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# EXPONENTIAL ESTIMATES FOR "NOT VERY LARGE DEVIATIONS" AND WAVE FRONT PROPAGATION FOR A CLASS OF REACTION-DIFFUSION EQUATIONS

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**Summary**
A Large Deviation Principle (LDP) for a class of random processes depending on a small parameter $\varepsilon > 0$ is established. This class of processes arises from a random perturbation of a dynamical system. Then, exponential estimates for events of the type "not very large deviations" (deviations of order $\varepsilon^{\kappa}$, $0 < \kappa < \frac{1}{2}$) are obtained. Finally, the wave front propagation, as $\varepsilon \downarrow 0$, of the solution of some initial-boundary value problems is analyzed; these problems are formulated in terms of a reaction-diffusion equation whose diffusion coefficient is of order $\frac{1}{\varepsilon}$ and the nonlinear term is of order $\frac{1}{\varepsilon^{1-2\kappa}}$. The wave front is characterized in terms of the action functional corresponding to the Large Deviation Principle initially obtained.

**Key Words**: Action functional; large deviation; "not very large deviations"; wave front propagation.

## 1 Introduction

This paper is concerned with a family of random processes $(X_t^{\varepsilon} : t \geq 0)$ depending on a small parameter $\varepsilon > 0$ and satisfying the system of differential equations

$$\dot{X}_t^\varepsilon = b(X_t^\varepsilon, Y_t^\varepsilon), \qquad X_0^\varepsilon = x \in \mathbb{R}^d, \tag{1.1}$$

where $b(x,y) = (b^1(x,y), \cdots, b^d(x,y))$, $x \in \mathbb{R}^d$, $y \in \mathbb{R}^l$, is bounded as well as are its first and second derivatives. We define $Y_t^\varepsilon \equiv Y_{\frac{t}{\varepsilon}}$ where $(Y_t : t \geq 0)$ is a random process whose trajectories are continuous with probability one or have a finite number of discontinuities of first kind on any finite interval. These conditions are sufficient (see Freidlin and Wentzell(1984, Chapter 7, §1)) for system (1.1) having a unique solution with probability one.

We assume that there exists a vector field $\bar{b}(x)$ in $\mathbb{R}^d$ such that

$$\lim_{T\to+\infty} \frac{1}{T}\int_0^T b(x, Y_s)\, ds = \bar{b}(x), \qquad \forall x \in \mathbb{R}^d, \tag{1.2}$$

uniformly in $x$, with probability one. Under (1.2) the trajectories of $(X_t^\varepsilon, t \geq 0)$ converge, as $\varepsilon \downarrow 0$, to the solution $(\bar{x}_t : t \geq 0)$ of

$$\dot{\bar{x}}_t = \bar{b}(\bar{x}_t), \qquad \bar{x}_0 = x \in \mathbb{R}^d. \tag{1.3}$$

The convergence is in the space $\left(C_{[0,T]}(\mathbb{R}^d); \|\cdot\|\right)$ of the continuous functions on $[0,T]$ with values in $\mathbb{R}^d$, with the supremum norm $\|\cdot\|$.

Problems related with deviations of order 1 (large deviations) have been studied by Freidlin (1985a,1985b, 1976, 1972). Under some additional conditions, he established a Large Deviation Principle (LDP) for the family of random processes $(X_t^\varepsilon : t \geq 0)$. In this set up the theory of Large Deviations is described through an "action functional" $S_{0T}(\varphi)$, $\varphi \in C_{[0,T]}(\mathbb{R}^d)$ which satisfies the following conditions:

**(A.0)** Compactness of the level sets: $\forall s > 0$, $\forall x \in \mathbb{R}^d$,

$$\Phi(s) = \left\{\varphi \in C_{[0,T]}(\mathbb{R}^d) : \mathrm{S_{0T}}(\varphi) \leq \mathrm{s},\ \varphi_0 = \mathrm{x}\right\}$$

are compact sets.

**(A.I)** Lower bound: $\forall \delta > 0$, $\forall \gamma > 0$, $\forall \varphi \in C_{[0,T]}(\mathbb{R}^d)$, $\exists \varepsilon_0 > 0$ such that

$$P\left\{\|X_\cdot^\varepsilon - \varphi\| < \delta\right\} \geq \exp\left\{-\frac{1}{\varepsilon}\left[S_{0T}(\varphi) + \gamma\right]\right\}, \quad 0 < \varepsilon \leq \varepsilon_0.$$

**(A.II)** Upper bound: $\forall \delta > 0$, $\forall \gamma > 0$, $\forall s > 0$, $\exists \varepsilon_0 > 0$ such that

$$P\left\{\rho_{0T}\left(X_\cdot^\varepsilon, \Phi(s)\right) \geq \delta\right\} \leq \exp\left\{-\frac{1}{\varepsilon}(s - \gamma)\right\}, \quad 0 < \varepsilon \leq \varepsilon_0.$$

Under conditions **(A.0)-(A.II)**, one says that $\frac{1}{\varepsilon}S_{oT}(\cdot)$ is the action functional for the family of random processes $(X_t^\varepsilon : t \geq 0)$. The term $\frac{1}{\varepsilon}$ is the normalizing coefficient and $S_{oT}(\cdot)$ is the normalized action functional.

Deviations of order $\sqrt{\varepsilon}$ (normal deviations) were studied by Khas'minskii (1966). Taking into account the smoothness of $b(x,y)$ and assuming strong mixing conditions for $(Y_t : t \geq 0)$, he proved that

$$\zeta_t^\varepsilon = \frac{X_t^\varepsilon - \bar{x}_t}{\sqrt{\varepsilon}} \tag{1.4}$$

converges weakly, as $\varepsilon \downarrow 0$, to a Gaussian Markov process on $[0,T]$. The precise assumptions for the function $b(x,y)$ and the process $(Y_t : t \geq 0)$ may be found in Khas'minskii (1966) or in Freidlin and Wentzell (1984, Theorem 3.1, Chapter 7).

In this paper we are mainly interested in the asymptotic behavior of $(Z_t^\varepsilon : t \geq 0)$, as $\varepsilon \downarrow 0$, where

$$Z_t^\varepsilon = \frac{X_t^\varepsilon - \bar{x}_t}{\varepsilon^\kappa}, \qquad 0 < \kappa < \frac{1}{2}. \tag{1.5}$$

It turns out that , $\forall \delta > 0$,

$$\lim_{\varepsilon \downarrow 0} P\{\|X_\cdot^\varepsilon - \bar{x}_\cdot\| > \delta \varepsilon^\kappa\} = \lim_{\varepsilon \downarrow 0} P\{\|Z_t^\varepsilon\| > \delta\} = 0. \tag{1.6}$$

Deviations of order $\varepsilon^\kappa$ of $X_\cdot^\varepsilon$ from $\bar{x}_\cdot$ are called “not very large deviations” or “moderated deviations”. Estimations for this kind of deviations are obtained from the LDP for $(Z_t^\varepsilon : t \geq 0)$, which is the main result of this paper.

Baîer and Freidlin (1977) and Freidlin and Wentzell (1984, Chapter 7, §7), considered “not very large deviations” when the initial condition is an equilibrium point of the system (1.3). They studied the stability of the solution of (1.1) in a neighborhood of order $\varepsilon^\kappa$ of the equilibrium point, as $\varepsilon \downarrow 0$. In this case, if $0$ is the initial point, then $\bar{b}(0) = 0$ and the process $Z_t^\varepsilon$ becomes

$$Z_t^\varepsilon = \frac{X_t^\varepsilon}{\varepsilon^\kappa}. \tag{1.7}$$

The stability of “0” is characterized by the first exit time of $(X_t^\varepsilon)_{t \geq 0}$ from $D_\varepsilon \equiv D_{\varepsilon^\kappa}$. For $0 < \kappa < \frac{1}{2}$, the behavior of such random variable is related to “not very large deviations” of $(X_t^\varepsilon)$ from $D_\varepsilon$ or, equivalently, “large deviations” of $Z_t^\varepsilon$ from $D$.

In Baîer and Freidlin (1977) or in Freidlin and Wentzell (1984) a LDP for this family of processes is enunciated and a suggestion for the proof of the lower and upper bounds is given.

By using the method suggested by Baîer and Freidlin (1977) we shall establish a LDP for the family $Z_t^\varepsilon$ in (1.5) when the initial point is not necessarily an equilibrium point. These large deviations results are important in the study of wave-type solutions for reaction-diffusions equations depending on a small parameter (see Freidlin (1985a,b) and some extensions in Carmona (1995 a)). In §5 of this paper we consider some examples.

In what follows we outline the main steps and state the main results related with LDP in the more general situation, when the initial point of the process in (1.1) is any $x \in \mathbb{R}^d$.

From the smoothness of $b(x,y)$ we may write

$$\begin{aligned} X_t^\varepsilon - \bar{x}_t &= \int_0^t b(X_s^\varepsilon, Y_s^\varepsilon)\, ds - \int_0^t \bar{b}(\bar{x}_s)\, ds \\ &= \int_0^t [b(\bar{x}_s, Y_s^\varepsilon) - \bar{b}(\bar{x}_s)]\, ds + \int_0^t B(\bar{x}_s, Y_s^\varepsilon)(X_s^\varepsilon - \bar{x}_s)\, ds \\ &+ \int_0^t r^2(X_s^\varepsilon - \bar{x}_s)\, ds; \end{aligned} \tag{1.8}$$

the matrix $B(x,y)$ is given by

$$B_k^i(x,y) = \frac{\partial b^i}{\partial x^k}(x,y), \qquad i,k \in \{1,2,\cdots,d\} \tag{1.9}$$

and $r^2(\cdot)$ is the rest of Lagrange in the Taylor's expansion of $b(x,y)$ in a neighborhood of $\bar{x}_t$.

Let us define

$$\eta_t^\varepsilon = \frac{1}{\sqrt{\varepsilon}} \int_0^t [b(\bar{x}_s, Y_s^\varepsilon) - \bar{b}(\bar{x}_s)]\, ds \equiv \frac{1}{\sqrt{\varepsilon}} \int_0^t \tilde{b}(\bar{x}_s, Y_s^\varepsilon)\, ds. \tag{1.10}$$

Then $Z_t^\varepsilon$ in (1.5) satisfies

$$\dot{Z}_t^\varepsilon = \varepsilon^{\frac{1}{2}-\kappa} \dot{\eta}_t^\varepsilon + B(\bar{x}_t, Y_t^\varepsilon)\, Z_t^\varepsilon + \frac{r^2(X_t^\varepsilon - \bar{x}_t)}{\varepsilon^\kappa}, \quad Z_0^\varepsilon = 0. \tag{1.11}$$

Our first result is the LDP for the family of random processes $\varepsilon^{\frac{1}{2}-\kappa}\eta_t^\varepsilon$; the action functional is given in Theorem 1. Besides (1.2), we shall assume the following conditions:

**Condition B- 1** *There exists a matrix $A(x) = (A_{ij}(x))_{i,j=1,\cdots,d}$ nonnegative definite, symmetric, bounded, continuous in $x$, invertible, such that for any step functions $\alpha$, $\psi : [0,T] \to \mathbb{R}^d$,*

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} \varepsilon^{1-2\kappa} \quad & \ln E \exp\left\{ \tfrac{1}{\varepsilon^{1-\kappa}} \int_0^T < \alpha_s, \tilde{b}(\psi_s, Y_s^\varepsilon) > \, ds \right\} \\ & = \tfrac{1}{2} \int_0^T < A(\psi_s)\alpha_s, \alpha_s > \, ds. \end{aligned} \tag{1.12}$$

**Condition B- 2** *$\exists t_0$, $0 < t_0 \le 1$ and a function $\sigma(t) > 0$ with $\sigma(t) \to 0$ as $t \downarrow 0$ such that*

$$\begin{aligned} \overline{\lim}_{\varepsilon\downarrow 0} \sup_{\substack{\varepsilon \le t \le t_0 \\ 0 \le h \le 1-t}} \quad & \varepsilon^{1-2\kappa} \left| \ln E \exp\left\{ \tfrac{1}{\varepsilon^{1-\kappa}\sigma(t)} \int_h^{h+t} \tilde{b}(\bar{x}_s, Y_s^\varepsilon)\, ds \right\} \right| \\ & = l_{+\infty} < +\infty \end{aligned} \tag{1.13}$$

*where for a d-dimensional vector b,*

$$\ln E\{\exp(\int b)\}$$

*means*

$$\begin{pmatrix} \ln E\{\exp(\int b_1)\} \\ \ln E\{\exp(\int b_2)\} \\ \cdots \\ \ln E\{\exp(\int b_d)\} \end{pmatrix}.$$

**Condition B- 3** *For some $\Delta > 1 - 2\kappa$, $\forall \delta > 0$,*

$$\begin{aligned} & \lim_{\varepsilon\downarrow 0} \varepsilon^\Delta \ln P \left\{ \sup_{0\le t\le T} \left| \varepsilon^{-\kappa} \int_0^t (B(\bar{x}_s, Y_s^\varepsilon) - \bar{B}(\bar{x}_s)) \right.\right. \\ & \left.\left. \times \int_0^s e^{\int_u^s \bar{B}(\bar{x}_v)\, dv} . \tilde{b}(\bar{x}_u, Y_u^\varepsilon)\, du\, ds \right| > \delta \right\} = -\infty \end{aligned} \tag{1.14}$$

*where $\bar{B}(x)$ satisfies*

$$\bar{B}_k^i(x) = \frac{\partial \bar{b}^i}{\partial x^k}(x) = \lim_{T\to+\infty} \frac{1}{T} \int_0^T B_k^i(x, Y_s)\, ds \tag{1.15}$$

*uniformly in $x$, with probability one.*

**Remark 1.1** *Condition* **B-1** *is equivalent to the existence of the limit in (1.12) for every continuous functions $\alpha$ and $\psi$.*

In §3 we shall prove the following theorem:

**Theorem 1.1** *Under conditions* **B-1** *and* **B-2***, the action functional for the family of random processes $\varepsilon^{\frac{1}{2}-\kappa}\eta_t^\varepsilon$ is given by $\frac{1}{\varepsilon^{1-2\kappa}} S_{0T}^1(\varphi)$, where*

$$S_{0T}^1(\varphi) = \begin{cases} \frac{1}{2}\int_0^T < A^{-1}(\bar{x}_s)\dot{\varphi}_s, \dot{\varphi}_s > ds, & \varphi \text{ a.c.} \\ +\infty, & \text{in the rest of } C_{[0,T]}(\mathbb{R}^d) \end{cases} \tag{1.16}$$

*where $A^{-1}(x)$ is the inverse matrix of $A(x)$.*

Theorem 1.1 is an extension of a result obtained by Gärtner (1976) where he has considered a family of random processes converging weakly, as $\varepsilon \downarrow 0$, to a Wiener process in $\mathbb{R}$. He established sufficient conditions for this family of random processes, conveniently re-scaled, to have the same action functional as of the limit process in the new scale. In Theorem 1.1 we extend Gärtner's result in two ways: The space variable has dimension $d \geq 1$ and the family of random processes $\eta_t^\varepsilon$ converges weakly, as $\varepsilon \downarrow 0$, to a Gaussian process $W_t^0$ with independent increments, $EW_t^0 = 0$ (if we assume the hypothesis for Khas'minskii's result being valid). It is worth to observe that the weak convergence above cited is not an hypothesis of Theorem 1.1.

