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THE MAXIMUM SIZE OF A CRITICAL 2-EDGE-CONNECTED GRAPH

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§ 1. INTRODUCTION

Let G be a simple connected graph. The vertex set of G is denoted by $V(G)$ and the edge set by $E(G)$; they will be abbreviated to V and E if there is no danger of ambiguity. The number of vertices in G, called the order of G, is denoted by $p(G)$ and the number of edges in G, called the size of G, is denoted by $q(G)$. Similarly, they will be abbreviated to p, q respectively. We denote by xy the edge joining vertices x and y.

If V_1, V_2 are two vertex disjoint subsets in V, we denote by V_1V_2 the edge set $\{uv \in E \mid u \in V_1, v \in V_2\}$. If $\{V_1, V_2\}$ is a partition of V (i.e. $V_1 \cup V_2 = V$, $V_1 \cap V_2 = \phi$), and $|V_1V_2| = 1$, then the unique edge in V_1V_2 is called a bridge in G.

If $V_1 \subseteq V$, the subgraph generated by V_1 is denoted by $G(V_1)$. Particularly, if $v \in V$, we write $G - v$ instead of $G(V \backslash \{v\})$. If $G - v$ is disconnected, the vertex v is called a cut-vertex in G.

We denote the set of vertices adjacent to v by $N(v)$. The number of vertices in $N(v)$ is called the degree of v, denoted by $d(v)$. If $d(v) = k$ we call v a k-vertex.

Suppose that G is a 2-edge-connected graph. A vertex v is said to be critical if $G - v$ either is disconnected or has at least one bridge. G is said to be critical 2-edge-connected if every vertex of G is critical.

In this paper we give the maximum size of a critical 2-edge-connected graph of order p, and construct the graphs with maximum size.

§ 2. MAIN THEOREM

Theorem. *If $f(p)$ denotes the maximum size of a critical 2-edge-connected graph of order p, then*

$$f(p) = \begin{cases} 7, & p = 6; \\ \frac{1}{8}(p^2 + 4p), & p \equiv 0 \pmod 4; \\ \frac{1}{8}(p^2 + 2p + 13), & p \equiv 1 \pmod 4; \\ \frac{1}{8}(p^2 + 28), & p \equiv 2 \pmod 4,\ p \neq 6; \\ \frac{1}{8}(p^2 + 2p + 9), & p \equiv 3 \pmod 4. \end{cases}$$

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The graphs exhibited in Fig. 1 are critical 2-edge-connected with maximum size.

