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A class of shrinkage estimators for variance of a normal population

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Abstract: This paper suggests a class of shrinkage estimators for scale parameter σ^2 in complete samples from normal population $N(\mu, \sigma^2)$ when some apriori or guessed value σ_0^2 (say) of σ^2 is available and analyses their properties. Some estimators are generated from the proposed class and compared with the usual unbiased estimator, MMSE estimator, MLE and Singh and Singh (1997) estimators. Numerical computations have been given to judge the merits of the suggested class of shrinkage estimators over the MMSE estimator, the MLE and Singh and Singh (1997) estimators.

Key words: Bias, guessed value, mean squared error, normal distribution, percent absolute relative bias, percent relative efficiency, scale parameter, shrinkage estimator.

1 Introduction

The normal distribution plays a very important role in statistical theory and methods. The problem of estimating variance plays a significant role in solving the allocation problem in stratified random sampling, particularly in Neyman allocation, giving a quite good fit for the failure time data in life testing and reliability problems and many more.

Let $x_1, x_2, \ldots, x_n$ be a random sample of size n from a normal population $N(\mu, \sigma^2)$, probability density function (p. d. f.) of which is given by:

$$f(x; \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}, \ -\infty < x < \infty, \ -\infty < \mu < \infty, \ \sigma > 0, \tag{1.1}$$

where μ being the population mean acts as a location parameter and σ^2 being the population variance acts as a scale parameter.

For a complete sample, i.e., for an uncensored data set,

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2 \tag{1.2}$$

is the minimum variance unbiased estimator (MVUE) of σ^2, with variance

$$\mathrm{Var}\,(s^2) = \frac{2\sigma^4}{n-1}\ , \tag{1.3}$$

where $\overline{x} = (1/n)\sum_{i=1}^{n} x_i$ is the sample mean.

The maximum likelihood estimator (MLE) of σ^2 is given by

$$\hat{\sigma}^2_{ml} = \frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})^2 \tag{1.4}$$

with

$$\text{Bias}(\hat{\sigma}^2_{ml}) = -\frac{\sigma^2}{n} \tag{1.5}$$

and

$$\text{MSE}(\hat{\sigma}^2_{ml}) = \left(\frac{2n-1}{n^2}\right)\sigma^4 \ . \tag{1.6}$$

Further, from a result of Goodman (1953), Singh, Pandey and Hirano (1973) and Searls and Intarapanich (1990), it follows that the minimum mean squared error (MMSE) estimator, among the class of estimators of the form gs^2, g being a constant for which the mean squared error (MSE) of gs^2 is least, is

$$\hat{\sigma}^2_{m} = \left(\frac{n-1}{n+1}\right)s^2 \ , \tag{1.7}$$

with

$$\text{Bias}(\hat{\sigma}^2_{m}) = \frac{-2\sigma^2}{(n+1)} \tag{1.8}$$

and

$$\text{MSE}(\hat{\sigma}^2_{m}) = \frac{2\sigma^4}{(n+1)} \ . \tag{1.9}$$

Thompson (1968) considered the problem of shrinking an unbiased estimator $\hat{\theta}$ of the parameter θ towards a natural origin θ_0 and suggested a shrinkage type estimator $K\hat{\theta} + (1-K)\theta_0$, where K is a constant. The beauty of such type of shrinkage estimators lies in the fact that, though perhaps they are biased, has smaller MSE than $\hat{\theta}$ for θ in some interval around θ_0 (the so called effective interval). A large number of estimators for estimating the population variance of normal distribution have been proposed with their properties by various authors including Pandey and Singh (1977), Pandey (1979), Singh and Singh (1997) etc., when guessed value σ_0^2 of the population variance σ^2 is available.