The main result in this paper is the LDP for the family $(Z_t^\varepsilon : t \geq 0)$ in (1.5). In §4 we shall prove the following theorem:

**Theorem 1.2** *If conditions* **B-1**, **B-2**, *and* **B-3** *are satisfied then the action functional for* $(Z_t^\varepsilon : t \geq 0)$ *is given by* $\frac{1}{\varepsilon^{1-2\kappa}} S_{0T}(\varphi)$ *with*

$$S_{0T}(\varphi) = \begin{cases} \frac{1}{2}\int_0^T < A^{-1}(\bar{x}_s)(\dot{\varphi}_s - \bar{B}(\bar{x}_s)\varphi_s), (\dot{\varphi}_s - \bar{B}(\bar{x}_s)\varphi_s) > ds, \\ \qquad\qquad\qquad\qquad \textit{if } \varphi \textit{ is a.c.} \\ +\infty, \qquad \textit{in the rest of } C_{[0,T]}(\mathbb{R}^{\mathrm{d}}) \end{cases} \tag{1.17}$$

*where* $\bar{B(x)}$ *satisfies (1.15).*

Theorem 1.2 provides the exponential estimates for probabilities of "not very large deviations". The asymptotics of such probabilities are essentially different of the corresponding to "large deviations". As in the case of "normal deviations", the study of deviations of order $\varepsilon^\kappa$ is reduced to the study of deviations of the same order of the linearized system obtained from (1.8).

Now we sketch the proof of Theorem 1.2. Firstly we consider the linearized system

$$\dot{\tilde{\Upsilon}}_t^\varepsilon = \tilde{b}(\bar{x}_t, Y_t^\varepsilon) + B(\bar{x}_t, Y_t^\varepsilon)\,\tilde{\Upsilon}_t^\varepsilon, \qquad \tilde{\Upsilon}_0^\varepsilon = 0. \tag{1.18}$$

We prove that, if $\frac{1}{\varepsilon^{1-2\kappa}} S_{0T}(\varphi)$ is the action functional for $\frac{\tilde{\Upsilon}_t^\varepsilon}{\varepsilon^\kappa}$ then it is the action functional for $Z_t^\varepsilon$. Then we take a simplified linearized system

$$\dot{\hat{\Upsilon}}_t^\varepsilon = \tilde{b}(\bar{x}_t, Y_t^\varepsilon) + \bar{B}(\bar{x}_t)\,\hat{\Upsilon}_t^\varepsilon, \qquad \hat{\Upsilon}_0^\varepsilon = 0. \tag{1.19}$$

It turns out that, under Condition **B-3**, $\frac{\tilde{\Upsilon}^\varepsilon}{\varepsilon^\kappa}$ and $\frac{\hat{\Upsilon}_t^\varepsilon}{\varepsilon^\kappa}$ have the same action functional. Finally, using Theorem 1.1, we prove that $\frac{\hat{\Upsilon}_t^\varepsilon}{\varepsilon^\kappa}$ has $\frac{1}{\varepsilon^{1-2\kappa}} S_{0T}(\varphi)$ as its action functional.

In §5 we study the asymptotics of the solution for a class of reaction-diffusion equations depending on a small parameter $\varepsilon > 0$, as $\varepsilon \downarrow 0$. Using

Theorem 1.1 and Theorem 1.2 we prove that the solution converges to a wave-type function.

Wave-type solutions for reaction-diffusion equations have been studied since 1930’s by Kolmogorov, Petrovskii, Piskounov (1937) (such equation is called KPP equation), Aronson & Weinberger (1975), by using classical methods and later, after 1970, by Freidlin (1985b), Gärtner (1982), McKean (1975), and others, via stochastic approach. Freidlin (1985b) introduced a small parameter $\varepsilon > 0$ in the generalized KPP equation whose diffusion coefficient became small, of order $\varepsilon$. He described the wave front for the solution of certain class of problems, as $\varepsilon \downarrow 0$, by using the Feynman-Kac formula and large deviations for some families of random processes.

Carmona (1995a) generalized Freidlin’s work in one direction by introducing a “fast variable” $y$ of order $\frac{1}{\varepsilon}$ in some initial-boundary value problems for the equation

$$\frac{\partial u^\varepsilon(t,x,y)}{\partial t} = \tfrac{1}{2\varepsilon}\tfrac{\partial^2 u^\varepsilon(t,x,y)}{\partial y^2} + \tfrac{\varepsilon}{2}a(x,y)\tfrac{\partial^2 u^\varepsilon(t,x,y)}{\partial x^2} + \tfrac{1}{\varepsilon}f(x,y,u^\varepsilon), \quad x \in \mathrm{I\!R}, \quad |\mathrm{y}| < \mathrm{a}, \quad \mathrm{t} > 0.$$

In §5 of this paper, we consider problems of the type

$$\begin{cases} \frac{\partial u^\varepsilon(t,x,y)}{\partial t} = \frac{1}{2\varepsilon}\frac{\partial^2 u^\varepsilon(t,x,y)}{\partial y^2} + \frac{1}{\varepsilon^{1-2\kappa}} f(\varepsilon^\kappa x, y, u^\varepsilon) + \frac{1}{\varepsilon^\kappa}\, b(\varepsilon^\kappa x, y)\, \frac{\partial u^\varepsilon(t,x,y)}{\partial x}, \\ x \in \mathrm{I\!R}^{\mathrm{d}}, \quad \mathrm{y} \in (-\mathrm{a},\mathrm{a}), \\ u^\varepsilon(0,x,y) = g(x) \\ \frac{\partial u^\varepsilon(t,x,y)}{\partial y}\big|_{y=\pm a} = 0 \end{cases} \tag{1.20}$$

where $0 < \kappa < \frac{1}{2}$, $b(x,y)$ satisfies the conditions specified in the introduction of this paper, the initial function is nonnegative, continuous in the interior of its support $G_0 \neq \mathrm{I\!R}^{\mathrm{d}}$, $[(G_0)] = [G_0]$ where $[A]$ is the closure of $A$ and $(A)$ its interior]. For each pair $x, y$, the nonlinear term $f(x,y,u)$ belongs to the class $\mathcal{F}_1$ (see Freidlin (1985a)), i.e., $\forall x, y$, $f(x,y,\cdot) \in C^1$, $c(x,y) = f'(x,y,0) = \sup_{0\le u\le 1}\frac{f(x,y,u)}{u} > 0$; we call $c(x,y,u) = \frac{f(x,y,u)}{u}$.

To analyze the solution $u^\varepsilon(t,x,y)$ to this type of problem we shall use the Feynman-Kac formula and “not very large deviations” for families of random processes as in (1.1) or, equivalently, large deviations for families of random processes as in (1.5) and (1.7). This is done, roughly speaking, in the following way: To the differential operator

$$L^\varepsilon = \frac{1}{2\varepsilon}\frac{\partial^2}{\partial y^2} + \frac{1}{\varepsilon^\kappa} b(\varepsilon^\kappa x, y)\frac{\partial}{\partial x} \tag{1.21}$$

it is associated a random process $(X_t^\varepsilon, Y_t^\varepsilon; P_{xy}^\varepsilon)$ where $(Y_t; \bar{P}_y)$ is a Brownian motion on $[-a,a]$ starting at $y \in (-a,a)$, with instantaneous reflection at $\pm a$, $Y_t^\varepsilon \equiv Y_{\frac{t}{\varepsilon}}$ and

$$X_t^\varepsilon = x + \frac{1}{\varepsilon^\kappa}\int_0^t b(\varepsilon^\kappa X_s^\varepsilon, Y_s^\varepsilon)\, ds, \qquad x \in \mathbb{R}^d. \tag{1.22}$$

Notice that the diffusion coefficient of the variable $y$ is of order $\frac{1}{\varepsilon}$; so it is called "fast variable".

The Feynman-Kac formula allows us to express the solution of (1.20) through the integral equation

$$u^\varepsilon(t,x,y) = E_{xy}^\varepsilon g(X_t^\varepsilon) \exp\left\{\frac{1}{\varepsilon^{1-2\kappa}}\int_0^t c(\varepsilon^\kappa X_s^\varepsilon, Y_s^\varepsilon, u^\varepsilon(t-s, X_s^\varepsilon, Y_s^\varepsilon))\, ds\right\}, \tag{1.23}$$

where $E_{xy}^\varepsilon$ is the expectation corresponding to the measure $P_{xy}^\varepsilon$. Using the action functional for certain families of random processes as in (1.5) and (1.7) one can verify that $u^\varepsilon(t,x,y)$ converges, as $\varepsilon \downarrow 0$, to a step function $u^0(t,x,y)$ given by

$$u^0(t,x,y) = \begin{cases} 1, & V(t,x) > 0, \quad |y| \le a \\ 0, & V(t,x) < 0, \quad |y| \le a, \end{cases}$$

for some function $V(t,x)$ which will be specified in §5.

We emphasize that Baîer and Freidlin(1977) and Freidlin and Wentzell (1984, Chapter 7) considered the case when the averaged system starts at an equilibrium position. We have considered a general starting point. Although a long time has passed since the publication of Baîer and Freidlin(1977) the complete proof has never been published, as far as we know. Only a short proof was provided by those authors. We give the complete proof to the extended result. On the way of obtaining such complete proof we needed extensions of LPD results in Gärtner(1976) in directions described in the manuscript. The proofs of such extensions are provided. Using those generalizations we study propagation of traveling waves for an equation of Kolmogorov-Petrovskii- Piskunov as extensions to the results in Carmona(1995a, b).

## 2 Auxiliary Results

**Proposition 2.1** *If condition* **B-1** *holds then* $\forall x, \alpha \in \mathbb{R}^d$,

$$\begin{aligned} \lim_{T\to+\infty} T^{2\kappa-1} \ln E \quad & \exp\left\{T^{-\kappa}\int_0^T < \alpha, \tilde{b}(x,Y_s) > \, ds\right\} \\ & = \frac{1}{2} < A(x)\alpha, \alpha > . \end{aligned} \tag{2.1}$$

**Proof:** This equality follows from Condition **B-1** by changing variables. ■

The proof of the following proposition is similar to the one of Lemma 4.3, Chapter 7, in Freidlin and Wentzell (1984) and we omit it.

**Proposition 2.2** *Suppose that* $(Y_t; \bar{P}_y)$ *is a homogeneous Markov process with values in a compact set* $D \subset \mathbb{R}^1$ *and (2.1) holds uniformly in the initial point* $y \in (D)$, *where* $(D)$ *is the interior of* $D$. *Then, Condition* **B-1** *is satisfied.*

Now we shall characterize a class of random processes $(Y_t : t \geq 0)$ which satisfies conditions **B-1** and **B-2**.

For sake of completeness, first, we announce the conditions (L.1)-(L.5) in Theorem 2.2 of Carmona (1995b) which are used next.

**(L.1).** $(Y_t; \bar{P}_y)$ *is a homogeneous Feller-Markov Process.*

**(L.2).** $(Y_t; \bar{P}_y)$ *is uniformly stochastically continuous, i.e.,*

$$\forall \varepsilon > 0, \bar{P}_y(|Y_s - Y_t| \geq \varepsilon) \to 0$$

as $t - s \to 0$, *uniformly in* $y \in D$ *and in* $s, t \in [0, +\infty)$.

**(L.3).** *The semigroup* $\{T_t\}_{t\geq 0}$ *in (2.1) is strongly positive with respect to the cone* $\{f \in C_D : \ f \geq 0\}$.

**(L.4).** *For each* $h \in C_D$, *the semigroup* $\{T_t^{(h)}\}_{t\geq 0}$ *in (I) below satisfies the Feller condition, i.e.,* $T^{(h)} C_D \subset C_D$.

**(L.5).** *For each* $h \in C_D, \{T_t^{(h)}\}_{t\geq 0}$ *is a compact semigroup. Then, for any* $\beta \in \mathbb{R}$,

$$\lim_{T\to\infty} \frac{1}{T} \log \bar{E}_y \exp\left(\int_0^t \beta h(Y_s) ds\right) = H(\beta)$$

*exists uniformly in* $y$. *Moreover,* $H(\beta)$ *is differentiable and convex in* $\beta$.

**Lemma 2.1** *Let* $(Y_t; \bar{P}_y)$ *be a homogeneous Markov process on the phase space* $(D, \mathcal{B}(D))$, $D \subset \mathbb{R}^1$ *compact, and* $\mathcal{B}(D)$ *the* $\sigma$*-field of the Borel subsets of* $D$ *in the topology inherited from the Euclidean norm in* $\mathbb{R}^1$. *Assume conditions* **(L.1)**-**(L.5)** *above. Then, Condition* **B-1** *is satisfied.*

**Proof:** Let us suppose that $\bar{b}(0) = 0$. For each $\alpha \in \mathbb{R}^d$ we introduce the semigroup of operators

$$T_t^\alpha f(y) = \bar{E}_y f(Y_t) \exp\left\{\int_0^t < \alpha, b(0, Y_s) > ds\right\}, \qquad (I)$$

where $f$ is a continuous numerical function on $D$.

As we assumed its conditions, from Theorem 2.2 in Carmona (1995b) we know that

$$\lim_{T\to+\infty}\frac{1}{T}\ln \bar{E}_y \exp\left\{\int_0^T <\alpha, b(0,Y_s)>\,ds\right\} = \lambda(\alpha), \qquad (2.2)$$

where $\lambda(\alpha)$ is the maximal eigenvalue of $\mathcal{A}^\alpha$, the infinitesimal generator of $T_t^\alpha$. It is real and simple, differentiable and convex; the corresponding eigenvector $\phi$ is positive, and $\|\phi\| = 1$. Moreover, $T_t^\alpha \phi(y) = e^{\lambda(\alpha)t}\phi(y)$.