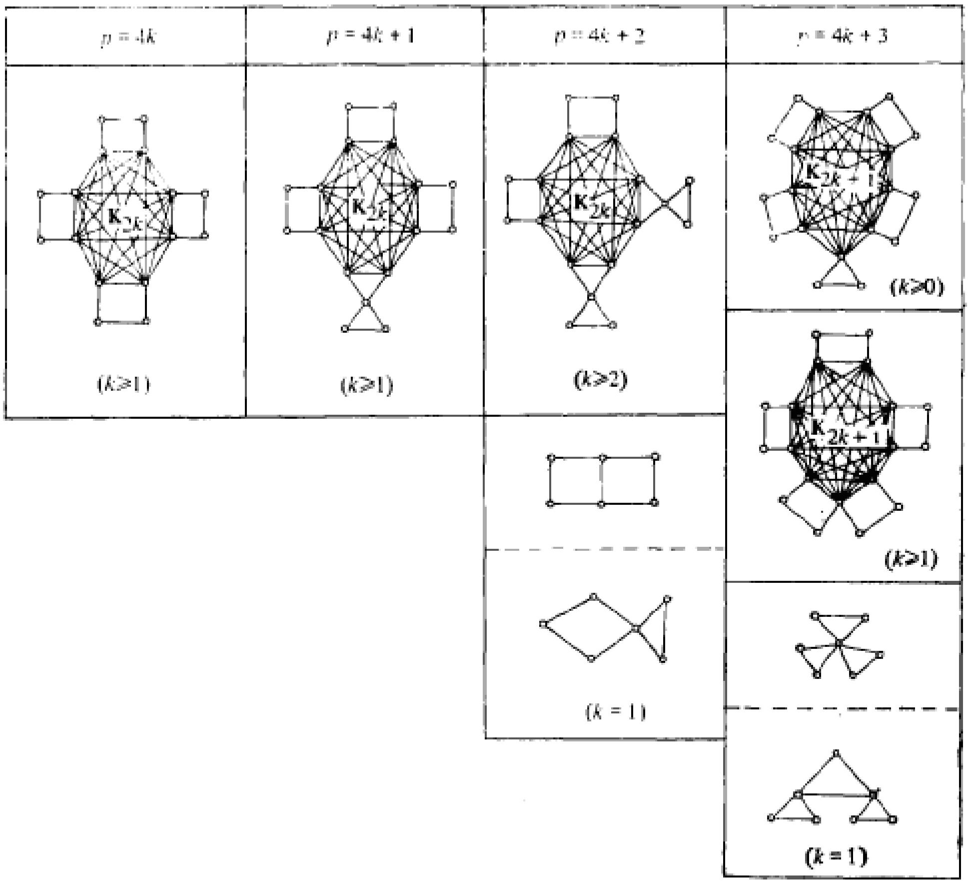

Fig. 1.

Proof. (1) Suppose that G is a critical 2-edge-connected graph of order p. When $p \leqslant 6$, the assertion of the theorem can be verified directly. So we shall deal with $p \geqslant 7$.

From the critical 2-edge-connectedness of G, it follows that

(A) $d(v) \geqslant 2$ for any $v \in V$;

(B) there is at least one partition $\{V'_v, V''_v\}$ of $V \setminus \{v\}$ such that $|V'_v V''_v| \leqslant 1$ for any $v \in V$.

It can be seen easily that the partition $\{V'_v, V''_v\}$ has the following properties:

(a) $N(v) \cap V'_v \neq \phi$, $N(v) \cap V''_v \neq \phi$;

(b) $G(V'_v \cup \{v\})$, $G(V''_v \cup \{v\})$ are both connected subgraphs;

(c) If $|V'_v V''_v| = 1$, let $V'_v V''_v = \{xy\}$ $(x \in V'_v,\ y \in V''_v)$. Then

$$|V'_v| \geqslant \max\{d(x) - 1, \max_{u \in V'_v \setminus \{x\}} d(u)\},$$

$$|V''_v| \geqslant \max\{d(y) - 1, \max_{u \in V''_v \setminus \{y\}} d(u)\}.$$

Denote by S_v the set of partitions $\{V'_v, V''_v\}$ mentioned above. Let

$$a(v) = \max_{\{V'_v, V''_v\} \in S_v} \{\min(|V'_v|, |V''_v|)\},$$

$$a(G) = \max_{v \in V} \{a(v)\}.$$

Obviously,

$$1 \leqslant a(v) \leqslant \frac{1}{2}(p-1), \qquad 1 \leqslant a(G) \leqslant \frac{1}{2}(p-1).$$

We shall show that $a(G) \neq 1$.

In fact, if $a(G) = 1$, then for every $v \in V$ we have $a(v) = 1$ and hence there is at least one 2-vertex adjacent to v. Let u be such a vertex. Let $N(u) = \{v, u_1\}$.

If $d(v) = 2$, then $a(u_1) \geqslant 2$ (because $p \geqslant 7$). It contradicts $a(G)1=$. If $d(v) \geqslant 3$, then the 2-vertex adjacent to u must be u_1. Thus $a(v) \geqslant 2$. This contradicts $a(G) = 1$, too.

Now we show the theorem by induction on p. Suppose (reductio ad absurdum) that G is a critical 2-edge-connected graph of minimum order such that $q(G) > f(p)$.

(2) Suppose $a(G) = 2$.

Let $T = \{v \in V | a(v) = 2\}$; $O = \{v \in V | a(v) = 1\}$. It follows that $T \neq \phi$ by (1).

First, we give the following propositions.

Proposition 1. Every 2-vertex is adjacent to at least one 2-vertex.

In fact, suppose that $d(x) = 2$. If $x \in O$, then the proposition holds obviously. If $x \in T$, let $V'_x = \{u, v\}$. Since $d(x) = 2$, x cannot be a cut-vertex. Thus $|V'_x V''_x| = 1$. Let $V'_x V''_x = \{uw\}$. Then v is not adjacent to any vertex in V''_x. Thus, from $d(v) \geqslant 2$, it follows that v is a 2-vertex adjacent to x.

Proposition 2. If $d(u) = d(v) = 2$ and $uv \in E$, then $u, v \in O$.