Singh and Singh (1997) considered a class of estimators for population variance σ^2 as

$$\tilde{\sigma}^2_{(p)} = \sigma_0^2\left[1 + w\left(\frac{s^2}{\sigma_0^2}\right)^p\right] \ , \tag{1.10}$$

where p is a non-zero real number and w is a constant such that the MSE of $\tilde{\sigma}^2_{(p)}$ is at a minimum value. The idea behind this estimator was that w improves the MVUE estimator σ^2. This yields a class of shrinkage estimators, viz.

$$\hat{\sigma}^2_{(p)} = \sigma_0^2 + w_{(p)}(s^2 - \sigma_0^2), \tag{1.11}$$

with

$$\text{Bias}(\hat{\sigma}^2_{(p)}) = \sigma^2(\lambda - 1)(1 - w_{(p)}) \tag{1.12}$$

and

$$\text{MSE}(\hat{\sigma}^2_{(p)}) = \sigma^4 \left[(\lambda - 1)^2 (1 - w_{(p)})^2 + \frac{2w^2_{(p)}}{n-1} \right], \tag{1.13}$$

where $\lambda = \sigma_0^2/\sigma^2$ and $w_{(p)} = K_{1(p)}/K_{2(p)}$, defined by (2.2).

In this paper, an effort has been made to propose a modified class of shrinkage estimators for scale parameter σ^2 by considering the reciprocal of s^2 in addition to s^2 and introducing another constant to its exponent such that this constant improves the reciprocal. The properties of suggested class of estimators are further studied theoretically and empirically.

2 Suggested class of shrinkage estimators

We consider a class of estimators $\tilde{\sigma}^2_{(p,q)}$ for σ^2 in model (1.1), which is defined as:

$$\tilde{\sigma}^2_{(p,q)} = \sigma_0^2 \left[1 + w_1 \left(\frac{s^2}{\sigma_0^2} \right)^p + w_2 \left(\frac{\sigma_0^2}{s^2} \right)^q \right], \tag{2.1}$$

where p and q are non-zero real numbers such that $p + q \neq 0$, w_1 and w_2 are constants to be chosen such that $\text{MSE}(\tilde{\sigma}^2_{(p,q)})$ is minimum and σ_0^2 is a prior point estimate or guessed value of σ^2. This value σ_0^2 may be obtained either from similar studies in the past or through a guess of the experimenter.

Using the result that $E\{(s^2)^{jk}\} = K_{j(k)}(\sigma^2)^{jk}$, $j = 1, 2$, $k = p, q$ or any function of p and/or q, where

$$K_{j(k)} = \left(\frac{2}{n-1} \right)^{jk} \frac{\Gamma\left(\frac{n+2jk-1}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \tag{2.2}$$

the MSE of $\tilde{\sigma}^2_{(p,q)}$ is given by

$$\begin{aligned} \text{MSE}\{\tilde{\sigma}^2_{(p,q)}\} &= \sigma^4[r^2 + w_1^2(r+1)^{2(1-p)}K_{2(p)} + w_2^2(r+1)^{2(1+q)}K_{2(-q)} \\ &\quad +2rw_1(r+1)^{(1-p)}K_{1(p)} + 2w_1w_2(r+1)^{2-p+q}K_{1(p-q)} \\ &\quad +2rw_2(r+1)^{(1+q)}K_{1(-q)}], \end{aligned} \tag{2.3}$$

where

$$r = \lambda - 1 . \tag{2.4}$$

Minimizing (2.3) with respect to w_1 and w_2, we get

$$w_1 = -r(r+1)^{(p-1)} \frac{K_{1(p)}K_{2(-q)} - K_{1(-q)}K_{1(p-q)}}{K_{2(p)}K_{2(-q)} - K^2_{1(p-q)}} \tag{2.5}$$

and

$$w_2 = -r(r+1)^{(-q-1)} \frac{K_{2(p)}K_{1(-q)} - K_{1(p)}K_{1(p-q)}}{K_{2(p)}K_{2(-q)} - K^2_{1(p-q)}} \ . \tag{2.6}$$

Since σ^2 is unknown therefore replacing σ^2 by its MVUE s^2 in (2.5) and (2.6), we get

$$\hat{w}_1 = -\left(\frac{\sigma_0^2}{s^2} - 1\right)\left(\frac{\sigma_0^2}{s^2}\right)^{(p-1)} C_1(p,q), \tag{2.7}$$

where

$$C_1(p,q) = \frac{K_{1(p)}K_{2(-q)} - K_{1(-q)}K_{1(p-q)}}{K_{2(p)}K_{2(-q)} - K^2_{1(p-q)}} \tag{2.8}$$

and

$$\hat{w}_2 = -\left(\frac{\sigma_0^2}{s^2} - 1\right)\left(\frac{\sigma_0^2}{s^2}\right)^{(-q-1)} C_2(p,q), \tag{2.9}$$

with

$$C_2(p,q) = \frac{K_{2(p)}K_{1(-q)} - K_{1(p)}K_{1(p-q)}}{K_{2(p)}K_{2(-q)} - K^2_{1(p-q)}} \ . \tag{2.10}$$

