty$ fault detection problem.," in *Proceedings of the IEEE 45th Conference on Decision and Control (CDC '06)*, pp. 2177–2182, San Diego, Calif, USA, December 2006.

[13] E. Frisk and L. Nielsen, "Robust residual generation for diagnosis including a reference model for residual behavior," in *Proceedings of the 14th World Congress International Federation of Automatic Control (IFAC '99)*, pp. 55–60, Beijing, China, July 1999.

[14] M. Zhong, S. X. Ding, J. Lam, and H. Wang, "An LMI approach to design robust fault detection filter for uncertain LTI systems," *Automatica*, vol. 39, no. 3, pp. 543–550, 2003.

[15] E. Frisk and L. Nielsen, "Robust residual generation for diagnosis including a reference model for residual behavior," *Automatica*, vol. 42, no. 3, pp. 437–445, 2006.

[16] E. Mazars, I. M. Jaimoukha, and Z. Li, "Linear matrix solutions to a robust fault detection and isolation problem," under review in *European Journal of Control*.

[17] D. J. Clements, B. D.O. Anderson, A. J. Laub, and J. B. Matson, "Spectral factorization with imaginary-axis zeros," *Linear Algebra and Its Applications*, vol. 250, pp. 225–252, 1997.

[18] P. M. Frank and X. Ding, "Survey of robust residual generation and evaluation methods in observer-based fault detection systems," *Journal of Process Control*, vol. 7, no. 6, pp. 403–424, 1997.

[19] S. P. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in Systems and Control Theory*, SIAM, Philadelphia, Pa, USA, 1994.

[20] W. H. Chung and J. L. Speyer, "A game theoretic fault detection filter," *IEEE Transactions on Automatic Control*, vol. 43, no. 2, pp. 143–161, 1998.

[21] R. K. Douglas and J. L. Speyer, "Robust fault detection filter design," in *Proceedings of the American Control Conference (ACC '95)*, vol. 1, pp. 91–96, Seattle, Wash, USA, June 1995.

[22] I. M. Jaimoukha, Z. Li, and V. Papakos, "A matrix factorization solution to the $\mathcal{H}_-/\mathcal{H}_\infty$ fault detection problem," *Automatica*, vol. 42, no. 11, pp. 1907–1912, 2006.

[23] H. B. Wang, L. Lam, S. X. Ding, and M. Y. Zhong, "Iterative linear matrix inequality algorithms for fault detection with unknown inputs," *Journal of Systems and Control Engineering*, vol. 219, no. 2, pp. 161–172, 2005.

[24] H. D. Tuan, P. Apkarian, and T. Q. Nguyen, "Robust filtering for uncertain nonlinearly parameterized plants," *IEEE Transactions on Signal Processing*, vol. 51, no. 7, pp. 1806–1815, 2003.

[25] L. El Ghaoui and J. P. Folcher, "Multiobjective robust control of LTI systems subject to unstructured perturbations," *Systems and Control Letters*, vol. 28, no. 1, pp. 23–30, 1996.

[26] C. Scherer, P. Gahinet, and M. Chilali, "Multi-objective output-feedback control via LMI optimization," *IEEE Transactions on Automatic Control*, vol. 42, no. 7, pp. 896–911, 1997.

[27] K.-C. Goh, M. G. Safonov, and G. P. Papavassilopoulos, "Global optimization approach for the BMI problem," in *Proceedings of the IEEE Conference on Decision and Control (CDC '94)*, vol. 3, pp. 2009–2014, Lake Buena Vista, Fla, USA, 1994.

[28] Yamada, Hara, and Fujioka, "Global optimization for constantly scaled $\mathcal{H}_\infty$ control problem," in *Proceedings of the American Control Conference (ACC '95)*, vol. 1, pp. 427–430, Seattle, Wash, USA, June 1995.

[29] J. C. Geromel, P. L. D. Peres, and S. R. De Souza, "Output feedback stabilization of uncertain systems through a min/max problem," in *Proceedings of the 12th World Congress International Federation of Automatic Control (IFAC '93)*, vol. 6, pp. 35–38, The Institute of Engineers, Sydney, Australia, July 1993, in preprints of papers.

[30] E. Beran and K. Grigoriadis, "A combined alternating projections and semidefinite programming algorithm for low-order control design," in *Proceedings of the 13th World Congress, International Federation of Automatic Control (IFAC '96)*, pp. 85–90, San Francisco, Calif, USA, July 1996.