In fact, let $N(u) = \{v, u_1\}$, $N(v) = \{u, v_1\}$. It is obvious that $d(u_1) \geqslant 3, d(v_1) \geqslant 3$. (Otherwise, we have $a(v_1) \geqslant 3$ or $a(u_1) \geqslant 3$.)

Now we consider graph $G - u$. Since $d(u) = 2$, u is not a cut-vertex in G. Thus, $G - u$ is a connected graph which contains at least one bridge.

We show that vv_1 is the unique bridge in $G - u$. Obviously, vv_1 is a bridge in $G - u$. If $G - u$ contains a bridge xy different from vv_1, then xy belongs to every chain connected u_1 and v_1 in $G - u$. From the property (c) in (1) and $d(u_1) \geqslant 3$, $d(v_1) \geqslant 3$, it follows that xy must be an edge incident with u_1 or v_1. If $xy = u_1u_2 (u_2 \in N(u_1)$, $u_2 \neq u)$, then $d(u_1) = 3$ because $a(G) = 2$. Let $N(u_1) = \{u, u_2, u_3\}$. Hence, there is a partition $\{V'_u, V''_u\}$ of $V \backslash \{u\}$ such that $V'_u = \{u_1, u_3\}$. Therefore $d(u_3) = 1$, which is impossible. If $xy = v_1v_2 (v_2 \in N(v_1), v_2 \neq v)$, the proof is similar.

Thus vv_1 is the unique bridge in $G - u$. Hence $a(u) = 1$, i.e. $u \in O$.

It can be shown similarly that $v \in O$.

According to Propositions 1,2, we can prove that graph G has the following properties.

Property 1. Let $V'_x = \{u, v\}(x \in T)$; then $d(u) = d(v) = 2$ and $u, v \in O$.

If x is a cut-vertex in G, then the assertion is obvious. So we suppose that x is not a cut-vertex. Let $vw \in E$ $(w \in V''_x)$. Thus xu, $uv \in E$ and $d(u) = 2$. If $xv \in E$ then xv must be a bridge in $G - u$. (Otherwise, vertex u is not critical.) Thus x is a cut-vertex in G, a contradiction. Therefore, $xv \bar{\in} E$ and $d(v) = 2$. We have $u, v \in O$ by Proposition 2. Meanwhile, we also find that $N(x) \cap O \neq \phi$.

Property 2. $d(y) = 2$ and $N(y) \cap O \neq \phi$ for any $y \in O$.

Since $y \in O$, there must exist $x \in N(y)$ such that $d(x) = 2$. Let $N(x) = \{y, z\}$. By Proposition 1, there must exist at least one 2-vertex in $N(x)$. If $d(z) = 2$ then $a(y) = 2$. This contradicts $y \in O$. Therefore, we have $d(y) = 2$. By Proposition 2, we have $x \in O$. So, $N(y) \cap O \neq \phi$.

Property 3. $N(y) \cap T \neq \phi$ for any $y \in O$.

Let $N(y) = \{x, v\}$ and $x \in O$. So $d(x) = 2$ by Property 2. Then $a(v) \geqslant 2$. Since $a(G) = 2$ we have $a(v) = 2$, i.e. $v \in T$. Hence $N(y) \cap T \neq \phi$.

From the above properties it is clear that $|O| \equiv 0 \pmod 2$, $|O| \geqslant |T|$, $|OT| = |O|$ and $q(G(O)) = \frac{1}{2}|O|$.

Furthermore, there must exist a vertex y such that $y \in N(v) \cap O$ and $d(y) = 2$ for any $v \in V$. So, G is still critical 2-edge-connected when we add an edge between two non-adjacent vertices in T. Thus we can suppose that $G(T)$ is a complete graph.

Let $|O| = 2t$ and $|T| = p - 2t$. Then $4t \geqslant p$ because $|O| \geqslant |T|$, and $2t \leqslant p-1$ because $|T| \geqslant 1$. Hence the integer t must be on the closed interval $\left[\left\lceil \frac{p}{4} \right\rceil, \left\lfloor \frac{p-1}{2} \right\rfloor\right]$, where $\lceil m \rceil$, $\lfloor n \rfloor$ denote respectively the minimum integer not less than m and the maximum integer not greater than n.

Since $q(G) = q(G(O)) + q(G(T)) + |OT| = t + \frac{1}{2}(p-2t)(p-2t-1) + 2t$ $= 3t + \frac{1}{2}(p-2t)(p-2t-1)$, we have $\frac{d^2q(G)}{dt^2} > 0$. Thus the maximum values of $q(G)$ must be attained at an end of the interval $\left[\left\lceil \frac{p}{4} \right\rceil, \left\lfloor \frac{p-1}{2} \right\rfloor\right]$.