Substituting (2.7) and (2.9) in (2.1) yields a class of shrinkage estimators for σ^2 in more feasible forms as

$$\hat{\sigma}^2_{(p,q)} = \sigma_0^2 + C(p,q)(s^2 - \sigma_0^2) = C(p,q)s^2 + \{1 - C(p,q)\}\sigma_0^2, \tag{2.11}$$

where

$$\begin{aligned} C(p,q) &= C_1(p,q) + C_2(p,q) \\ &= \frac{K_{1(p)}[K_{2(-q)} - K_{1(p-q)}] - K_{1(-q)}[K_{1(p-q)} - K_{2(p)}]}{K_{2(p)}K_{2(-q)} - K^2_{1(p-q)}} \ . \end{aligned} \tag{2.12}$$

It is apparent that (2.11) becomes the convex combination of s^2 and σ_0^2 if $C(p,q) > 0$ and $1 - C(p,q) > 0$.

Since we are dealing with the problem of estimating variance which cannot be negative, obviously it is necessary that $\hat{\sigma}^2_{(p,q)} > 0$. Thus, irrespective of the values of s^2 and σ_0^2 this immediately leads to impose the constraint

$$0 < C(p,q) < 1 \ . \tag{2.13}$$

Therefore, acceptable range of values of (p,q) for all n is given by

$$\{(p,q) \mid 0 < C(p,q) < 1\} \ . \tag{2.14}$$

If $C(p,q) = 1$, the proposed class of shrinkage estimators turns into the MVUE, otherwise it is biased with

$$\text{Bias}\{\hat{\sigma}^2_{(p,q)}\} = \sigma^2(\lambda - 1)\{1 - C(p,q)\} \ . \tag{2.15}$$

The mean squared error of $\hat{\sigma}^2_{(p,q)}$ is given by

$$\mathrm{MSE}\{\hat{\sigma}^2_{(p,q)}\} = \sigma^4 \left[(\lambda-1)^2\{1-C(p,q)\}^2 + \frac{2\{C(p,q)\}^2}{n-1}\right] . \tag{2.16}$$

It is quite evident in expressions (2.15) and (2.16) that if $\lambda = 1$, i.e., if the guessed value σ_0^2 coincides exactly with the true value σ^2, the proposed class of shrinkage estimators $\hat{\sigma}^2_{(p,q)}$ becomes unbiased and possesses minimum MSE, which is given by

$$\min \mathrm{MSE}\{\hat{\sigma}^2_{(p,q)}\} = \sigma^4 \left[\frac{2\{C(p,q)\}^2}{(n-1)}\right]. \tag{2.17}$$

The quantity $\lambda = (\sigma_0^2/\sigma^2)$ represents the departure of natural origin σ_0^2 from the true value σ^2. But in practical situations it is hardly possible to get an idea about λ. Consequently, an unbiased estimator of λ is proposed as a guideline to know in practice whether λ is within its acceptable range of dominance or not. Using application of the result of Mishra (1985), an unbiased estimator of λ is given by

$$\hat{\lambda} = \left(\frac{n-3}{n-1}\right)\frac{\sigma_0^2}{s^2},\ n > 3, \tag{2.18}$$

with variance

$$\mathrm{Var}\,(\hat{\lambda}) = \frac{2\sigma_0^4}{\sigma^4(n-5)} . \tag{2.19}$$

In inequalities (3.1), (3.4), (3.7) and (3.10) the upper and lower bounds of λ are functions of known quantities n, p and q, hence can be easily determined. Once this range is made known one can judge whether the estimated value of λ, i.e., $\hat{\lambda}$, lies in the calculated range of dominance or not. Such ranges are reckoned in Tables 1 to 5. While observing these tables it is quite evident that the ranges of dominance of λ are wide enough depicting that even if $\hat{\lambda}$ departs much from λ there is enough possibility of lying λ in its range of dominance.