Taking into account that 1 is the maximal eigenvalue of $T_t^\alpha$ for $\alpha = 0$, we have $\lambda(0) = 0$. On the other hand, from (2.2),

$$\begin{aligned}\lambda(\alpha) &\geq \lim_{T\to+\infty}\frac{1}{T}\bar{E}_y\int_0^T <\alpha, b(0,Y_s)>\,ds \\ &= <\alpha, \lim_{T\to+\infty}\frac{1}{T}\int_0^T \bar{E}_y b(0,Y_s)\,ds> = <\alpha, \bar{b}(0)>.\end{aligned}$$

the last equality following from (1.2). Since $\bar{b}(0) = 0$ we have $\lambda(\alpha) \geq 0$, for all $\alpha \in \mathrm{I\!R}^{\mathrm{d}}$. Therefore $\lambda'(0) = 0$ and

$$\lambda(\alpha) = \frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial^2\lambda}{\partial\alpha_i\partial\alpha_j}(0)\alpha_i\alpha_j + o(\alpha^2) \quad \text{as } |\alpha|\to 0. \qquad (2.3)$$

Now, the compactness of $D$ implies that $\exists K > 0$ such that $0 < K \leq \phi(y) \leq 1$, $\forall y \in D$. Then,

$$\begin{aligned}&t^{2\kappa-1}\ln K + t^{2\kappa-1}\ln(T_t^{t^{-\kappa}\alpha}1)(y) \leq t^{2\kappa-1}\ln(T_t^{t^{-\kappa}\alpha}\phi)(y) \\ &= t^{2\kappa}\lambda(t^{-\kappa}\alpha) + t^{2\kappa-1}\ln\phi(y) \leq t^{2\kappa-1}\ln(T_t^{t^{-\kappa}\alpha}1)(y).\end{aligned}$$

Hence, using (2.3) we get

$$\begin{aligned}\lim_{t\to+\infty} t^{2\kappa-1}\ln\left(T_t^{t^{-\kappa}\alpha}1\right)(y) &= \frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial^2\lambda}{\partial\alpha_i\partial\alpha_j}(0)\alpha_i\alpha_j \\ &\equiv \frac{1}{2} < A\alpha, \alpha >\end{aligned}$$

which is Condition **B-1** in the case $\bar{b}(0) = 0$.

When the initial point in (1.3) is not an equilibrium point, the arguments are the same as above if one recall that

$$\lim_{t\to+\infty}\frac{1}{t}\int_0^t \tilde{b}(x, Y_s)\,ds = 0, \qquad \forall x \in \mathrm{I\!R}^{\mathrm{d}}, \ \text{ w.p. } 1$$

and then $\lambda(x,\alpha) \geq 0$, $\forall x, \alpha$. The matrix $A(x)$ in Condition **B-1** is given by

$$A(x) = \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial^2 \lambda}{\partial \alpha_i \partial \alpha_j}(x,0)\alpha_i \alpha_j. \tag{2.4}$$

■

Let $\psi \in C_{[0,T]}(\mathrm{I\!R}^{\mathrm{d}})$ be a step function, constant in $[\frac{j}{r}T, \frac{j+1}{r}T]$, $j = 0, 1, 2, \cdots, r-1$. For each $\alpha = (\alpha^1, \cdots, \alpha^r) \in (\mathrm{I\!R}^{\mathrm{d}})^{\mathrm{r}}$, define

$$H^{\psi}(\alpha) = \frac{1}{2r} \sum_{j=0}^{r-1} < A(\psi_{\frac{jT}{r}})\alpha^{j+1}, \alpha^{j+1} > . \tag{2.5}$$

This function is convex, lower semi-continuous in $\alpha$, $H^{\psi}(0) = 0$, $H^{\psi}(\alpha) < +\infty$, $\forall \alpha$. Let $L^{\psi}(\beta)$ be its Legendre transform:

$$\begin{aligned} L^{\psi}(\beta) &= \sup_{\alpha} \left\{ < \alpha, \beta > - H^{\psi}(\alpha) \right\} \\ &= \tfrac{r}{2} \textstyle\sum_{j=0}^{r-1} < A^{-1}(\psi_{\frac{jT}{r}})\beta^{j+1}, \beta^{j+1} >, \qquad \beta \in (\mathrm{I\!R}^{\mathrm{d}})^{\mathrm{r}}. \end{aligned} \tag{2.6}$$

This function is convex, lower semi-continuous, assuming values in $(-\infty, +\infty]$, and it is not identically equal to $+\infty$.
Define for each $s > 0$,

$$\begin{aligned} \Phi^r(s) &= \left\{ \beta \in (\mathrm{I\!R}^{\mathrm{d}})^{\mathrm{r}} : \mathrm{L}^{\psi}(\beta) \leq \mathrm{s} \right\} \\ &= \left\{ \beta \in (\mathrm{I\!R}^{\mathrm{d}})^{\mathrm{r}} : \tfrac{\mathrm{r}}{2} \textstyle\sum_{\mathrm{j}=0}^{\mathrm{r}-1} < \mathrm{A}^{-1}(\psi_{\frac{\mathrm{jT}}{\mathrm{r}}})\beta^{\mathrm{j}+1}, \beta^{\mathrm{j}+1} > \leq \mathrm{s} \right\}. \end{aligned} \tag{2.7}$$

The following proposition is similar to Theorem 1.1, Chapter 5, in Freidlin and Wentzell (1984) and we omit its proof.

**Proposition 2.3** $\forall \delta > 0, \forall s > 0, \exists \alpha_1, \cdots, \alpha_N \in (\mathrm{I\!R}^{\mathrm{d}})^{\mathrm{r}}$ *such that*

$$\Phi^r(s) \subset \bigcap_{i=1}^{N} \left\{ \beta : < \alpha_i, \beta > - H^{\psi}(\alpha_i) \leq s \right\} \subset \Phi^r_{+\delta}(s),$$

*where* $\Phi^r_{+\delta}(s) = \{\beta : dist(\beta, \Phi^r(s)) < \delta\}$.

Let us define, for each $x, \alpha \in \mathrm{I\!R}^{\mathrm{d}}$

$$H(x,\alpha) = \frac{1}{2} < A(x)\alpha, \alpha > = \frac{1}{2} \sum_{i,j=1}^{d} A_{ij}(x)\alpha_i \alpha_j. \tag{2.8}$$

This function is convex in the second argument and jointly continuous (by the hypothesis $A(x)$ is continuous). Let $L(x,\beta)$ be its Legendre transform:

$$L(x,\beta) = \sup_{\alpha}\{<\alpha,\beta> - H(x,\alpha)\} = \frac{1}{2} < A^{-1}(x)\beta,\beta>, \quad \beta\in \mathbb{R}^{\mathrm{d}}; \tag{2.9}$$

it is convex in $\beta$ and jointly lower semi-continuous in all variables.

For $\alpha : [0,T] \to \mathbb{R}^{\mathrm{d}}$, let us define

$$\begin{aligned} G_\varepsilon(\alpha) &\equiv \ln \bar{E}_y \exp\left\{\int_0^T \alpha_t\, d\eta_t^\varepsilon\right\} \\ &= \ln \bar{E}_y \exp\left\{\frac{1}{\sqrt{\varepsilon}}\int_0^T <\alpha_t, \tilde{b}(\bar{x}_t, Y_t^\varepsilon)> dt\right\}. \end{aligned} \tag{2.10}$$

The process $\eta_t^\varepsilon$ was introduced in (1.10). For $\psi_t = \bar{x}_t$, Condition **B-1** may be written as

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{1-2\kappa} G_\varepsilon\left(\frac{1}{\varepsilon^{\frac{1}{2}-\kappa}}\alpha\right) = \frac{1}{2}\int_0^T < A(\bar{x}_t)\alpha_t, \alpha_t > dt. \tag{2.11}$$

# 3 Proof of Theorem 1.1

In this part we shall prove that, under **B-1** and **B-2**, conditions **(A.O)-(A.II)** are fulfilled for the family of processes $\varepsilon^{\frac{1}{2}-\kappa}\eta_t^\varepsilon$ and the functional in (1.16). This result is an extension of results obtained by Gärtner (1976). We shall use the same approach for proving it.

It is well known (see e.g., Freidlin and Wentzell (1984), Lemma 4.2, Chapter 7) that the level sets of the functional $S_{0T}^1(\cdot)$ in (1.16) are compact sets. So condition **(A.0)** is verified. The following theorem gives condition **(A.I)** (the lower bound).

**Theorem 3.1** *If conditions* **B-1** *and* **B-2** *are satisfied then* $\forall\gamma > 0$, $\forall\delta > 0$, $\forall\varphi \in C_{[0,T]}(\mathbb{R}^{\mathrm{d}})$, $\varphi_0 = 0$, $\exists\varepsilon_0 > 0$ *such that*

$$P\{\|\varepsilon^{\frac{1}{2}-\kappa}\eta_\cdot^\varepsilon - \varphi\| < \delta\} \geq \exp\left\{-\frac{1}{\varepsilon^{1-2\kappa}}\left[S_{0T}^1(\varphi)+\gamma\right]\right\}, \quad 0<\varepsilon\leq\varepsilon_0, \tag{3.1}$$

*where* $\eta_t^\varepsilon$ *is the process introduced in (1.10).*

**Proof:** For sake of simplifying notation we assume $T = 1$. Let $r > 0$ be an integer and $\bar{\eta}_t^\varepsilon$ be the random polygonal line with vertices at the points $\frac{j}{r}$ and $\bar{\eta}_{\frac{j}{r}}^\varepsilon = \eta_{\frac{j}{r}}^\varepsilon$, $j = 1,\cdots,r$. Let $n \equiv n(\varepsilon) = r[\frac{1}{\varepsilon r}]$ and $\tilde{\eta}_t^\varepsilon$ be the random polygonal line with vertices at $\frac{j}{n}$ with $\tilde{\eta}_{\frac{j}{n}}^\varepsilon = \eta_{\frac{j}{n}}^\varepsilon$, $j = 1,\cdots,n$. Notice that $\tilde{\eta}_{\frac{j}{r}}^\varepsilon = \bar{\eta}_{\frac{j}{r}}^\varepsilon$, $j=1,\cdots,r$.

Let $(Q_m)_{m=1,2,\dots}$ be a sequence of sets in $C_{[0,1]}(\mathbb{R}^d)$ which will be defined later. Then for any $\varphi \in C_{[0,1]}(\mathbb{R}^d)$ and $\delta > 0$,

$$\begin{aligned} P\{\|\varepsilon^{\frac{1}{2}-\kappa}\eta^\varepsilon_\cdot - \varphi\| < \delta\} &\\ \geq P\left\{\|\varepsilon^{\frac{1}{2}-\kappa}\bar{\eta}^\varepsilon_\cdot - \varphi\| < \tfrac{\delta}{2}\right\} - P\left\{\|\varepsilon^{\frac{1}{2}-\kappa}\eta^\varepsilon_\cdot - \varepsilon^{\frac{1}{2}-\kappa}\bar{\eta}^\varepsilon_\cdot\| \geq \tfrac{\delta}{2}\right\} &\\ \geq P\{\|\varepsilon^{\frac{1}{2}-\kappa}\bar{\eta}^\varepsilon_\cdot - \varphi\| < \tfrac{\delta}{2}\} - P\{\varepsilon^{\frac{1}{2}-\kappa}\tilde{\eta}^\varepsilon_\cdot \notin Q_{n(\varepsilon)}\} &\\ -P\{\|\varepsilon^{\frac{1}{2}-\kappa}\eta^\varepsilon_\cdot - \varepsilon^{\frac{1}{2}-\kappa}\bar{\eta}^\varepsilon_\cdot\| \geq \tfrac{\delta}{2},\, \varepsilon^{\frac{1}{2}-\kappa}\tilde{\eta}^\varepsilon_\cdot \in Q_{n(\varepsilon)}\} &\\ \equiv I_1 - I_2 - I_3. & \end{aligned} \tag{3.2}$$

Since $\varphi$ is continuous, then for $r$ sufficiently large and $0 < \delta' < \frac{\delta}{2}$ we have

$$I_1 \equiv P\left\{\|\varepsilon^{\frac{1}{2}-\kappa}\bar{\eta}^\varepsilon_\cdot - \varphi\| < \frac{\delta}{2}\right\} \geq P\left\{\max_{k=1,\cdots,r} \|\varepsilon^{\frac{1}{2}-\kappa}\eta^\varepsilon_{\frac{k}{r}} - \varphi_{\frac{k}{r}}\| < \delta'\right\}.$$

Let $\varphi \in C_{[0,1]}(\mathbb{R}^d)$ with $S^1_{01}(\varphi) < +\infty$ and $\bar{\varphi}_t$ be the polygonal line with step $\frac{1}{r}$ such that $\bar{\varphi}_{\frac{k}{r}} = \varphi_{\frac{k}{r}}$, $k = 0, 1, \cdots, r$. Then, $\dot{\bar{\varphi}}_t$ is a step function. Let us define

$$\alpha(t,x) = \frac{\partial L}{\partial \beta}(x, \dot{\bar{\varphi}}_t), \tag{3.3}$$

where $L(x,\beta)$ was introduced in (2.9). Then

$$\frac{\partial H}{\partial \alpha}(x, \alpha(t,x)) = \dot{\bar{\varphi}}_t, \tag{3.4}$$

where $H(x,\alpha)$ is given in (2.8). Since $\dot{\bar{\varphi}}_t$ is a step function then $\alpha(\cdot,x)$ is also a step function. Besides, $\alpha(t,x)$ is bounded because the matrix $A(x)$ and $\dot{\bar{\varphi}}_t$ are bounded.

