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THE MAXIMAL SIZE OF A DIGRAPH WITH MAXIMAL LOCAL CONNECTIVITY 1

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A digraph $D=(V, A)$ consists of a vertex set $V(D)$ and an arc set $A(D)$ such that each arc (x, y) has head y and tail x in $V(D)$. In this paper we consider only the strict digraphs, that is, the digraphs without loops and two arcs having the same heads and tails. Such a digraph will be denoted by $D=(V, A)$. The number of arcs in D is called the size of D and denoted by $e(D)$. Let $d^-(x)$, $d^+(x)$, $N^-(x)$ and $N^+(x)$ denote the indegree, outdegree, in-neighbour set and out-neighbour set of a vertex x in D, respectively. For every $x, y \in V(D)$ (in this order), $x \neq y$, the local connectivity $\kappa(x, y)$ (resp. arc-connectivity $\lambda(x, y)$) is defined to be the maximum number of vertex-disjoint (resp. arc-disjoint) paths from x to y. Put $\bar{\kappa}(D)=\max\{\kappa(x, y): x, y \in V(D), x \neq y\}$ and $\bar{\lambda}(D)=\max\{\lambda(x, y): x, y \in V(D), x \neq y\}$. $[r]$ denotes the greatest integer $\leqslant r$ and $\{r\}$ the least integer $\geqslant$ r. Most graphical terms and notations used here can be found in [1].

The purpose of this paper is to determine the maximal size of a digraph D with order n and $\bar{\kappa}(D) \leqslant 1$ (resp. $\bar{\lambda}(D) \leqslant 1$).

First we establish an upper bound on the size $e(D)$.

Theorem 1. *Let $D=(V, A)$ be a digraph of order n with $\bar{\kappa}(D) \leqslant 1$. Then*

$$e(D) \leqslant \begin{cases} 2(n-1) & \text{if } n \leqslant 7, \\ \left[\dfrac{n^2}{4}\right] & \text{if } n \geqslant 7. \end{cases}$$

Proof. We apply induction on n. For $n=1, 2$, there is nothing to prove; so suppose $n \geqslant 2$ and the result holds for smaller values of n.

Suppose first that D contains two oppositely oriented arcs with the same ends, say (x, y), $(y, x) \in A(D)$. Then we contract (x, y) and (y, x), that is, delete (x, y) and (y, x) from D and identify x and y. The resulting digraph is denoted by D'. Then D' has $n-1$ vertices, $e(D)=e(D')+2$, and $\bar{\kappa}(D') \leqslant 1$. By the induction hypothesis, we have

$$e(D') \leqslant \begin{cases} 2(n-2) & \text{if } n-1 \leqslant 7, \\ \left[\dfrac{(n-1)^2}{4}\right] & \text{if } n-1 \geqslant 7. \end{cases}$$

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It follows that $e(D) \leqslant 2(n-1)$ if $n \leqslant 7$ and $e(D) < \left[\frac{n^2}{4}\right]$ if $n > 7$.

Suppose next that D contains a directed cycle of length $k+1$, $k \geqslant 2$. We obtain by contracting the directed cycle a new digraph, denoted by D'. Then D' has $n-k$ vertices, $e(D) = e(D') + k + 1$, and $\bar{\kappa}(D') \leqslant 1$. By the induction hypothesis,

$$e(D') \leqslant \begin{cases} 2(n-k-1) & \text{if } n-k \leqslant 7, \\ \left[\frac{(n-k)^2}{4}\right] & \text{if } n-k \geqslant 7. \end{cases}$$

Therefore

$$e(D) < \begin{cases} 2(n-1) & \text{if } n \leqslant 7, \\ \left[\frac{n^2}{4}\right] & \text{if } n \geqslant 7. \end{cases}$$

Note that $\left[\frac{n^2}{4}\right] \leqslant 2(n-1)$ for $n \leqslant 7$. To complete the proof it suffices to prove the following

Lemma. *If the digraph $D = (V, A)$ in Theorem 1 contains neither a directed cycle nor two oppositely oriented arcs with the same ends, then $e(D) \leqslant \left[\frac{n^2}{4}\right]$.*

Proof. First we show there is a vertex v in $V(D)$ such that $d^-(v) + d^+(v) \leqslant \left[\frac{n}{2}\right]$. Since D contains neither a directed cycle nor two oppositely oriented arcs with the same ends, D has a vertex x such that $d^-(x) = 0$. If $d^+(x) \leqslant \left[\frac{n}{2}\right]$, we are done. So we may assume $d^+(x) > \left[\frac{n}{2}\right]$, then there is no arc with both head and tail in the out-neighbour set $N^+(x)$, for otherwise there would be two vertex-disjoint paths from x to some y in $N^+(x)$. Thus for any $z \in N^+(x)$, $N^-(z) \cup N^+(z) \subseteq V(D) - N^+(x)$, which implies that $d^-(z) + d^+(z) \leqslant n - d^+(x) \leqslant n - \left[\frac{n}{2}\right] - 1 \leqslant \left[\frac{n}{2}\right]$, as required.