3 Comparison of estimators

It is generally accepted that minimum MSE is a highly desirable property, and it is therefore used as a criterion to compare different estimators with each other, for instance see James and Stein (1961). The conditions under which the proposed class of estimators is better than the conventional estimators and Singh and Singh (1997) class of estimators are given below:

(i) The $\hat{\sigma}^2_{(p,q)}$ has smaller MSE than s^2 if

$$1 - \sqrt{T} \le \lambda \le 1 + \sqrt{T} \tag{3.1}$$

or equivalently,

$$\sigma_0^2(1+\sqrt{T})^{-1} \le \sigma^2 \le \sigma_0^2(1-\sqrt{T})^{-1}, \tag{3.2}$$

where

$$T = \frac{2}{n-1}\left[\frac{1+C(p,q)}{1-C(p,q)}\right] . \tag{3.3}$$

(ii) $\mathrm{MSE}(\hat{\sigma}^2_{(p,q)})$ does not exceed $\mathrm{MSE}(\hat{\sigma}^2_m)$ if

$$1-\sqrt{G} \le \lambda \le 1+\sqrt{G} \tag{3.4}$$

or, an equivalent condition for the true value of the population variance is

$$\sigma_0^2(1+\sqrt{G})^{-1} \le \sigma^2 \le \sigma_0^2(1-\sqrt{G})^{-1}, \tag{3.5}$$

where

$$G = \frac{2}{\{1-C(p,q)\}^2}\left[\frac{1}{n+1} - \frac{\{C(p,q)\}^2}{n-1}\right] . \tag{3.6}$$

(iii) $\mathrm{MSE}(\hat{\sigma}^2_{(p,q)})$ does not overstep $\mathrm{MSE}(\hat{\sigma}^2_{ml})$ if

$$1-\sqrt{R} \le \lambda \le 1+\sqrt{R} \tag{3.7}$$

or equivalently,

$$\sigma_0^2(1+\sqrt{R})^{-1} \le \sigma^2 \le \sigma_0^2(1-\sqrt{R})^{-1} , \tag{3.8}$$

where

$$R = \frac{1}{\{1-C(p,q)\}^2}\left[\frac{2n-1}{n^2} - \frac{2\{C(p,q)\}^2}{n-1}\right] . \tag{3.9}$$

(iv) $\mathrm{MSE}(\hat{\sigma}^2_{(p,q)}) \le \mathrm{MSE}(\hat{\sigma}^2_{(p)})$ if

$$\lambda \notin (1-\sqrt{M},\ 1+\sqrt{M}), \tag{3.10}$$

where

$$M = \frac{2}{n-1}\left[\frac{C(p,q)+w_{(p)}}{2-C(p,q)-w_{(p)}}\right] . \tag{3.11}$$

Besides minimum MSE criterion, minimum bias is also important and therefore should be considered under study. The conditions under which the proposed class of estimators has smaller absolute relative bias (ARB) than the other competitive estimators are given below:

(i) $\mathrm{ARB}(\hat{\sigma}^2_{(p,q)})$ does not exceed $\mathrm{ARB}(\hat{\sigma}^2_m)$, if

$$\left[1-\frac{2}{(n+1)\{1-C(p,q)\}}\right] < \lambda < \left[1+\frac{2}{(n+1)\{1-C(p,q)\}}\right] . \tag{3.12}$$

(ii) ARB($\hat{\sigma}^2_{(p,q)}$) does not overstep ARB($\hat{\sigma}^2_{ml}$) if

$$\left[1-\frac{1}{n\{1-C(p,q)\}}\right]<\lambda<\left[1+\frac{1}{n\{1-C(p,q)\}}\right]. \tag{3.13}$$

(iii) The $\hat{\sigma}^2_{(p,q)}$ has smaller ARB than $\hat{\sigma}^2_{(p)}$ if

$$\{1-C(p,q)\}^2<\{1-w_{(p)}\}^2. \tag{3.14}$$

4 Numerical illustrations

An exact analytical study about the performance of the proposed class of estimators is not possible because the expressions for the proposed class of estimators appear to be too complicated to obtain in nice compact forms. A similar remark applies to the expressions for ARBs and MSEs of the proposed class of estimators in order to obtain the corresponding relative efficiencies and bias performance. Therefore, we are left with no other better choice than empirical study.