Now we apply Cramér's method by introducing a new probability measure $\tilde{P}^\varepsilon$ defined by

$$\tilde{P}^\varepsilon(A) = E\mathcal{X}_A \exp\left\{\frac{1}{\varepsilon^{\frac{1}{2}-\kappa}} \int_0^1 \alpha(t, \bar{x}_t)\, d\eta^\varepsilon_t - G_\varepsilon\left(\frac{1}{\varepsilon^{\frac{1}{2}-\kappa}} \alpha(\cdot, \bar{x}_\cdot)\right)\right\},$$

where $G_\varepsilon(\alpha)$ is given in (2.10). Hence,

$$P\left\{\max_{k=1,\cdots,r}\|\varepsilon^{\frac{1}{2}-\kappa}\eta^\varepsilon_{\frac{k}{r}}-\varphi_{\frac{k}{r}}\|<\delta'\right\}$$
$$=\tilde{E}^\varepsilon\mathcal{X}_{[\max_{k=1,\cdots,r}\|\varepsilon^{\frac{1}{2}-\kappa}\eta^\varepsilon_{\frac{k}{r}}-\varphi_{\frac{k}{r}}\|<\delta']}\exp\left\{-\frac{1}{\varepsilon^{\frac{1}{2}-\kappa}}\int_0^1\alpha(t,\bar{x}_t)\,d\eta^\varepsilon_t\right.$$
$$\left.+G_\varepsilon\left(\frac{1}{\varepsilon^{\frac{1}{2}-\kappa}}\alpha(\cdot,\bar{x}_.)\right)\right\}$$
$$=\tilde{E}^\varepsilon\mathcal{X}_{[\max_{k=1,\cdots,r}\|\varepsilon^{\frac{1}{2}-\kappa}\eta^\varepsilon_{\frac{k}{r}}-\varphi_{\frac{k}{r}}\|<\delta']}\exp\left\{-\frac{1}{\varepsilon^{\frac{1}{2}-\kappa}}\left[\int_0^1\alpha(t,\bar{x}_t)\,d(\eta^\varepsilon_t-\frac{1}{\varepsilon^{\frac{1}{2}-\kappa}}\bar{\varphi}_t)\right]\right\}$$
$$\times\exp\left\{-\frac{1}{\varepsilon^{1-2\kappa}}\left[\int_0^1\alpha(t,\bar{x}_t)\,\dot{\bar{\varphi}}_t\,dt-\varepsilon^{1-2\kappa}G_\varepsilon\left(\frac{1}{\varepsilon^{\frac{1}{2}-\kappa}}\alpha(\cdot,\bar{x}_.)\right)\right]\right\},\tag{3.5}$$

where $\tilde{E}^\varepsilon$ is the expectation corresponding to the measure $\tilde{P}^\varepsilon$. From Condition **B-1** and (2.8) we have

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}\varepsilon^{1-2\kappa}G_\varepsilon\left(\frac{1}{\varepsilon^{\frac{1}{2}-\kappa}}\alpha(\cdot,\bar{x}_.)\right) &= \frac{1}{2}\int_0^1 <A(\bar{x}_t)\alpha(t,\bar{x}_t),\alpha(t,\bar{x}_t)>\,dt\\ &= \int_0^1 H(\bar{x}_t,\alpha(t,\bar{x}_t))\,dt.\end{aligned}$$

Then $\forall\gamma>0$, $\exists\varepsilon_0>0$ such that

$$\frac{1}{\varepsilon^{1-2\kappa}}\left[\int_0^1 H(\bar{x}_t,\alpha(t,\bar{x}_t)\,dt-\frac{\gamma}{3}\right]<G_\varepsilon\left(\frac{1}{\varepsilon^{\frac{1}{2}-\kappa}}\alpha(\cdot,\bar{x}_.)\right)$$
$$<\frac{1}{\varepsilon^{1-2\kappa}}\left[\int_0^1 H(\bar{x}_t,\alpha(t,\bar{x}_t))\,dt+\frac{\gamma}{3}\right],\qquad 0<\varepsilon\le\varepsilon_0.$$

Since $\alpha(t,x)\dot{\bar{\varphi}}_t-H(x,\alpha(t,x))=L(x,\dot{\bar{\varphi}}_t)$ and taking into account that

$$\int_0^1 L(\bar{x}_t,\dot{\bar{\varphi}}_t)\,dt\le\int_0^1 L(\bar{x}_t,\dot{\varphi}_t)\,dt,$$

the second exponential in (3.5) is greater or equal to

$$\exp\left\{-\frac{1}{\varepsilon^{1-2\kappa}}\left[\int_0^1 L(\bar{x}_t,\dot{\varphi}_t)\,dt+\frac{\gamma}{3}\right]\right\},\qquad 0<\varepsilon\le\varepsilon_0.$$

On the other hand, if $\max_{k=1,\cdots,r}|\varepsilon^{\frac{1}{2}-\kappa}\eta^\varepsilon_{\frac{k}{r}}-\varphi_{\frac{k}{r}}|<\delta$", $0<\delta$" $<\delta'$ and $\delta$" sufficiently small, we have

$$\int_0^1\alpha(t,\bar{x}_t)\,d(\varepsilon^{\frac{1}{2}-\kappa}\eta^\varepsilon_t-\bar{\varphi}_t)<\frac{\gamma}{3}$$

because $\alpha(t,\bar{x}_t)$ is bounded. Hence, returning to (3.5), we obtain

$$\begin{aligned} & I_1 \geq \exp\left\{-\tfrac{1}{\varepsilon^{1-2\kappa}}\tfrac{\gamma}{3}\right\}\exp\left\{-\tfrac{1}{\varepsilon^{1-2\kappa}}\left[\int_0^1 L(\bar{x}_t,\dot{\varphi}_t)\,dt+\tfrac{\gamma}{3}\right]\right\} \\ & \times\tilde{P}^\varepsilon\left\{\max_{k=1,\cdots,r}\|\varepsilon^{\frac12-\kappa}\eta^\varepsilon_{\frac{k}{r}}-\varphi_{\frac{k}{r}}\|<\delta"\right\},\qquad 0<\varepsilon\leq\varepsilon_0. \end{aligned} \tag{3.6}$$

Now we shall prove that

$$\lim_{\varepsilon\downarrow 0}\tilde{P}^\varepsilon\left\{\max_{k=1,\cdots,r}\|\varepsilon^{\frac12-\kappa}\eta^\varepsilon_{\frac{k}{r}}-\varphi_{\frac{k}{r}}\|\geq\delta"\right\}=0.$$

For this, it is sufficient to show that for $t\in\{\frac1r,\frac2r,\cdots,\frac rr\}$, the following relations are valid:

$$\begin{aligned} & \lim_{k=1,\cdots,r}\tilde{P}^\varepsilon\{\varepsilon^{\frac12-\kappa}\eta^{\varepsilon,i}_t-\bar{\varphi}^i_t-\delta"\geq 0\}=0, \qquad \text{and} \\ & \lim_{k=1,\cdots,r}\tilde{P}^\varepsilon\{-\varepsilon^{\frac12-\kappa}\eta^{\varepsilon,i}_t+\bar{\varphi}^i_t-\delta"\geq 0\}=0, \quad \text{for } i=1,\cdots,d. \end{aligned} \tag{3.7}$$

From the Chebyshev's exponential inequality we can write, for all $\gamma^*>0$,

$$\begin{aligned} \tilde{P}^\varepsilon&\left\{\varepsilon^{\frac12-\kappa}\eta^{\varepsilon,i}_t-\bar{\varphi}^i_t-\delta"\geq 0\right\}\leq\tilde{E}^\varepsilon\exp\left\{\tfrac{\gamma^*}{\varepsilon^{1-2\kappa}}[\varepsilon^{\frac12-\kappa}\eta^{\varepsilon,i}_t-\bar{\varphi}^i_t-\delta"]\right\} \\ &=E^\varepsilon\exp\left\{\tfrac{\gamma^*}{\varepsilon^{1-2\kappa}}[\varepsilon^{\frac12-\kappa}\eta^{\varepsilon,i}_t-\bar{\varphi}^i_t-\delta"]\right\} \\ &\times\exp\left\{\tfrac{1}{\varepsilon^{\frac12-\kappa}}\int_0^1\alpha(s,\bar{x}_s)\,d\eta^\varepsilon_s-G_\varepsilon\left(\tfrac{1}{\varepsilon^{\frac12-\kappa}}\alpha(\cdot,\bar{x}_\cdot)\right)\right\} \\ &=E^\varepsilon\exp\left\{\tfrac{1}{\varepsilon^{\frac12-\kappa}}\int_0^1[\alpha(s,\bar{x}_s)+\gamma^*\mathcal{X}_{[0,t]}e^{(i)}]\,d\eta^\varepsilon_s-\tfrac{\gamma^*}{\varepsilon^{1-2\kappa}}(\bar{\varphi}^i_t+\delta")\right. \\ &\qquad\left.-G_\varepsilon\left(\tfrac{1}{\varepsilon^{\frac12-\kappa}}\alpha(\cdot,\bar{x}_\cdot)\right)\right\}, \end{aligned}$$

where $e^{(i)}$ is the component of order $i$ of the canonical basis of $\mathbb{R}^d$. Hence,

$$\begin{aligned} \tilde{P}^\varepsilon\{\varepsilon^{\frac12-\kappa}\eta^{\varepsilon,i}_t-\bar{\varphi}^i_t-\delta"\geq 0\}&\leq\exp\left\{-\tfrac{1}{\varepsilon^{1-2\kappa}}\left[\varepsilon^{1-2\kappa}G_\varepsilon\left(\tfrac{1}{\varepsilon^{\frac12-\kappa}}\alpha(\cdot,\bar{x}_\cdot)\right)\right.\right. \\ &\left.\left.-\varepsilon^{1-2\kappa}G_\varepsilon\left(\tfrac{1}{\varepsilon^{\frac12-\kappa}}\left[\alpha(\cdot,\bar{x}_\cdot)+\gamma^*\mathcal{X}_{[0,t]}e^{(i)}\right]\right)+\gamma^*(\bar{\varphi}^i_t+\delta")\right]\right\}. \end{aligned} \tag{3.8}$$

From Condition **B-1**,

$$\lim_{\varepsilon\downarrow 0}\varepsilon^{1-2\kappa}G_\varepsilon\left(\frac{1}{\varepsilon^{\frac12-\kappa}}\alpha(\cdot,\bar{x}_\cdot)\right)=\frac12\int_0^1 <A(\bar{x}_s)\alpha(s,\bar{x}_s),\alpha(s,\bar{x}_s)>\,ds.$$

But

$$\alpha(s,\bar{x}_s)+\gamma^*\mathcal{X}_{[0,t]}e^{(i)}=\begin{cases}\alpha(s,\bar{x}_s)+\gamma^*e^{(i)}, & \text{if } s\le t\\ \alpha(s,\bar{x}_s), & \text{if } s>t.\end{cases}$$

Then,

$$\begin{aligned}&\int_0^1 < A(\bar{x}_s)(\alpha(s,\bar{x}_s)+\gamma^*\mathcal{X}_{[0,t]}e^{(i)}),(\alpha(s,\bar{x}_s)+\gamma^*\mathcal{X}_{[0,t]}e^{(i)}) > \,ds\\ &=\int_0^1 < A(\bar{x}_s)\alpha(s,\bar{x}_s),\alpha(s,\bar{x}_s) > \,ds+2\int_0^t\sum_{j=1}^d A_{ij}(\bar{x}_s)\alpha^j(s,\bar{x}_s)\gamma^*\,ds\\ &\quad+\gamma^{*2}\int_0^t A_{ii}(\bar{x}_s)\,ds.\end{aligned}$$

Hence, Condition **B-1** implies that

$$\begin{aligned}&\lim_{\varepsilon\downarrow 0}\varepsilon^{1-2\kappa}G_\varepsilon\left(\frac{1}{\varepsilon^{\frac{1}{2}-\kappa}}[\alpha(\cdot,\bar{x}_\cdot)+\gamma^*\mathcal{X}_{[0,t]}e^{(i)}]\right)\\ &=\frac{1}{2}\int_0^1 < A(\bar{x}_s)\alpha(s,\bar{x}_s),\alpha(s,\bar{x}_s) > \,ds\\ &+\gamma^*\int_0^t\sum_{j=1}^d A_{ij}(\bar{x}_s)\alpha^j(s,\bar{x}_s)\,ds+\frac{1}{2}\gamma^{*2}\int_0^t A_{ii}(\bar{x}_s)\,ds.\end{aligned}$$

Therefore, the expression in brackets in (3.8) converges, as $\varepsilon\downarrow 0$, to

$$\begin{aligned}&-\gamma^*\int_0^t\sum_{j=1}^d A_{ij}(\bar{x}_s)\alpha^j(s,\bar{x}_s)\,ds-\frac{1}{2}\gamma^{*2}\int_0^t A_{ii}(\bar{x}_s)\,ds+\gamma^*(\bar{\varphi}_t^i+\delta")\\ &=\gamma^*\left[\delta"+\bar{\varphi}_t^i-\int_0^t\sum_{j=1}^d A_{ji}(\bar{x}_s)\alpha^j(s,\bar{x}_s)\,ds-\frac{\gamma^*}{2}\int_0^t A_{ii}(\bar{x}_s)\,ds\right].\end{aligned}$$

We should find $\gamma^*>0$ such that the above expression be strictly positive. For such $\gamma^*$ we get (3.7) from (3.8).

Notice that

$$\frac{\partial H}{\partial\alpha^k}(x,\alpha)=\sum_{i=1}^d A_{ik}(x)\,\alpha^i$$

and

$$\frac{\partial^2 H}{\partial\alpha^k\partial\alpha^l}(x,\alpha)=\frac{\partial}{\partial\alpha^l}\left[\sum_{i=1}^d A_{ik}(x)\,\alpha^i\right]=A_{lk}(x).$$

From (3.4) we have

$$\frac{\partial H}{\partial\alpha^k}(x,\alpha(t,x))=\sum_{i=1}^d A_{ik}(x)\alpha^i(t,x)=\dot{\bar{\varphi}}_t^k.$$

Therefore the limit in (3.9) reduces to

$$\gamma^*\left[\delta"-\frac{\gamma^*}{2}\int_0^t A_{ii}(\bar{x}_s)\,ds\right].$$

From the hypothesis on $A(x)$ we have $\int_0^t A_{ii}(\bar{x}_s)\,ds \geq 0$. We choose $\gamma^* > 0$ such that $\delta" - \frac{\gamma^*}{2}\int_0^t A_{ii}(\bar{x}_s)\,ds > 0$.

Now we can say that $\forall \bar{\gamma} > 0$, $\exists \varepsilon_0 > 0$ such that

$$\tilde{P}^\varepsilon\{\varepsilon^{\frac{1}{2}-\kappa}\eta_t^{\varepsilon,i} - \bar{\varphi}_t^i - \delta" \geq 0\} \leq \exp\left\{-\frac{1}{\varepsilon^{1-2\kappa}}\left[\gamma^*\delta" - \frac{1}{2}\gamma^{*2}\int_0^t A_{ii}(\bar{x}_s)\,ds - \bar{\gamma}\right]\right\},$$

for all $0 < \varepsilon \leq \varepsilon_0$. Choose $0 < \bar{\gamma} < \gamma^*\delta" - \frac{1}{2}\gamma^{*2}\int_0^t A_{ii}(\bar{x}_s)\,ds$ and we get

$$\tilde{P}^\varepsilon\{\varepsilon^{\frac{1}{2}-\kappa}\eta_t^{\varepsilon,i} - \bar{\varphi}_t^i - \delta" \geq 0\} \leq \exp\left\{-\frac{1}{\varepsilon^{1-2\kappa}}C\right\}, \qquad 0 < \varepsilon \leq \varepsilon_0,$$

for some $C > 0$. Then, $\forall \delta > 0$, $\exists \varepsilon_0 > 0$ such that

$$\tilde{P}^\varepsilon\left\{\max_{k=1,\cdots,r}\|\varepsilon^{\frac{1}{2}-\kappa}\eta_{\frac{k}{r}}^\varepsilon - \varphi_{\frac{k}{r}}\| < \delta"\right\} > 1 - \delta, \qquad 0 < \varepsilon \leq \varepsilon_0.$$

But, for $\varepsilon$ sufficiently small, $1 - \delta > \exp\{-\frac{\frac{\gamma}{3}}{\varepsilon^{1-2\kappa}}\}$. Therefore, returning to (3.6), we conclude that $\forall \gamma > 0$, $\exists \varepsilon_0 > 0$ such that

$$I_1 > \exp\left\{-\frac{1}{\varepsilon^{1-2\kappa}}\left[S_{01}^1(\varphi) + \gamma\right]\right\}, \qquad 0 < \varepsilon \leq \varepsilon_0.$$

For estimating

$$I_2 \equiv P\left\{\varepsilon^{\frac{1}{2}-\kappa}\tilde{\eta}_\cdot^\varepsilon \notin Q_{n(\varepsilon)}\right\}$$

we shall use arguments analogous to the ones in Gärtner (1976), which we outline next.