After calculation, we can see that $q(G)$ attains its maximum value when $t = \left\lceil \frac{p}{4} \right\rceil$. The max $q(G)$, for four cases $p \equiv 0, 1, 2, 3 \pmod 4$, are listed as follows.

p	$4k(k \geqslant 2)$	$4k+1(k \geqslant 2)$	$4k+2(k \geqslant 2)$	$4k+3(k \geqslant 1)$
max $q(G)$	$2k(k+1) = \frac{1}{8}(p^2+4p)$	$2k^2+4 = \frac{1}{8}(p^2-2p+33)$	$2k(k+1)+3 = \frac{1}{8}(p^2+20)$	$2k^2+4k+3 = \frac{1}{8}(p^2+2p+9)$

It follows that $q(G) \leqslant f(p)$, a contradiction.

(3) Now we suppose that $a(G) \geqslant 3$. Hence there will exist a vertex v in V such that $a(v) \geqslant 3$. Let $\{V'_v, V''_v\}$ be the partition of $V \setminus \{v\}$ such that $|V'_v| \geqslant 3$, $|V''_v| \geqslant 3$.

According to whether $V'_v V''_v$ is empty or not we construct two graphs $G_i = (V_i, E_i)$ of order p_i and size $q_i (i = 1, 2)$ as follows.

If $V'_v V''_v = \phi$, let

$$V_1 = V'_v \cup \{v, x_1, x_2\};$$
$$E_1 = E(G(V'_v \cup \{v\})) \cup \{vx_1, x_1x_2, x_2v\};$$
$$V_2 = V''_v \cup \{v, y_1, y_2\};$$
$$E_2 = E(G(V''_v \cup \{v\})) \cup \{vy_1, y_1y_2, y_2v\}.$$

It is obvious that $p_1 + p_2 = p + 5$, $q_1 + q_2 = q + 6$.

If $V'_v V''_v \neq \phi$, let $V'_v V''_v = \{xy\} (x \in V'_v, y \in V''_v)$. Then let

$$V_1 = V'_v \cup \{v, x_1, x_2\};$$
$$E_1 = E(G(V'_v \cup \{v\})) \cup \{xx_1, x_1x_2, x_2v\};$$
$$V_2 = V''_v \cup \{v, y_1, y_2\};$$
$$E_2 = E(G(V''_v \cup \{v\})) \cup \{yy_1, y_1y_2, y_2v\}.$$

It is obvious that $p_1 + p_2 = p + 5$, $q_1 + q_2 = q + 5$.

Evidently, both G_1 and G_2 are critical 2-edge-connected graphs and $p_1 < p$, $p_2 < p$. (We suppose that $p_1 \leqslant p_2$ in the following.)

By induction, $q_i \leqslant f(p_i) (i = 1, 2)$. Hence $q(G) \leqslant f(p_1) + f(p_2) - l$, where $l = 5$ or 6.

By the assumption $q(G) > f(p)$,

$$f(p) < f(p_1) + f(p_2) - l.$$

We prove the theorem by considering three cases:

Case 1. $p_1 \geqslant 8$. In this case, $p \geqslant 11$. We consider two subcases.

(a) $p \geqslant 13$. It can be easily found that

$$\frac{1}{8}(p^2 + 28) \leqslant f(p) \leqslant \frac{1}{8}(p^2 + 4p).$$

So,

$$\frac{1}{8}(p^2 + 28) \leqslant f(p) < f(p_1) + f(p_2) - l$$
$$\leqslant \frac{1}{8}(p_1^2 + 4p_1) + \frac{1}{8}(p_2^2 + 4p_2) - l.$$

Since $p_2 = p - p_1 + 5$, we have

$$F(p_1)=2p_1^2-10p_1-2p_1p+14p+17-8l>0.$$

$F(p_1)$ is a decreasing function of p_1 because $\frac{dF(p_1)}{dp_1}\leqslant 0$. (Note that $2p_1\leqslant p+5$.) Hence, $F(p_1)$ attains its maximum value when $p_1=8$. But, since $p\geqslant 13$ and $l\geqslant 5$,

$$0<F(p_1)\leqslant F(8)=65-2p-8l\leqslant 39-8l\leqslant -1,$$

a contradiction.