To illustrate the efficiency performance of the proposed class of estimators $\hat{\sigma}^2_{(p,q)}$ with the MMSE estimator $\hat{\sigma}^2_m$ (which is theoretically better than the MVU estimator s^2), the MLE and Singh and Singh (1997) class of estimators, we have computed the Percent Relative Efficiencies (PREs) of $\hat{\sigma}^2_{(p,q)}$ with respect to $\hat{\sigma}^2_m$, $\hat{\sigma}^2_{ml}$ and $\hat{\sigma}^2_{(p)}$ by the formulae:

$$\mathrm{PRE}\{\hat{\sigma}^2_{(p,q)},\hat{\sigma}^2_m\}=\frac{2(n-1)}{(n+1)[(n-1)(\lambda-1)^2\{1-C(p,q)\}^2+2\{C(p,q)\}^2]}\times 100, \tag{4.1}$$

$$\mathrm{PRE}\{\hat{\sigma}^2_{(p,q)},\hat{\sigma}^2_{ml}\}=\frac{2n^2-3n+1}{n^2[(n-1)(\lambda-1)^2\{1-C(p,q)\}^2+2\{C(p,q)\}^2]}\times 100, \tag{4.2}$$

$$\mathrm{PRE}\{\hat{\sigma}^2_{(p,q)},\hat{\sigma}^2_{(p)}\}=\frac{(n-1)(\lambda-1)^2\{1-w_{(p)}\}^2+2w^2_{(p)}}{(n-1)(\lambda-1)^2\{1-C(p,q)\}^2+2\{C(p,q)\}^2}\times 100. \tag{4.3}$$

The range of dominance of λ in which the proposed estimator $\hat{\sigma}^2_{(p,q)}$ is better than the MMSE estimator, the MLE and Singh and Singh (1997) class of estimators can be reckoned with the help of (3.4), (3.7) and (3.10) respectively.

To elucidate the bias performance of the envisaged class of estimators $\hat{\sigma}^2_{(p,q)}$ with the MMSE estimator $\hat{\sigma}^2_m$, the MLE $\hat{\sigma}^2_{ml}$ and Singh and Singh (1997) class of estimators $\hat{\sigma}^2_{(p)}$, the Percent Absolute Relative Bias (PARBs) are computed by the formulae:

$$\mathrm{PARB}\{\hat{\sigma}^2_{(p,q)},\hat{\sigma}^2_m\}=\left|\frac{2}{(n+1)(\lambda-1)\{1-C(p,q)\}}\right|\times 100, \tag{4.4}$$

$$\mathrm{PARB}\{\hat{\sigma}^2_{(p,q)},\hat{\sigma}^2_{ml}\}=\left|\frac{1}{n(\lambda-1)\{1-C(p,q)\}}\right|\times 100, \tag{4.5}$$

$$\mathrm{PARB}\{\sigma^2_{(p,q)},\hat{\sigma}^2_{(p)}\}=\frac{\{1-w_{(p)}\}^2}{\{1-C(p,q)\}^2}\times 100. \tag{4.6}$$

The findings of PREs are set down in Tables 1, 3, 5 and that of PARBs in Tables 2, 4, 6 for $n = 5(5)20$, $p = \pm 2$, $q = 1.00$, 1.25, 1.75 and different values of λ.

Table 1 *PREs of proposed estimator $\hat{\sigma}^2_{(p,q)}$ with respect to MMSE estimator $\hat{\sigma}^2_m$*