[31] A. Saberi, P. Sannuti, and B. M. Chen, H_2 *optimal control system and control engineering*, Prentice Hall, London, UK, 1995.

[32] T. D. Curry and E. G. Collins Jr., "Robust fault detection and isolation using robust ℓ 1 estimation," *Journal of Guidance, Control, and Dynamics*, vol. 28, no. 6, pp. 1131–1139, 2005.

[33] X. C. Ding and P. M. Frank, "Frequency domain approach and threshold selector for robust model-based fault detection and isolation," in *IFAC/IMACS Symposium on Fault Detection, Supervision and Safety for Technical Processes—SAFEPROCESS '91 (IFAC '92)*, vol. 6, pp. 271–276, Baden-Baden, Germany, 1992.

[34] F. Rambeaux, F. Hamelin, and D. Sauter, "Optimal thresholding for robust fault detection of uncertain systems," *International Journal of Robust and Nonlinear Control*, vol. 10, no. 14, pp. 1155–1173, 2000.

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Since its conception in the mid 1950s, artificial intelligence with its great ambition to understand intelligence, its origin and creation, in natural and artificial environments alike, has been a truly multidisciplinary field that reaches out and is inspired by a great diversity of other fields in perpetual motion. Rapid advances in research and technology in various fields have created environments into which artificial intelligence could embed itself naturally and comfortably. Neuroscience with its desire to understand nervous systems of biological organisms and system biology with its longing to comprehend, holistically, the multitude of complex interactions in biological systems are two such fields. They target ideals artificial intelligence has dreamt about for a long time including the computer simulation of an entire biological brain or the creation of new life forms from manipulations on cellular and genetic information in the laboratory.

The scope for artificial intelligence, neuroscience, and systems biology is extremely wide. The motivation of this special issue is to create a bird-eye view on areas and challenges where these fields overlap in their defining ambitions and where these fields may benefit from a synergetic mutual exchange of ideas. The rationale behind this special issue is that a multidisciplinary approach in modern artificial intelligence, neuroscience, and systems biology is essential and that progress in these fields requires a multitude of views and contributions from a wide spectrum of contributors. This special issue, therefore, aims to create a centre of gravity pulling together researchers and industry practitioners from a variety of areas and backgrounds to share results of current research and development and to discuss existing and emerging theoretical and applied problems in artificial intelligence, neuroscience, and systems biology transporting them beyond the event horizon of their individual domains.

Before submission authors should carefully read over the journal's Author Guidelines, which are located at http://www.hindawi.com/journals/aai/guidelines.html. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/ according to the following timetable:

Manuscript Due	September 1, 2009
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Special Issue on

Selected Papers from Workshop on Synergies in Communications and Localization (SyCoLo 2009)

Call for Papers

In conjunction with the IEEE International Conference on Communications (ICC) 2009 in Dresden, Germany, the International Workshop on Synergies in Communications and Localization (SyCoLo 2009) will be held.

The main objective of this workshop is to show how wireless communications and navigation/localization techniques can benefit from each other. With respect to these synergies the workshop aims at the following fundamental questions:

- How can navigation systems benefit from existing communications systems?
- How can communication systems benefit from positioning information of mobile terminals?

This workshop, whose proposal was jointly generated by the EU Research Projects WHERE and NEWCOM++, aims at inspiring the development of new position-aware procedures to enhance the efficiency of communication networks, and of new positioning algorithms based both on (outdoor or indoor) wireless communications and on satellite navigation systems.

The SyCoLo 2009 is, therefore, well in agreement with the new IJNO journal aims at promoting and diffusing the aims of joint communications and navigation among universities, research institutions, and industries.

This proposed IJNO Special Issue focuses all the research themes related to the timing aspects of joint communications and navigation, and starts from the SyCoLo 2009 where the Guest Editors will attend the different sessions and directly invite the authors of the most promising papers to submit an extended version of their papers to the journal.

The proposed Guest Editors are also part of the Scientific Committees of the SyCoLo 2009, therefore, directly involved in the evaluation of submitted papers.

Topics of interest will include, but are not limited to:

- Hybrid positioning using both wireless communications and satellite navigation systems
- Resource management with positioning information
- Location-aware PHY/MAC algorithms/procedures
- Indoor positioning combined with short-range communications
- Signal processing techniques for (seamless) indoor/ outdoor localization

Before submission authors should carefully read over the journal's Author Guidelines, which are located at http://www .hindawi.com/journals/ijno/guidelines.html. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/ according to the following timetable:

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