For each number $I > 0$ there exists a monotonic function $\Gamma(t)$ with $\Gamma(t) \downarrow 0$ as $t \downarrow 0$ such that

$$\inf_{k=1,\cdots,r}\frac{\Gamma(t)}{\sigma(t)\sqrt{d}} - l_\infty > I,$$

where $t_0$, $l_\infty$, and $\sigma(t)$ come from Condition **B-2** in (1.13). We choose $r > 0$ sufficiently large such that $\Gamma(\frac{1}{r}) < \frac{\delta}{12}$ and $\frac{1}{r} < t_0$.

Define

$$Q_n = \bigcap_{j=1}^{[nt_0]}\bigcap_{k=0}^{n-j}\left\{f \in C_{[0,1]}(\mathrm{I\!R}^{\mathrm{d}}) : \left\|\mathrm{f}(\frac{\mathrm{k}+\mathrm{j}}{\mathrm{n}}) - \mathrm{f}(\frac{\mathrm{k}}{\mathrm{n}})\right\| < \Gamma(\frac{\mathrm{j}}{\mathrm{n}})\right\}. \tag{3.10}$$

Let $n(\varepsilon) = r[\frac{1}{\varepsilon r}]$. Then,

$$I_2 \leq \sum_{j=1}^{[nt_0]} \sum_{k=0}^{n-j} P\left\{ \|\varepsilon^{\frac{1}{2}-\kappa}\eta^{\varepsilon}_{\frac{k+j}{n}} - \varepsilon^{\frac{1}{2}-\kappa}\eta^{\varepsilon}_{\frac{k}{n}}\| \geq \Gamma(\frac{j}{n}) \right\}$$
$$\leq \sum_{j=1}^{[nt_0]} \sum_{k=0}^{n-j} \sum_{i=1}^{d} P\left\{ |\varepsilon^{\frac{1}{2}-\kappa}\eta^{\varepsilon,i}_{\frac{k+j}{n}} - \varepsilon^{\frac{1}{2}-\kappa}\eta^{\varepsilon,i}_{\frac{k}{n}}| > \frac{\Gamma(\frac{j}{n})}{\sqrt{d}} \right\}.$$

From Chebyshev's inequality one obtain

$$I_2 \leq d\, n^2(\varepsilon)\, t_0\, 2\exp\left\{ -\tfrac{1}{\varepsilon^{1-2\kappa}} \left[ \inf_{0\leq t\leq t_0} \tfrac{\Gamma(t)}{\sigma(t)\sqrt{d}} \right.\right.$$
$$\left.\left. - \sup_{\substack{0<h\leq 1-t \\ \varepsilon<t\leq t_0}} \varepsilon^{1-2\kappa} \left\| \ln E\exp\left\{ \pm \tfrac{1}{\varepsilon^{\frac{1}{2}-\kappa}\sigma(t)} \int_h^{h+t} d\eta^{\varepsilon,i}_s \right\} \right\| \right] \right\}.$$

But $n^2(\varepsilon) = (r[\frac{1}{\varepsilon r}])^2 \leq \frac{1}{\varepsilon^2}$, $\frac{1}{\varepsilon^2} = \exp\{-\frac{2\varepsilon^{1-2\kappa}\ln\varepsilon}{\varepsilon^{1-2\kappa}}\}$ and $\varepsilon^{1-2\kappa}\ln\varepsilon \to 0$ as $\varepsilon \downarrow 0$. Then one may conclude that

$$I_2 \leq 2dt_0\exp\left\{ -\frac{1}{\varepsilon^{1-2\kappa}}(2\varepsilon^{1-2\kappa}\ln\varepsilon) \right\} \exp\left\{ -\frac{1}{\varepsilon^{1-2\kappa}} \left[ \inf_{0<t\leq t_0} \frac{\Gamma(t)}{\sigma(t)\sqrt{d}} - l_\infty - \frac{I}{2} \right] \right\}$$
$$\leq \exp\left\{ -\frac{I}{\varepsilon^{1-2\kappa}} \right\}, \quad 0<\varepsilon\leq\varepsilon_0.$$

For estimating

$$I_3 \equiv P\left\{ \|\varepsilon^{\frac{1}{2}-\kappa}\eta^{\varepsilon}_{\cdot} - \varepsilon^{\frac{1}{2}-\kappa}\bar{\eta}^{\varepsilon}_{\cdot}\| > \frac{\delta}{2}, \varepsilon^{\frac{1}{2}-\kappa}\tilde{\eta}^{\varepsilon}_{\cdot} \in Q_{n(\varepsilon)} \right\},$$

one can prove that $\forall I > 0$, $\exists \varepsilon_0 > 0$ such that

$$I_3 < \exp\left\{ -\frac{I}{\varepsilon^{1-2\kappa}} \right\}, \qquad 0<\varepsilon\leq\varepsilon_0.$$

Returning to (3.2) the result follows.

■

Now we shall prove the upper bound **(A.II)**.

**Theorem 3.2** $\forall\delta>0$, $\forall\gamma>0$, $\forall s>0$, $\exists\varepsilon_0>0$ *such that*

$$P\left\{ \rho_{0T}\left( \varepsilon^{\frac{1}{2}-\kappa}\eta^{\varepsilon}_{\cdot}, \Phi(s) \right) \geq \delta \right\} \leq \exp\left\{ -\frac{1}{\varepsilon^{1-2\kappa}}(s-\gamma) \right\}, \qquad 0<\varepsilon\leq\varepsilon_0, \tag{3.11}$$

*where*

$$\Phi(s) = \left\{ \varphi \in C_{[0,T]}(\mathrm{I\!R}^{\mathrm{d}}) : \mathrm{S}^1_{0\mathrm{T}}(\varphi) \leq \mathrm{s},\, \varphi_0 = 0 \right\}$$

*and* $S^1_{0T}(\varphi)$ *is defined in (1.16).*

**Proof:** Again we take $T = 1$.

$$\begin{aligned} &P\left\{\rho_{01}\left(\varepsilon^{\frac{1}{2}-\kappa}\eta^{\varepsilon}_{\cdot},\Phi(s)\right) > \delta\right\} \\ &\quad \le P\left\{\rho_{01}\left(\varepsilon^{\frac{1}{2}-\kappa}\bar{\eta}^{\varepsilon}_{\cdot},\Phi(s)\right) > \tfrac{\delta}{2}\right\} + P\left\{\|\varepsilon^{\frac{1}{2}-\kappa}\eta^{\varepsilon}_{\cdot} - \varepsilon^{\frac{1}{2}-\kappa}\bar{\eta}^{\varepsilon}_{\cdot}\| \le \tfrac{\delta}{2}\right\} \\ &\quad \le P\left\{\rho_{01}\left(\varepsilon^{\frac{1}{2}-\kappa}\bar{\eta}^{\varepsilon}_{\cdot},\Phi(s)\right) > \tfrac{\delta}{2}\right\} + P\left\{\varepsilon^{\frac{1}{2}-\kappa}\tilde{\eta}^{\varepsilon}_{\cdot} \notin Q_{n(\varepsilon)}\right\} \\ &\quad +P\left\{\|\varepsilon^{\frac{1}{2}-\kappa}\eta^{\varepsilon}_{\cdot} - \varepsilon^{\frac{1}{2}-\kappa}\bar{\eta}^{\varepsilon}_{\cdot}\| > \tfrac{\delta}{2}, \varepsilon^{\frac{1}{2}-\kappa}\tilde{\eta}^{\varepsilon}_{\cdot} \in Q_{n(\varepsilon)}\right\} \\ &\quad \equiv I_4 + I_2 + I_3, \end{aligned} \tag{3.12}$$

where $Q_{n(\varepsilon)}$ is defined in (3.10).

We have seen in the proof of Theorem 3.1 that for all $I > 0$, there exists a sequence $\{Q_m\}_{m=1,2,\cdots}$, a number $r$, and $\varepsilon_0 > 0$ such that

$$I_2 + I_3 < \exp\left\{-\frac{I}{\varepsilon^{1-2\kappa}}\right\}, \qquad 0 < \varepsilon \le \varepsilon_0. \tag{3.13}$$

Now we shall estimate

$$I_4 \equiv P\left\{\rho_{01}\left(\varepsilon^{\frac{1}{2}-\kappa}\bar{\eta}^{\varepsilon}_{\cdot},\Phi(s)\right) > \frac{\delta}{2}\right\}.$$

It is known that $L(x,\beta)$ is jointly semi-continuous in all variables. Then, the functional

$$\int_0^1 L(\psi_t,\dot{\varphi}_t)\,dt = \frac{1}{2}\int_0^1 < A^{-1}(\psi_t)\dot{\varphi}_t,\dot{\varphi}_t > \,dt$$

is lower semi-continuous in $\psi$ and $\varphi$. Let $\psi^n \to \psi$ as $n \to +\infty$. Then, for $\varphi$ fixed and using Fatou's Lemma,

$$\liminf_{n\to+\infty}\int_0^1 L(\psi^n_t;\dot{\varphi}_t)\,dt \ge \int_0^1 \liminf_{n\to+\infty} L(\psi^n_t;\dot{\varphi}_t)\,dt \ge \int_0^1 L(\psi_t;\dot{\varphi}_t)\,dt.$$

Then, $\forall\Delta > 0$, $\exists\delta^* > 0$ such that if $\|\bar{x}_{\cdot} - \psi\| < \delta^*$,

$$\frac{1}{2}\int_0^1 < A^{-1}(\psi_t)\dot{\varphi}_t,\dot{\varphi}_t > \,dt > \frac{1}{2}\int_0^1 < A^{-1}(\bar{x}_t)\dot{\varphi}_t,\dot{\varphi}_t > \,dt - \Delta. \tag{3.14}$$

We choose $\psi$ as a step function with $\psi_{\frac{j}{r}} = \bar{x}_{\frac{j}{r}}$, $j = 0,\cdots,r-1$ satisfying $\|\bar{x}_{\cdot} - \psi\| < \delta^*$ and, for $s > 0$, we define

$$\Phi^{\psi}(s) = \left\{\varphi \in C_{[0,1]}(\mathrm{I\!R}^{\mathrm{d}}) : \frac{1}{2}\int_0^1 < \mathrm{A}^{-1}(\psi_{\mathrm{t}})\dot{\varphi}_{\mathrm{t}},\dot{\varphi}_{\mathrm{t}} > \,\mathrm{dt} \le \mathrm{s}\right\}.$$

Since $\Phi^{\psi}(s-\Delta)\subset\Phi(s)$ we have

$$P\left\{\rho_{01}\left(\varepsilon^{\frac{1}{2}-\kappa}\bar{\eta}^{\varepsilon}_{\cdot},\Phi(s)\right)>\frac{\delta}{2}\right\}\leq P\left\{\rho_{01}\left(\varepsilon^{\frac{1}{2}-\kappa}\bar{\eta}^{\varepsilon}_{\cdot},\Phi^{\psi}(s-\Delta)\right)>\frac{\delta}{2}\right\}\equiv\mathcal{P}.$$

Define

$$\lambda^{\varepsilon}=(\lambda^{\varepsilon}_1,\cdots,\lambda^{\varepsilon}_r)=\varepsilon^{\frac{1}{2}-\kappa}(\bar{\eta}^{\varepsilon}_{\frac{1}{r}},\bar{\eta}^{\varepsilon}_{\frac{2}{r}}-\bar{\eta}^{\varepsilon}_{\frac{1}{r}},\bar{\eta}^{\varepsilon}_{\frac{3}{r}}-\bar{\eta}^{\varepsilon}_{\frac{2}{r}},\cdots,\bar{\eta}^{\varepsilon}_{\frac{r}{r}}-\bar{\eta}^{\varepsilon}_{\frac{r-1}{r}}).$$

Notice that $\lambda^{\varepsilon}\in(\mathrm{I\!R}^{\mathrm{d}})^{\mathrm{r}}$. Fix $\alpha=(\alpha^1,\cdots,\alpha^r)\in(\mathrm{I\!R}^{\mathrm{d}})^{\mathrm{r}}$. Then,

$$\begin{aligned}&\lim_{\varepsilon\to0}\varepsilon^{1-2\kappa}\ln E\exp\left\{\tfrac{1}{\varepsilon^{1-2\kappa}}<\alpha,\lambda^{\varepsilon}>\right\}\\&=\lim_{\varepsilon\to0}\varepsilon^{1-2\kappa}\ln E\exp\left\{\varepsilon^{\kappa-1}\textstyle\sum_{j=1}^{r}<\alpha^j,\int_{\frac{j-1}{2}}^{\frac{j}{r}}\tilde{b}(\bar{x}_t,Y^{\varepsilon}_t)\,dt>\right\}\\&=\lim_{\varepsilon\downarrow0}\varepsilon^{1-2\kappa}\ln E\exp\left\{\varepsilon^{\kappa-1}\textstyle\int_0^1<\alpha_t,\tilde{b}(\bar{x}_t,Y^{\varepsilon}_t)>\,dt\right\}\\&=\tfrac{1}{2}\textstyle\int_0^1<A(\bar{x}_t)\alpha_t,\alpha_t>\,dt,\end{aligned}\tag{3.15}$$

where the last equality follows from Condition **B-1** and $\alpha_t=\alpha^j$, $\frac{j-1}{r}\leq t<\frac{j}{r}$, $j=1,\cdots,r$. Now,

$$\mathcal{P}\leq P\left\{\frac{1}{2}\int_0^1<A^{-1}(\psi_t)\varepsilon^{\frac{1}{2}-\kappa}\dot{\bar{\eta}}^{\varepsilon}_t,\varepsilon^{\frac{1}{2}-\kappa}\dot{\bar{\eta}}^{\varepsilon}_t>\,dt>s-\Delta\right\}.$$

From (2.6) and (2.7) we get

$$\mathcal{P}\leq P\left\{L^{\psi}(\lambda^{\varepsilon})>s-\Delta\right\}\leq P\left\{\lambda^{\varepsilon}\notin\Phi^r(s-\Delta)\right\}.$$