$q \downarrow$	$p \rightarrow$	-2				2			
	$\lambda\downarrow$ $n\rightarrow$	5	10	15	20	5	10	15	20
1.00	0.05	98.65	91.29	100.33	95.05	76.92	96.39	100.17	100.69
	0.25	105.22	112.72	103.27	95.19	113.11	114.08	109.17	103.80
	0.50	112.09	143.74	106.12	95.53	199.29	137.23	119.00	112.24
	0.75	116.65	172.18	107.91	95.67	367.16	156.26	125.79	116.06
	1.00	118.26	184.33	108.52	95.71	510.48	163.82	128.23	117.39
	1.25	116.65	172.18	107.91	95.67	367.16	156.26	125.79	116.06
	1.50	112.09	143.74	106.12	95.53	199.29	137.23	119.00	112.24
	1.75	105.22	112.72	103.27	95.30	113.11	114.08	109.17	106.40
	2.00	96.91	86.57	99.52	94.98	70.45	92.29	97.86	99.17
	Range of λ	(0.09, 1.91)	(0.14, 1.86)	(0.03, 1.97)	–	(0.19, 1.81)	(0.09, 1.91)	(0.05, 1.95)	(0.03, 1.97)
	$C(p.q)$	0.7508	0.6662	0.8980	0.9722	0.3614	0.7067	0.8261	0.8779
1.25	0.05	103.07	71.91	100.82	98.50	66.34	84.17	95.03	98.11
	0.25	115.69	99.65	108.59	99.10	101.50	109.04	109.37	103.10
	0.50	130.37	154.39	116.87	100.59	197.88	149.68	126.99	117.66
	0.75	141.11	230.31	122.47	101.21	459.91	192.80	140.58	124.81
	1.00	145.10	275.46	124.45	101.42	823.32	213.28	145.78	127.39
	1.25	141.11	230.31	122.47	101.21	459.91	192.80	140.58	124.81
	1.50	130.37	154.39	116.87	100.59	197.88	149.68	126.99	117.66
	1.75	115.69	99.65	108.59	99.58	101.50	109.04	109.37	107.41
	2.00	99.94	66.59	98.80	98.20	60.35	79.01	91.58	95.73
	Range of λ	(0.00, 2.00)	(0.25, 1.75)	(0.03, 1.97)	(0.34, 1.66)	(0.24, 1.76)	(0.18, 1.82)	(0.12, 1.88)	(0.09, 1.91)
	$C(p.q)$	0.6778	0.5450	0.8385	0.9445	0.2846	0.6194	0.7747	0.8428
1.75	0.05	100.71	44.52	81.85	100.27	58.54	61.20	78.42	87.23
	0.25	125.38	68.66	103.11	103.84	91.50	89.01	100.76	95.74
	0.50	161.82	136.89	135.44	113.69	189.66	152.89	136.50	125.60
	0.75	196.00	338.97	166.84	118.23	532.24	268.48	173.39	143.79
	1.00	210.84	667.36	180.81	119.83	1337.63	358.94	190.56	151.07
	1.25	196.00	338.97	166.84	118.23	532.24	268.48	173.39	143.79
	1.50	161.82	136.89	135.44	113.69	189.66	152.89	136.50	125.60
	1.75	125.38	68.66	103.11	106.84	91.50	89.01	100.76	103.74
	2.00	95.33	40.44	77.28	98.53	53.06	56.16	73.74	83.42
	Range of λ	(0.04, 1.96)	(0.40, 1.60)	(0.22, 1.78)	(0.04, 1.96)	(0.29, 1.71)	(0.31, 1.69)	(0.24, 1.76)	(0.21, 1.79)
	$C(p.q)$	0.5623	0.3501	0.6957	0.8689	0.2232	0.4774	0.6776	0.7739

Table 2 *PARBs of proposed estimator $\hat{\sigma}^2_{(p,q)}$ with respect to MMSE estimator $\hat{\sigma}^2_m$*