(b) $p=11$ or 12. In this subcase, we can find that $p_1=p_2=8$ when $p=11$, and $p_1=8$, $p_2=9$ when $p=12$.

Thus

$$f(p)\geqslant\frac{1}{8}(p^2+2p+9)\quad(\text{because } p\equiv 0,3 \pmod 4);$$

$$f(p_1)=f(8)=12;$$

$$f(p_2)\leqslant\frac{1}{8}(p_2^2+4p_2)\quad(\text{because } p_2\equiv 0,1 \pmod 4).$$

We have

$$\frac{1}{8}(p^2+2p+9)\leqslant f(p)<q(G)\leqslant f(8)+f(p_2)-l$$

$$\leqslant 12+\frac{1}{8}(p_2^2+4p_2)-l,$$

i. e.

$$p^2+2p<87+p_2^2+4p_2-8l.$$

Since $p=p_2+3$, $l\geqslant 5$ and $p_2\geqslant 8$, we have

$$40\leqslant 8l<72-4p_2\leqslant 72-32=40,$$

a contradiction.

Case 2. $p_1=7$. In this case $p\geqslant 9$. It is clear that

$$f(p)\geqslant\frac{1}{8}(p^2+28),$$

$$f(p_1)=f(7)=9,$$

$$f(p_2)\leqslant\frac{1}{8}(p_2^2+4p_2)\ (\text{because } p_2\geqslant 7).$$

Thus

$$\frac{1}{8}(p^2+28)<9+\frac{1}{8}(p_2^2+4p_2)-l,$$

which contradicts $p=p_2+2$ and $l\geqslant 5$.

Case 3. $p_1=6$. In this case $p\geqslant 7$.

(a) If $p=4k(k\geqslant 2)$, then $p_2=4k-1$. Since $f(4k)<f(6)+f(4k-1)-l$,

we have $2k^2 + 2k < 7 + 2(k-1)^2 + 4(k-1) + 3 - l$, i.e. $l < 8 - 2k$. This contradicts $l \geq 5$.

(b) If $p = 4k + 1 (k \geq 2)$ then $p_2 = 4k$. Since $f(4k+1) < f(6) + f(4k) - l$, we have $2k^2 + 2k + 2 < 7 + 2k^2 + 2k - l$, i.e. $l < 5$, a contradiction.

(c) If $p = 4k + 2 (k \geq 2)$, then $p_2 = 4k + 1$. Since $f(4k+2) < f(6) + f(4k+1) - l$, we have $2k^2 + 2k + 4 < 2k^2 + 2k + 2 + 7 - l$, which contradicts $l \geq 5$.

(d) If $p = 4k + 3 (k \geq 1)$, then $p_2 = 4k + 2$. Note that $f(4k+3) < f(6) + f(4k+2) - l$. When $k \geq 2$, we have $2k^2 + 4k + 3 < 7 + 2k^2 + 2k + 4 - l$, i.e. $l < 8 - 2k$, a contradiction.

When $k = 1$, we have $p_1 = p_2 = 6$ and $f(7) = 9 < f(6) + f(6) - 1 = 7 + 7 - l$, i.e. $l < 5$, which contradicts $l \geq 5$.

The proof of the theorem is completed.

Note. In the above proof, it can be found that $2 \leqslant a(G) \leqslant 5$ if G is a critical 2-edge-connected graph with maximum size. Furthermore, the authors believe it is impossible that $a(G) \geqslant 4$, and so conjecture that all of the critical 2-edge-connected graphs with maximum size have been discovered in Fig. 1. (If two graphs are isomorphic, we do not regard them as different.)

p 阶临界 2-边连通图的最大边数

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摘 要

设 $G=(V,E)$ 是 2-边连通图，若对每个点 $v \in V$，$G-v$ 不是 2-边连通图，则称 G 是临界 2-边连通图.

本文证明了 p 阶临界 2-边连通图的最大边数是

$$f(p)=\begin{cases} 7, & p=6; \\ \frac{1}{8}(p^2+4p), & p \equiv 0 \pmod 4; \\ \frac{1}{8}(p^2+2p+13), & p \equiv 1 \pmod 4; \\ \frac{1}{8}(p^2+28), & p \equiv 2 \pmod 4,\ p \neq 6; \\ \frac{1}{8}(p^2+2p+9), & p \equiv 3 \pmod 4. \end{cases}$$

并且给出了达到最大边数的极值图.