Since $\Phi^r(s-2\Delta)\subset\Phi^r(s-\Delta)$, $\partial\Phi^r(s-\Delta)=\{\beta\in(\mathrm{I\!R}^{\mathrm{d}})^{\mathrm{r}}:\mathrm{L}^{\psi}(\beta)=\mathrm{s}-\Delta\}$ is a compact set, and $\Phi^r(s-2\Delta)\cap\partial\Phi^r(s-\Delta)=\emptyset$, we have $d\equiv\mathrm{dist}(\phi^r(s-2\Delta),\partial\Phi^r(s-\Delta))>0$. Then, from Proposition 2.3 and Chebyshev's inequality, there exist $\alpha_1,\cdots,\alpha_N\in(\mathrm{I\!R}^{\mathrm{d}})^{\mathrm{r}}$ such that

$$\begin{aligned}\mathcal{P}&\leq P\left\{\lambda^{\varepsilon}\notin\Phi^r(s-\Delta)\right\}\leq P\left\{\rho(\lambda^{\varepsilon},\Phi^r(s-2\Delta))\geq d\right\}\\&\leq P\left\{\lambda^{\varepsilon}\in\cup_{i=1}^{N}\{\beta:<\alpha_i,\beta>-H^{\psi}(\alpha_i)>s-2\Delta\}\right\}\\&\leq \textstyle\sum_{i=1}^{N}P\left\{\exp\left\{\tfrac{1}{\varepsilon^{1-2\kappa}}[<\alpha_i,\lambda^{\varepsilon}>-H^{\psi}(\alpha_i)]\right\}>\exp\left\{\tfrac{s-2\Delta}{\varepsilon^{1-2\kappa}}\right\}\right\}\\&\leq \textstyle\sum_{i=1}^{N}\exp\left\{\tfrac{s-2\Delta}{\varepsilon^{1-2\kappa}}\right\}E\exp\left\{\tfrac{1}{\varepsilon^{1-2\kappa}}\left[<\alpha_i,\lambda^{\varepsilon}>-H^{\psi}(\alpha_i)\right]\right\}\\&= \exp\left\{-\tfrac{s-2\Delta}{\varepsilon^{1-2\kappa}}\right\}\textstyle\sum_{i=1}^{N}\exp\left\{-\tfrac{H^{\psi}(\alpha_i)}{\varepsilon^{1-2\kappa}}\right\}\\&\quad\exp\left\{\tfrac{1}{\varepsilon^{1-2\kappa}}\left[\varepsilon^{1-2\kappa}\ln E\exp\left\{\tfrac{1}{\varepsilon^{1-2\kappa}}<\alpha_i,\lambda^{\varepsilon}>\right\}\right]\right\}.\end{aligned}\tag{3.16}$$

From (3.15) we know that

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{1-2\kappa} \ln E \exp\left\{\frac{1}{\varepsilon^{1-2\kappa}} < \alpha_i, \lambda^\varepsilon >\right\} = \frac{1}{2}\int_0^1 < A(\bar{x}_t)\alpha_{t,i}, \alpha_{t,i} > dt,$$

where $\alpha_{t,i} = \alpha_i^j$, $\frac{j-1}{r} \le t \le \frac{j}{r}$, $j = 1, \cdots, r$. Since $\frac{1}{2}\int_0^1 < A(\psi_t)\alpha_t, \alpha_t > dt$ is lower semi-continuous in $\psi$, we take the same step function $\psi_t$ (with $\psi_{\frac{j}{r}} = \bar{x}_{\frac{j}{r}}$, $j = 0, \cdots, r-1$), and we write for every $\gamma > 0$,

$$\begin{aligned}\lim_{\varepsilon\downarrow 0} \varepsilon^{1-2\kappa} \ln E \exp\left\{\frac{1}{\varepsilon^{1-2\kappa}} < \alpha_i, \lambda^\varepsilon >\right\} &< \frac{1}{2}\int_0^1 < A(\psi_t)\alpha_{t,i}, \alpha_{t,i} > dt + \Delta \\ &= H^\psi(\alpha_i) + \Delta < H^\psi(\alpha_i) + \frac{\gamma}{4},\end{aligned}$$

for $\Delta > 0$ sufficiently small. Returning to (3.16) we get

$$\begin{aligned}\mathcal{P} &\le \exp\left\{-\tfrac{s-2\Delta}{\varepsilon^{1-2\kappa}}\right\} \textstyle\sum_{i=1}^N \exp\left\{-\tfrac{H^\psi(\alpha_i)}{\varepsilon^{1-2\kappa}}\right\} \exp\left\{\tfrac{1}{\varepsilon^{1-2\kappa}}\left[H^\psi(\alpha_i) + \gamma\right]\right\} \\ &= \exp\left\{-\tfrac{s}{\varepsilon^{1-2\kappa}}\right\} \exp\left\{\tfrac{2\Delta}{\varepsilon^{1-2\kappa}}\right\} N \exp\left\{\tfrac{\frac{\gamma}{4}}{\varepsilon^{1-2\kappa}}\right\} \\ &\le \exp\left\{-\tfrac{s-\frac{\gamma}{2}}{\varepsilon^{1-2\kappa}}\right\}, \qquad 0 < \varepsilon \le \varepsilon_0.\end{aligned}$$

and we can say that $\forall \gamma > 0$, $\exists \varepsilon_0 > 0$ such that

$$I_4 \le \exp\left\{-\frac{s - \frac{\gamma}{2}}{\varepsilon^{1-2\kappa}}\right\}, \qquad 0 < \varepsilon \le \varepsilon_0. \tag{3.17}$$

Therefore, by taking $I = s - \frac{\gamma}{2}$ in (3.13), there exists $\varepsilon_0 > 0$ such that

$$\begin{aligned}P\left\{\rho_{01}\left(\varepsilon^{1-2\kappa}\eta^\varepsilon_\cdot, \Phi(s)\right) > \delta\right\} &\le I_4 + I_2 + I_3 \\ &\le \exp\left\{-\frac{1}{\varepsilon^{1-2\kappa}}(s-\gamma)\right\}, \qquad 0 < \varepsilon \le \varepsilon_0.\end{aligned}$$

■

**Remark 3.1** *$A(x)$ being Lipschitz continuous, there exists a matrix $\sigma(x)$ such that $A(x) = \sigma(x)\sigma^*(x)$, $x \in \mathbb{R}^d$. Define $W_t^0 = \int_0^t \sigma(\bar{x}_s)\, dW_s$ where $W_t$ is a d-dimensional Wiener process starting at zero, $\bar{x}_t$ is the function introduced in (1.3), and $W_t^0$ is a Gaussian process with independent increments, $EW_t^0 = 0$, and correlation matrix $(R^{ij}(t))_{i,j=1,\cdots,d}$ given by*

$$R^{ij}(t) = EW_t^{0,i}W_t^{0,j} = \int_0^t A_{ij}(\bar{x}_s)\, ds.$$

*It is known (see Freidlin and Wentzell (1984)) that the action functional for* $\varepsilon^{\frac{1}{2}-\kappa}W_t^0$ *is given by* $\frac{1}{\varepsilon^{1-2\kappa}}S_{0T}^1(\varphi)$ *where* $S_{0T}^1(\varphi)$ *is the functional in (1.16). Then, from Theorem 3.1 and Theorem 3.2 we conclude that* $\varepsilon^{\frac{1}{2}-\kappa}\eta_t^\varepsilon$ *and* $\varepsilon^{\frac{1}{2}-\kappa}W_t^0$ *have the same action functional. Moreover, under Khas'minskii's conditions (see Khas'minskii (1966)),* $\eta_t^\varepsilon$ *converges weakly to* $W_t^0$ *where* $A(x)$ *satisfies*

$$A_{ki}(x) = \lim_{T\to+\infty} \frac{1}{T}\int_0^T \int_0^T A_{ki}(x,s,t)\,ds\,dt$$

*and*

$$A_{ki}(x,s,t) = E[b^k(x,Y_s) - Eb^k(x,Y_s)][b^i(x,Y_t) - Eb^i(x,Y_t)].$$

**Remark 3.2** *When* $\bar{b}(0) = 0$, *Theorem 1.1 gives the normalized action functional for* $\varepsilon^{\frac{1}{2}-\kappa}\eta_t^\varepsilon = \int_0^t \frac{f(Y_s^\varepsilon)}{\varepsilon^\kappa}\,ds$ *which is*

$$S_{0T}^1(\varphi) = \begin{cases} \frac{1}{2}\int_0^T < A^{-1}\dot{\varphi}_s, \dot{\varphi}_s > \,ds, & \varphi \text{ a.c.} \\ +\infty, & \text{in the rest of } C_{[0,T]}(\mathbb{R}^d), \end{cases}$$

*with normalizing coefficient* $\frac{1}{\varepsilon^{1-2\kappa}}$ *where* $A$ *is the matrix in Condition* **B-1**. *This is the case dealt in Baîer and Freilin(1977) and in Freidlin and Wentzell(1984, Chapter 7). It is easy to verify that the action functional for the family of processes*

$$V_t^\varepsilon = x + \frac{\int_0^t f(Y_s^\varepsilon)\,ds}{\varepsilon^\kappa}, \qquad x \in \mathbb{R}^d$$

*is also given by* $\frac{1}{\varepsilon^{1-2\kappa}}S_{0T}^1(\varphi)$. *However, in this case, the level sets are*

$$\Phi(s) = \{\varphi \in C_{[0,T]}(\mathbb{R}^d) : \mathrm{S}_{0\mathrm{T}}^1(\varphi) \le \mathrm{s},\ \varphi_0 = \mathrm{x}\}.$$

## 4 Proof of Theorem 1.2

Now we shall prove Theorem 1.2 in the most general situation, when the initial point is not necessarily an equilibrium point for the system (1.3).

We consider $X_t^\varepsilon$ satisfying (1.1) with $X_0^\varepsilon = x \in \mathbb{R}^d$ and $\bar{x}_t$ the solution of (1.3). Then,

$$X_t^\varepsilon = x + \int_0^t b(X_s^\varepsilon, Y_s^\varepsilon)\,ds. \tag{4.1}$$

We define

$$\Upsilon_t^\varepsilon = \int_0^t [b(X_s^\varepsilon, Y_s^\varepsilon) - \bar{b}(\bar{x}_s)]\, ds. \tag{4.2}$$

The process $Z_t^\varepsilon$ in (1.5) can be written as

$$Z_t^\varepsilon = \frac{\Upsilon_t^\varepsilon}{\varepsilon^\kappa}, \qquad 0 < \kappa < \frac{1}{2}.$$

We define

$$\tilde{Z}_t^\varepsilon = \frac{\tilde{\Upsilon}_t^\varepsilon}{\varepsilon^\kappa} \quad \text{and} \quad \hat{Z}_t^\varepsilon = \frac{\hat{\Upsilon}_t^\varepsilon}{\varepsilon^\kappa},$$

where $\tilde{\Upsilon}_t^\varepsilon$ and $\hat{\Upsilon}_t^\varepsilon$ were introduced in (1.18) and (1.19).

First we recall that the action functional for $\varepsilon^{\frac{1}{2}-\kappa}\eta_t^\varepsilon$ with $\eta_t^\varepsilon$ given in (1.10) is $\frac{1}{\varepsilon^{1-2\kappa}} S_{0T}^1(\varphi)$ with $S_{0T}^1$ given in (1.16). The contraction principle implies that the normalized action functional for $\hat{Z}_t^\varepsilon$ is $S_{0T}(\varphi)$ in (1.17) with normalizing coefficient $\frac{1}{\varepsilon^{1-2\kappa}}$. Now we shall prove that $\hat{Z}_t^\varepsilon$ and $\tilde{Z}_t^\varepsilon$ have the same action functional.

**Proposition 4.1** *If Condition* **B-3** *in (1.14) holds then $\hat{Z}_t^\varepsilon$ and $\tilde{Z}_t^\varepsilon$ have the same action functional.*

**Proof:** Given $\gamma > 0$, $\delta > 0$, and $\varphi \in C_{[0,T]}(\mathbb{R}^d)$, $\varphi_0 = x$,

$$\begin{aligned} P\{\|\tilde{Z}_\cdot^\varepsilon - \varphi\| < \delta\} &\geq P\{\|\tilde{Z}_\cdot^\varepsilon - \hat{Z}_\cdot^\varepsilon\| < \tfrac{\delta}{2}, \|\hat{Z}_\cdot^\varepsilon - \varphi\| < \tfrac{\delta}{2}\} \\ &\geq P\{\|\hat{Z}_\cdot^\varepsilon - \varphi\| < \tfrac{\delta}{2}\} - P\{\|\tilde{Z}_\cdot^\varepsilon - \hat{Z}_\cdot^\varepsilon\| \geq \tfrac{\delta}{2}\} \\ &\equiv I_1 - I_2. \end{aligned} \tag{4.3}$$

Since $\frac{1}{\varepsilon^{1-2\kappa}} S_{0T}(\varphi)$ in (1.17) is the action functional for $\hat{Z}_t^\varepsilon$, $\exists \varepsilon_0 > 0$ such that

$$I_1 \geq \exp\left\{-\frac{1}{\varepsilon^{1-2\kappa}}[S_{0T}(\varphi) + \gamma]\right\}, \qquad 0 < \varepsilon \leq \varepsilon_0.$$

For estimating $I_2$ we recall that the processes $\tilde{\Upsilon}_t^\varepsilon$ and $\hat{\Upsilon}_t^\varepsilon$ satisfy the linear differential equations

$$\begin{aligned} \dot{\tilde{\Upsilon}}_t^\varepsilon - B(\bar{x}_t, Y_t^\varepsilon)\, \tilde{\Upsilon}_t^\varepsilon = \tilde{b}(\bar{x}_t, Y_t^\varepsilon), \quad \text{and} \\ \dot{\hat{\Upsilon}}_t^\varepsilon - \bar{B}(\bar{x}_t)\, \hat{\Upsilon}_t^\varepsilon = \tilde{b}(\bar{x}_t, Y_t^\varepsilon), \end{aligned}$$

which implies that

$$\tilde{\Upsilon}_t^\varepsilon = \exp\left\{\int_0^t B(\bar{x}_s, Y_s^\varepsilon)\, ds\right\} \int_0^t \exp\{-\int_0^s B(\bar{x}_u, Y_u^\varepsilon)\, du\}\, \tilde{b}(\bar{x}_s, Y_s^\varepsilon)\, ds$$

and

$$\hat{\Upsilon}^\varepsilon_t = \exp\left\{\int_0^t \bar{B}(\bar{x}_s)\,ds\right\}\int_0^t \exp\left\{-\int_0^s \bar{B}(\bar{x}_u)\,du\right\}\tilde{b}(\bar{x}_s, Y^\varepsilon_s)\,ds.$$