Now we prove the lemma by induction on n. To avoid the trivial cases let us proceed to the induction step. Delete a vertex v with $d^-(v) + d^+(v) \leqslant \left[\frac{n}{2}\right]$. The resulting digraph D' has $n-1$ vertices, $e(D) = e(D') + d^-(v) + d^+(v)$, and $\bar{\kappa}(D') \leqslant 1$. By the induction hypothesis,

$$e(D') \leqslant \left[\frac{(n-1)^2}{4}\right],$$

implying

$$e(D) \leqslant \left[\frac{n^2}{4}\right].$$

This completes the proof.

Our next aim is to characterize the extremal digraphs. Let D_1 be the associated digraph of a tree T with order n, that is, the digraph obtained from T by replacing each edge e of T by two oppositely oriented arcs with the same ends as e. Obviously, $\bar{\kappa}(D_1) = 1$, $e(D_1) = 2(n-1)$. Therefore D_1 is an extremal digraph if $n \leqslant 7$. Let D_2 be a digraph obtained from the complete bipartite graph $K_{[\frac{n}{2}], \{\frac{n}{2}\}}$ by orienting all edges from one part to the other. Such a digraph D_2 is said to be an extremal orientation of $K_{[\frac{n}{2}], \{\frac{n}{2}\}}$. Then $\bar{\kappa}(D_2) = 1$, $e(D_2) = \left[\frac{n^2}{4}\right]$, and D_2 is an extremal digraph if $n \geqslant 7$. Now we state a result in the converse direction.

Theorem 2. *Let $D = (V, A)$ be an extremal digraph of Theorem 1 with order n. Then D is the associated digraph of a tree when $n \leqslant 7$ or an extremal orientation of the complete bipartite graph $K_{[\frac{n}{2}], \{\frac{n}{2}\}}$ when $n \geqslant 7$.*

The proof of Theorem 2 can be easily completed by examining the three cases in the proof of Theorem 1. We leave it to the reader.

Note that neither extremal orientation of $K_{[\frac{n}{2}], \{\frac{n}{2}\}}$ is strongly connected. If the considered digraphs are restricted to strongly connected digraphs, then the extremal digraphs of Theorem 1 are only the associated digraphs of trees with order n.

Finally let us present the analogues of the above theorems for the maximal local arc-connectivity $\bar{\lambda}(D)$.

Theorem 3. *If $D = (V, A)$ is a digraph of order n such that $\bar{\lambda}(D) \leqslant 1$, then*

$$e(D) \leqslant \begin{cases} 2(n-1) & \text{if } n \leqslant 7, \\ \left[\frac{n^2}{4}\right] & \text{if } n \geqslant 7. \end{cases}$$

The equality holds if and only if D is an extremal digraph of Theorem 1.

Proof. As $\bar{\kappa}(D) \leqslant \bar{\lambda}(D) \leqslant 1$, it follows from Theorem 1 that

$$e(D) \leqslant \begin{cases} 2(n-1) & \text{if } n \leqslant 7, \\ \left[\frac{n^2}{4}\right] & \text{if } n \geqslant 7. \end{cases}$$

It is clear that every extremal digraph of Theorem 3 is an extremal digraph of Theorem 1. The converse implication is also true because $\bar{\lambda}(D) = 1$ for every extremal digraph D of Theorem 1.

Reference

[1] Bondy, J. A. and Murty, U. S. R., Graph Theory with Applications, MacMillan, London, 1976.

局部连通度至多是 1 的有向图的最大弧数

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摘 要

设 $D=(V, A)$ 是局部连通度至多是 1 的严格有向图，令 n 和 $e(D)$ 分别表示 D 的点数和弧数．本文证明：若 $n\leqslant 7$，则 $e(D)\leqslant 2(n-1)$；若 $n\geqslant 7$，则 $e(D)\leqslant\left[\frac{n^2}{4}\right]$；并且给出极图的完全刻画．