Then,

$$\begin{aligned}
|\tilde{Z}^\varepsilon_t - \hat{Z}^\varepsilon_t| &= \left|\int_0^t B(\bar{x}_s, Y^\varepsilon_s)\,\tilde{Z}^\varepsilon_s\,ds - \int_0^t \bar{B}(\bar{x}_s)\hat{Z}^\varepsilon_s\,ds\right| \\
&\le \left|\int_0^t B(\bar{x}_s, Y^\varepsilon_s)\tilde{Z}^\varepsilon_s\,ds - \int_0^t B(\bar{x}_s, Y^\varepsilon_s)\hat{Z}^\varepsilon_s\,ds\right| \\
&+ \left|\int_0^t B(\bar{x}_s, Y^\varepsilon_s)\hat{Z}^\varepsilon_s\,ds - \int_0^t \bar{B}(\bar{x}_s)\hat{Z}^\varepsilon_s\,ds\right| \\
&\le K\int_0^t |\tilde{Z}^\varepsilon_s - \hat{Z}^\varepsilon_s|\,ds + \frac{1}{\varepsilon^\kappa}\left|\int_0^t \left(B(\bar{x}_s, Y^\varepsilon_s) - \bar{B}(\bar{x}_s)\right)\,\hat{\Upsilon}^\varepsilon_s\,ds\right| \\
&= K\int_0^t |\tilde{Z}^\varepsilon_s - \hat{Z}^\varepsilon_s|\,ds \\
&+ \frac{1}{\varepsilon^\kappa}\left|\int_0^t \left(B(\bar{x}_s, Y^\varepsilon_s) - \bar{B}(\bar{x}_s)\right)\int_0^s e^{\int_u^s \bar{B}(\bar{x}_v)\,dv}\tilde{b}(\bar{x}_u, Y^\varepsilon_u)\,du\,ds\right|,
\end{aligned}$$

for some $K > 0$. Using Lemma 1.1, Chapter 2, in Freidlin and Wentzell (1984) we obtain

$$\begin{aligned}
&|\tilde{Z}^\varepsilon_t - \hat{Z}^\varepsilon_t| \\
&\le e^{Kt}\tfrac{1}{\varepsilon^\kappa}\left|\int_0^t \left(B(\bar{x}_s, Y^\varepsilon_s) - \bar{B}(\bar{x}_s)\right)\int_0^s e^{\int_u^s \bar{B}(\bar{x}_v)\,dv}\,\tilde{b}(\bar{x}_u, Y^\varepsilon_u)\,du\,ds\right|.
\end{aligned}$$

Hence, $\forall\delta > 0$

$$\begin{aligned}
&P\left\{\|\tilde{Z}^\varepsilon_\cdot - \hat{Z}_\cdot\| \ge \frac{\delta}{2}\right\} \le e^{KT} \\
&\times P\left\{\sup_{0\le t\le T}\frac{1}{\varepsilon^\kappa}\left|\int_0^t \left(B(\bar{x}_s, Y^\varepsilon_s) - \bar{B}(\bar{x}_s)\right)\int_0^s e^{\int_u^s \bar{B}(\bar{x}_v)\,dv}\,\tilde{b}(\bar{x}_u, Y^\varepsilon_u)\,du\,ds\right| \ge \frac{\delta}{2}\right\}.
\end{aligned}$$

From Condition **B-3** we have that $\forall M > 0$, $\exists\varepsilon_0 > 0$ such that

$$I_2 \equiv P\left\{\|\tilde{Z}^\varepsilon_\cdot - \hat{Z}^\varepsilon_\cdot\| \ge \frac{\delta}{2}\right\} \le \exp\left\{-\frac{M}{\varepsilon^{1-2\kappa}}\right\}, \quad 0 < \varepsilon \le \varepsilon_0. \tag{4.4}$$

Returning to (4.3) we get

$$P\left\{\|\tilde{Z}^\varepsilon_\cdot - \varphi\| < \delta\right\} \ge \exp\left\{-\frac{1}{\varepsilon^{1-2\kappa}}\left[S_{0T}(\varphi) + \gamma\right]\right\}, \quad 0 < \varepsilon \le \varepsilon_0,$$

which is the lower bound in **(A.I)**.

The upper bound is easily obtained. Let $s > 0$ and

$$\Phi(s) = \left\{\varphi \in C_{0T}(\mathrm{I\!R}^{\mathrm{d}}) : \mathrm{S}_{0\mathrm{T}}(\varphi) \le \mathrm{s},\ \varphi_0 = \mathrm{x} \in \mathrm{I\!R}^{\mathrm{d}}\right\}.$$

Then,

$$\begin{aligned} & P\left\{\rho_{0T}\left(\tilde{Z}^{\varepsilon}_{\cdot}, \Phi(s)\right) \ge \delta\right\} \\ \le\ & P\left\{\rho_{0T}\left(\tilde{Z}^{\varepsilon}_{\cdot}, \Phi(s)\right) \ge \delta, \|\tilde{Z}^{\varepsilon}_{\cdot} - \hat{Z}_{\cdot}\| < \tfrac{\delta}{4}\right\} + P\left\{\|\tilde{Z}^{\varepsilon}_{\cdot} - \hat{Z}^{\varepsilon}_{\cdot}\| \ge \tfrac{\delta}{4}\right\} \\ \le\ & P\left\{\rho_{0T}\left(\hat{Z}^{\varepsilon}_{\cdot}, \Phi(s)\right) \ge \tfrac{\delta}{2}\right\} + P\left\{\|\tilde{Z}^{\varepsilon}_{\cdot} - \hat{Z}^{\varepsilon}_{\cdot}\| \ge \tfrac{\delta}{4}\right\} \\ \le\ & \exp\left\{-\tfrac{1}{\varepsilon^{1-2\kappa}}\left(s - \tfrac{\gamma}{2}\right)\right\} + \exp\left\{-\tfrac{M}{\varepsilon^{1-2\kappa}}\right\}, \end{aligned}$$

where the last inequality follows from (4.4) and the fact that $S_{0T}(\varphi)$ is the normalized action functional for $\hat{Z}^{\varepsilon}_t$. By taking $M = s$,

$$P\left\{\rho_{0T}\left(\tilde{Z}^{\varepsilon}_{\cdot}, \Phi(s)\right) \ge \delta\right\} \le \exp\left\{-\frac{1}{\varepsilon^{1-2\kappa}}(s - \gamma)\right\}, \qquad 0 < \varepsilon \le \varepsilon_0.$$

■

**Proposition 4.2** *If $S_{0T}(\varphi)$ is the normalized action functional for $\tilde{Z}^{\varepsilon}_t$ with normalizing coefficient $\frac{1}{\varepsilon^{1-2\kappa}}$ then it is the normalized action functional for $Z^{\varepsilon}_t$ with the same normalizing coefficient.*

**Proof:** Given $\delta > 0$ and $\gamma > 0$,

$$\begin{aligned} P\left\{\|Z^{\varepsilon}_{\cdot} - \varphi\| < \delta\right\} & \ge P\left\{\|Z^{\varepsilon}_{\cdot} - \tilde{Z}^{\varepsilon}_{\cdot}\| < \tfrac{\delta}{2}, \|\tilde{Z}^{\varepsilon}_{\cdot} - \varphi\| < \tfrac{\delta}{2}\right\} \\ & \ge P\left\{\|\tilde{Z}^{\varepsilon}_{\cdot} - \varphi\| < \tfrac{\delta}{2}\right\} - P\left\{\|Z^{\varepsilon}_{\cdot} - \tilde{Z}^{\varepsilon}_{\cdot}\| \ge \tfrac{\delta}{2}\right\} \\ & \equiv I_1 - I_2. \end{aligned} \tag{4.5}$$

A lower bound for $I_1$ is obtained from the hypothesis that $\frac{1}{\varepsilon^{1-2\kappa}} S_{0T}(\varphi)$ is the action functional for $\tilde{Z}^{\varepsilon}_t$. For estimating $I_2$ we introduce a new process

$$V^{\varepsilon}_t = \int_0^t b(\tilde{\Upsilon}^{\varepsilon}_s + \bar{x}_s, Y^{\varepsilon}_s)\, ds - \int_0^t \bar{b}(\bar{x}_s)\, ds.$$

Then,

$$V^{\varepsilon}_t + \int_0^t \bar{b}(\bar{x}_s)\, ds = \int_0^t b(\tilde{\Upsilon}^{\varepsilon}_s + \bar{x}_s, Y^{\varepsilon}_s)\, ds.$$

From the smoothness of $b(x,y)$ we get

$$b(\tilde{\Upsilon}_s^\varepsilon + \bar{x}_s, Y_s^\varepsilon) = b(\bar{x}_s, Y_s^\varepsilon) + B(\bar{x}_s, Y_s^\varepsilon)\,\tilde{\Upsilon}_s^\varepsilon + r^{(2)}(\tilde{\Upsilon}_s^\varepsilon),$$

where $r^2(\cdot)$ is the rest of Lagrange of order 2. Then,

$$V_t^\varepsilon \quad = \textstyle\int_0^t \tilde{b}(\bar{x}_s, Y_s^\varepsilon)\,ds + \int_0^t B(\bar{x}_s, Y_s^\varepsilon)\,\tilde{\Upsilon}_s^\varepsilon\,ds + \int_0^t r^{(2)}(\tilde{\Upsilon}_s^\varepsilon)\,ds$$

which implies that

$$\tilde{\Upsilon}_t^\varepsilon = V_t^\varepsilon - \int_0^t r^{(2)}(\tilde{\Upsilon}_s^\varepsilon)\,ds.$$

Therefore, taking into account (4.1) and (4.2),

$$\begin{aligned} |Z_t^\varepsilon - \tilde{Z}_t^\varepsilon| &= \tfrac{1}{\varepsilon^\kappa}|\Upsilon_t^\varepsilon - \tilde{\Upsilon}_t^\varepsilon| \\ &= \tfrac{1}{\varepsilon^\kappa}\left|\textstyle\int_0^t b(X_s^\varepsilon, Y_s^\varepsilon)\,ds - \int_0^t b(\tilde{\Upsilon}_s^\varepsilon + \bar{x}_s, Y_s^\varepsilon)\,ds + \int_0^t r^{(2)}(\tilde{\Upsilon}_s^\varepsilon)\,ds\right| \\ &\le \tfrac{1}{\varepsilon^\kappa}\left[K\textstyle\int_0^t |X_s^\varepsilon - (\tilde{\Upsilon}_s^\varepsilon + \bar{x}_s)|\,ds + \int_0^t |r^{(2)}(\tilde{\Upsilon}_s^\varepsilon)|\,ds\right] \\ &= \tfrac{1}{\varepsilon^\kappa}\left[K\textstyle\int_0^t |\Upsilon_s^\varepsilon - \tilde{\Upsilon}_s^\varepsilon|\,ds + \int_0^t |r^{(2)}(\tilde{\Upsilon}_s^\varepsilon)|\,ds\right] \\ &= K\textstyle\int_0^t |Z_s^\varepsilon - \tilde{Z}_s^\varepsilon|\,ds + \tfrac{1}{\varepsilon^\kappa}\int_0^t |r^{(2)}(\tilde{\Upsilon}_s^\varepsilon)|\,ds. \end{aligned}$$

From Lemma 1.1, Chapter 2, in Freidlin and Wentzell (1984) we get

$$|Z_t^\varepsilon - \tilde{Z}_t^\varepsilon| \le e^{Kt}\frac{1}{\varepsilon^\kappa}\int_0^t |r^{(2)}(\tilde{\Upsilon}_s^\varepsilon)|\,ds$$

for some $K > 0$. Since the second order derivatives of $b(x,y)$ are bounded, we get

$$|r^{(2)}(\tilde{\Upsilon}_s^\varepsilon)| \le \frac{M}{2}|\tilde{\Upsilon}_s^\varepsilon|^2.$$

for some $M > 0$. Therefore,

$$\begin{aligned} \sup_{0\le t\le T}|Z_t^\varepsilon - \tilde{Z}_t^\varepsilon| &\le \frac{1}{\varepsilon^\kappa}e^{KT}\int_0^T |r^{(2)}(\tilde{\Upsilon}_s^\varepsilon)|\,ds \\ &\le \frac{1}{\varepsilon^\kappa}e^{KT}\frac{M}{2}\int_0^T |\tilde{\Upsilon}_s^\varepsilon|^2\,ds \le \frac{1}{2}e^{KT}M\varepsilon^\kappa T\|\tilde{Z}_\cdot^\varepsilon\|^2 \end{aligned}$$

which implies that

$$\begin{aligned} I_2 \equiv P\left\{\|Z_\cdot^\varepsilon - \tilde{Z}_\cdot^\varepsilon\| \ge \frac{\delta}{2}\right\} &\le P\left\{\|\tilde{Z}_\cdot^\varepsilon\|^2 \ge \tfrac{\delta}{2}\tfrac{1}{\varepsilon^\kappa}\tfrac{2e^{-KT}}{TM}\right\} \\ &= P\left\{\|\tilde{Z}_\cdot^\varepsilon\| \ge \tfrac{1}{\varepsilon^{\frac{\kappa}{2}}}\sqrt{\delta\tfrac{e^{-KT}}{TM}}\right\}. \end{aligned}$$

On the other hand, for $s > 0$, $\Phi(s) = \{\varphi : S_{0T}(\varphi) \leq s, \varphi_0 = 0\}$ is compact. Moreover, $\varphi \equiv 0 \in \Phi(s)$, $\forall s > 0$ because $S_{0T}(0) = 0$. Let $\rho = \operatorname{dist}(0; \partial\Phi(s))$. Notice that $\rho > 0$ since $\partial\Phi(s)$ is compact. Besides, for $\varepsilon > 0$ sufficiently small, $\sqrt{\frac{\delta e^{-KT}}{TM}} \frac{1}{\varepsilon^{\frac{\kappa}{2}}} > 2\rho$. Then, $\exists \varepsilon_0 > 0$ such that

$$\begin{aligned} I_2 &\leq P\left\{\|\tilde{Z}^\varepsilon_\cdot\| \geq 2\rho\right\} \leq P\left\{\rho_{0T}\left(\tilde{Z}^\varepsilon_\cdot, \Phi(s)\right) \geq \rho\right\} \\ &\leq \exp\left\{-\tfrac{1}{\varepsilon^{1-2\kappa}}\left(s - \tfrac{\gamma}{2}\right)\right\}, \quad 0 < \varepsilon \leq \varepsilon_0. \end{aligned} \tag{4.6}$$

The last inequality follows from the properties of the action functional. By choosing $s = S_{0T}(\varphi) + \gamma$, we have

$$I_2 \geq \frac{1}{2}\exp\left\{-\frac{1}{\varepsilon^{1-2\kappa}}\left[S_{0T}(\varphi) + \frac{\gamma}{2}\right]\right\}, \quad 0 < \varepsilon \leq \varepsilon_0.$$

From (4.5) we get

$$\begin{aligned} P\{\|Z^\varepsilon_\cdot - \varphi\| < \delta\} &\geq \exp\left\{-\frac{1}{\varepsilon^{1-2\kappa}}\left[S_{0T}(\varphi) + \frac{\gamma}{2}\right]\right\} \\ &- \frac{1}{2}\exp\left\{-\frac{1}{\varepsilon^{1-2\kappa}}\left[S_{0T}(\varphi) + \frac{\gamma}{2}\right]\right\} \\ &\geq \exp\left\{-\frac{1}{\varepsilon^{1-2\kappa}}\left[S_{0T}(\varphi) + \gamma\right]\right\}, \quad 0 < \varepsilon \leq \varepsilon_0. \end{aligned}$$

The upper bound follows easily from (4.6): $\forall s > 0$, $\forall \delta > 0$,

$$\begin{aligned} & P\left\{\rho_{0T}\left(Z^\varepsilon_\cdot, \Phi(s)\right) \geq \delta\right\} \\ \leq\, & P\left\{\rho_{0T}\left(Z^\varepsilon_\cdot, \Phi(s)\right) \geq \delta, \|Z^\varepsilon_\cdot - \tilde{Z}^\varepsilon_\cdot\| < \tfrac{\delta}{4}\right\} + P\left\{\|Z^\varepsilon_\cdot - \tilde{Z}^\varepsilon_\cdot\| \geq \tfrac{\delta}{4}\right\} \\ \leq\, & P\left\{\rho_{0T}\left(\tilde{Z}^\varepsilon_\cdot, \Phi(s)\right) \geq \tfrac{\delta}{2}\right\} + P\left\{\|Z^\varepsilon_\cdot - \tilde{Z}^\varepsilon_\cdot\| \geq \tfrac{\delta}{4}\right\} \\ \leq\, & \exp\left\{-\tfrac{1}{\varepsilon^{1-2\kappa}}(s - \gamma)\right\}, \quad 0 < \varepsilon \leq \varepsilon_0. \end{aligned}$$

■

# 5 Wave front Propagation

In this part we shall describe the wave front for the solution of the initial-boundary value problem introduced in (1.20). The main results may be proved by using the same approach as in Freidlin (1985a, Chapter VI, or 1985b).

The conditions under $b(x,y)$, the initial function $g(x)$, and the non-linear term $f(x,y,u)$ were specified in the introduction. We assume an additional condition: $\bar{b}(0)=0$ where $\bar{b}(x)$ satisfies (1.2).

Using the Feynman-Kac formula, the solution $u^\varepsilon(t,x,y)$ of (1.20) satisfies the equality (1.23). The properties of $f(x,y,u)$ imply that

$$u^\varepsilon(t,x,y) \le E^\varepsilon_{xy} g(X^\varepsilon_t) \exp\left\{ \frac{1}{\varepsilon^{1-2\kappa}} \int_0^t c(\varepsilon^\kappa X^\varepsilon_s, Y^\varepsilon_s)\, ds \right\} \tag{5.1}$$

where $(X^\varepsilon_t)_{t\ge 0}$ is the process in (1.22), $c(x,y,u) = \frac{f(x,y,u)}{u}$, and $c(x,y) = \sup_{0\le u\le 1} c(x,y,u)$.

Define

$$\Upsilon^\varepsilon_t \equiv \frac{1}{\varepsilon^\kappa} \int_0^t c(\varepsilon^\kappa X^\varepsilon_s, Y^\varepsilon_s)\, ds = \frac{1}{\varepsilon^\kappa} \int_0^t c(Z^\varepsilon_s, Y^\varepsilon_s)\, ds,$$

where

$$Z^\varepsilon_t = \varepsilon^\kappa X^\varepsilon_t = \varepsilon^\kappa x + \int_0^t b(Z^\varepsilon_s, Y^\varepsilon_s)\, ds, \quad x \in \mathbb{R}^{\mathrm{d}}.$$

Notice that

$$\dot{Z}^\varepsilon_t = b(Z^\varepsilon_t, Y^\varepsilon_t), \qquad Z^\varepsilon_0 = \varepsilon^\kappa x.$$

Moreover,

$$\frac{1}{t} \int_0^t b(z, Y_s)\, ds \to_{t\to+\infty} \bar{b}(z)$$

with probability 1. The Averaging Principle implies that $Z^\varepsilon_t \to_{\varepsilon\downarrow 0} \bar{z}_t \equiv 0$ where $\bar{z}_t$ satisfies (1.3) with $\bar{z}_0 = 0$. On the other hand, it is known that there exists a function $\bar{c}(z)$ such that $\frac{1}{t}\int_0^t c(z,Y_s)\, ds \to_{t\to+\infty} \bar{c}(z)$ with probability one, for all $z \in \mathbb{R}^{\mathrm{d}}$. Since $\bar{b}(0)=0$ we conclude that

$$\left( \int_0^t b(Z^\varepsilon_s, Y^\varepsilon_s)\, ds, \int_0^t c(Z^\varepsilon_s, Y^\varepsilon_s)\, ds \right) \to_{\varepsilon\downarrow 0} (0, \bar{c}(0)t)$$

with probability one.

Define

$$\eta^\varepsilon_t = \left( \varepsilon^\kappa x + \int_0^t b(0, Y^\varepsilon_s)\, ds, \int_0^t c(0, Y^\varepsilon_s)\, ds - \bar{c}(0)t \right).$$

It is not difficult to show that the action functional for $\frac{Z^\varepsilon_t}{\varepsilon^\kappa}$ does not change if we start with $Z^\varepsilon_0 = 0$. Then Theorem 1.1 gives the normalized action functional for $\frac{\eta^\varepsilon_t}{\varepsilon^\kappa}$:

$$S^1_{0T}(\varphi,\eta) = \begin{cases} \frac{1}{2}\int_0^T < A^{-1}(0,\bar{c}(0)t)(\dot{\varphi}_t, \dot{\eta}_t), (\dot{\varphi}_t, \dot{\eta}_t) > \, dt, \\ \varphi, \eta \text{ a.c. }, \\ +\infty, \quad \text{in the rest of } C_{[0,T]}(\mathbb{R}^{\mathrm{d}} \times \mathbb{R}^{\mathrm{d}}) \end{cases} \tag{5.2}$$

with normalizing coefficient $\frac{1}{\varepsilon^{1-2\kappa}}$, where $A(0,z)$ is the matrix satisfying

$$< A(0,z)(\alpha,\beta),(\alpha,\beta) >$$
$$= \lim_{T\to+\infty} \frac{1}{T^{1-2\kappa}} \ln \bar{E}_y \exp\left\{T^{-\kappa}\int_0^T < (\alpha,\beta),(b(0,Y_t),c(0,Y_t)-z > dt\right\}.$$

Using Theorem 1.2, we obtain the action functional for $\left(X_t^\varepsilon, \Upsilon_t^\varepsilon - \frac{\bar{c}(0)t}{\varepsilon^\kappa}\right)$ with initial point $(x,0)$ which is given by $\frac{1}{\varepsilon^{1-2\kappa}} S_{0T}(\varphi,\eta)$ where

$$S_{0T}(\varphi,\eta) = \begin{cases} \frac{1}{2}\int_0^T < A^{-1}(0,\bar{c}(0)t)\left((\dot\varphi_t,\dot\eta_t) - \bar{B}(0,\bar{c}(0)t)(\varphi_t,\eta_t)\right), \\ \left((\dot\varphi_t,\dot\eta_t) - \bar{B}(0,\bar{c}(0)t)(\varphi_t,\eta_t)\right) > dt, \\ \varphi,\eta \text{ a.c.} \\ +\infty, \qquad \text{in the rest of } C_{[0,T]}(\mathrm{I\!R^d}\times\mathrm{I\!R^d}). \end{cases} \tag{5.3}$$

Let us define, for each $t>0$ and $x\in\mathrm{I\!R^d}$,

$$V(t,x)$$
$$= \sup\left\{\bar{c}(0)t - S_{0t}(\varphi,\eta) : \varphi,\eta\in C_{[0,+\infty)}(\mathrm{I\!R^d}),\ \varphi_0 = \mathrm{x},\ \varphi_\mathrm{t}\in \mathrm{G}_0,\ \eta_0 = 0\right\}.$$

By using the properties of the action functional, one can prove, similarly to Lemma 6.2.1 in Freidlin (1985a), that

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{1-2\kappa} \ln E_{xy}^\varepsilon g(X_t^\varepsilon) \exp\left\{\frac{1}{\varepsilon^{1-2\kappa}}\int_0^t c(Z_s^\varepsilon, Y_s^\varepsilon)\, ds\right\} = V(t,x). \tag{5.4}$$

Using (5.1) and (5.4) we obtain

$$\lim_{\varepsilon\downarrow 0} u^\varepsilon(t,x,y) = 0 \quad \text{if } V(t,x)<0 \text{ and } |y|\le a.$$

For proving that $\lim_{\varepsilon\downarrow 0} u^\varepsilon(t,x,y) = 1$ in the region $V(t,x)>0$ and $|y|\le a$, we shall assume that Condition (N) (see Freidlin (1985a)) holds:

**Condition (N):** $\forall(t,x)$ such that $V(t,x)=0$,

$$V(t,x) = \sup\left\{\bar{c}(0)t - S_{0t}(\varphi,\eta) : (\varphi,\eta)\in C_{[0,+\infty)}(\mathrm{I\!R^d}\times\mathrm{I\!R^d}),\ \varphi_0 = \mathrm{x},\right.$$
$$\left. \varphi_t\in G_0,\ V(t-s,\varphi_s)<0 \text{ for } s\in(0,t),\ \eta_0=0\right\}.$$

Similarly to the proof of Theorem 6.2.1 in Freidlin (1985a) one can prove that, under Condition (N), $\lim_{\varepsilon\downarrow 0} u^\varepsilon(t,x,y) = 1$ uniformly in any compact subset of $\{(t,x,y) : V(t,x)>0,\ |y|\le a\}$.

The following examples show the form of the wave front in some particular cases of Problem (1.20).

**Example 5.1** Assume $b(x,y) \equiv b(y)$ and $f(x,y,u) \equiv f(u)$. The differential operator in (1.21) becomes

$$L^\varepsilon = \frac{1}{2\varepsilon}\frac{\partial^2}{\partial y^2} + \frac{1}{\varepsilon^\kappa} b(y)\frac{\partial}{\partial x}$$

and $(X_t^\varepsilon : t \geq 0)$ in (1.22) is given by

$$X_t^\varepsilon = x + \frac{1}{\varepsilon^\kappa}\int_0^t b(Y_s^\varepsilon)\, ds.$$

In this case, $\bar{b}(x) \equiv \bar{b} = 0$. Let $c(u) = uf(u)$, $\bar{c} \equiv f'(0) = \sup_{0\leq u\leq 1} \frac{f(u)}{u}$, $\bar{c} > 0$.

Now,

$$V(t,x)$$
$$= \sup\left\{\bar{c}t - \frac{1}{2}\int_0^t < A^{-1}\dot{\varphi}_s, \dot{\varphi}_s > \, ds : \varphi \in C_{[0,+\infty)},\, \varphi_0 = x,\, \varphi_t \in G_0\right\}.$$

For simplifying this function we assume $G_0 = \{x \in \mathrm{I\!R}^{\mathrm{d}} : \|\mathrm{x}\| < 0\}$. From the Euler-Lagrange equation (see Arnold (1989)) we obtain

$$V(t,x) = \bar{c}t - \frac{1}{2t} < A^{-1}x, x >, \quad x \in \mathrm{I\!R}^{\mathrm{d}}.$$

The set $\{(t,x,y) : 2t^2\bar{c} =< A^{-1}x, x >\}$ describes the position of the wave front, as $\varepsilon \downarrow 0$. If $x \in \mathrm{I\!R}$, then the velocity of propagation is $\alpha^* = \sqrt{2A\bar{c}}$. It is not difficult to show that Condition (N) is satisfied.

**Example 5.2** Assume $f \equiv f(y,u)$, $b \equiv b(y)$. The operator $L^\varepsilon$ remains the same as in Example 5.1 as well as the process $(X_t^\varepsilon : t \geq 0)$. Define $c(y,u) = \frac{f(y,u)}{u}$, $c(y) = \sup_{0\leq u\leq 1} c(y,u)$. We assume that $\exists k_1, k_2$ such that $0 < k_1 \leq c(y) \leq k_2$, $\forall y$ and $\lim_{t\to+\infty} \frac{1}{t}\int_0^t c(Y_s)\, ds = \bar{c} > 0$. According to Remark 3.1, the family of processes $\left(X_t^\varepsilon, \frac{1}{\varepsilon^\kappa}\left[\int_0^t c(Y_s^\varepsilon)\, ds - \bar{c}t\right]\right\}$ has the same action functional as $\left(\frac{1}{\varepsilon^\kappa}\int_0^t b(Y_s^\varepsilon)\, ds, \frac{1}{\varepsilon^\kappa}[\int_0^t c(Y_s^\varepsilon)\, ds - \bar{c}t]\right)$. The function $V(t,x)$ becomes

$$\begin{aligned} V(t,x) &= \sup\left\{\bar{c}t - \frac{1}{2}\int_0^t < A^{-1}(\dot{\varphi}_s, \dot{\eta}_s), (\dot{\varphi}_s, \dot{\eta}_s) > \, ds : \right. \\ &\qquad \left. (\varphi, \eta) \in C_{[0,+\infty)}(\mathrm{I\!R}^{\mathrm{d}} \times \mathrm{I\!R}^{\mathrm{d}}),\, \varphi_0 = \mathrm{x},\, \eta_0 = 0, \varphi_{\mathrm{t}} \in \mathrm{G}_0\right\} \\ &= \bar{c}t - \inf_{\gamma\in\mathrm{I\!R}^{\mathrm{d}}} \frac{1}{2t} < A^{-1}(x,\gamma), (x,\gamma) > . \end{aligned}$$

Let $\gamma^*$ be the point of minimum. Then,

$$V(t,x) = \bar{c}t - \frac{1}{2t} < A^{-1}(x,\gamma^*),(x,\gamma^*) > .$$

Condition (N) is satisfied and the position of the wave front is

$$\Big\{(t,x,y) : 2\bar{c}t =< A^{-1}(x,\gamma^*),(x,\gamma^*) > \Big\}.$$

**Example 5.3:** In (1.20) assume $f \equiv f(u)$. The differential operator is $L^\varepsilon$ in (1.21) and the process satisfies (1.22). The function $V(t,x)$ is given by

$$V(t,x) = \sup\Big\{\bar{c}t - \tfrac{1}{2}\int_0^t < A^{-1}(\dot{\varphi}_s - \bar{B}\varphi_s),(\dot{\varphi}_s - \bar{B}\varphi_s) > ds : \varphi \in C_{[0,+\infty)}(\mathrm{I\!R}^{\mathrm{d}}),\ \varphi_0 = \mathrm{x},\ \varphi_{\mathrm{t}} \in \mathrm{G}_0\Big\},$$

where $A$ is the matrix in Condition **B-1** and $\bar{B}$ is given in Remark 4.1.


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