$q\downarrow$	$p \rightarrow$	-2				2			
	$\lambda\downarrow$ $n\rightarrow$	5	10	15	20	5	10	15	20
1.00	0.05	140.81	57.34	128.95	361.25	54.94	65.25	75.64	82.11
	0.25	178.36	72.63	163.33	457.58	69.59	82.65	95.81	104.00
	0.50	267.55	108.95	245.00	686.37	104.39	123.98	143.72	156.01
	0.75	535.09	217.90	490.00	1372.73	208.78	247.96	287.44	312.01
	1.00	u.b.	u.b.	u.b.	u.b.	u.b.	u.b.	u.b.	u.b.
	1.25	535.09	217.90	490.00	1372.73	208.78	247.96	287.44	312.01
	1.50	267.55	108.95	245.00	686.37	104.39	123.98	143.72	156.01
	1.75	178.36	72.63	163.33	457.58	69.59	82.65	95.81	104.00
	2.00	133.77	54.47	122.50	343.18	52.20	61.99	71.86	78.00
	Range of λ	(0.00, 2.34)	(0.46, 1.54)	(0.00, 2.22)	(0.00, 4.43)	(0.48, 1.52)	(0.38, 1.62)	(0.28, 1.72)	(0.22, 1.78)
	$C(p.q)$	0.7508	0.6662	0.8980	0.9722	0.3614	0.7067	0.8261	0.8779
1.25	0.05	108.91	42.06	81.47	180.71	49.04	50.28	58.41	63.76
	0.25	137.96	53.28	103.19	228.90	62.12	63.69	73.99	80.76
	0.50	206.94	79.92	154.79	343.35	93.18	95.54	110.98	121.14
	0.75	413.87	159.84	309.58	686.70	186.36	191.07	221.97	242.27
	1.00	u.b.	u.b.	u.b.	u.b.	u.b.	u.b.	u.b.	u.b.
	1.25	413.87	159.84	309.58	686.70	186.36	191.07	221.97	242.27
	1.50	206.94	79.92	154.79	343.35	93.18	95.54	110.98	121.14
	1.75	137.96	53.28	103.19	228.90	62.12	63.69	73.99	80.76
	2.00	103.47	39.96	77.40	171.67	46.59	47.77	55.49	60.57
	Range of λ	(0.00, 2.03)	(0.60, 1.40)	(0.23, 1.77)	(0.00, 2.72)	(0.53, 1.47)	(0.52, 1.48)	(0.45, 1.55)	(0.39, 1.61)
	$C(p.q)$	0.6778	0.5450	0.8385	0.9445	0.2846	0.6194	0.7747	0.8428
1.75	0.05	80.17	29.45	43.23	76.49	45.17	36.62	40.82	44.33
	0.25	101.54	37.30	54.76	96.88	57.22	46.39	51.70	56.16
	0.50	152.31	55.96	82.15	145.32	85.83	69.59	77.55	84.24
	0.75	304.63	111.91	164.29	290.65	171.65	139.17	155.10	168.47
	1.00	u.b.	u.b.	u.b.	u.b.	u.b.	u.b.	u.b.	u.b.
	1.25	304.63	111.91	164.29	290.65	171.65	139.17	155.10	168.47
	1.50	152.31	55.96	82.15	145.32	85.83	69.59	77.55	84.24
	1.75	101.54	37.30	54.76	96.88	57.22	46.39	51.70	56.16
	2.00	76.16	27.98	41.07	72.66	42.91	34.79	38.77	42.12
	Range of λ	(0.24, 1.76)	(0.72, 1.28)	(0.59, 1.41)	(0.27, 1.73)	(0.57, 1.43)	(0.65, 1.35)	(0.61, 1.39)	(0.58, 1.42)
	$C(p.q)$	0.5623	0.3501	0.6957	0.8689	0.2232	0.4774	0.6776	0.7739

** u.b. stands for unbiased

Table 3 *PREs of proposed estimator $\hat{\sigma}^2_{(p,q)}$ with respect to MLE $\hat{\sigma}^2_{ml}$*

$q \downarrow$	$p \rightarrow$	-2				2			
	$\lambda\downarrow$ $n\rightarrow$	5	10	15	20	5	10	15	20
	0.05	106.54	95.39	103.45	97.31	83.07	100.73	103.29	103.09
	0.25	113.64	117.79	106.48	97.56	122.16	119.22	112.57	108.92
	0.50	121.05	150.21	109.42	97.80	215.24	143.41	122.70	114.90
	0.75	125.99	179.92	111.26	97.94	396.53	163.29	129.70	118.82
	1.00	127.72	192.62	111.89	97.99	551.32	171.20	132.22	120.18
1.00	1.25	125.99	179.92	111.26	97.94	396.53	163.29	129.70	118.82
	1.50	121.05	150.21	109.42	97.80	215.24	143.41	122.70	114.90
	1.75	113.64	117.79	106.48	97.56	122.16	119.22	112.57	108.92
	2.00	104.66	90.46	102.62	97.24	76.09	96.44	100.90	101.53
	Range of λ	(0.00, 2.12)	(0.09, 1.91)	(0.00, 2.15)	—	(0.15, 1.85)	(0.04, 1.96)	(0.00, 2.02)	(0.00, 2.05)
	$C(p.q)$	0.7508	0.6662	0.8980	0.9722	0.3614	0.7067	0.8261	0.8779
	0.05	111.32	75.14	103.96	100.84	71.65	87.96	97.98	100.44
	0.25	124.95	104.13	111.97	101.95	109.62	113.95	112.78	109.96
	0.50	140.80	161.34	120.50	102.98	213.71	156.42	130.94	120.45
	0.75	152.40	240.67	126.28	103.61	496.70	201.48	144.95	127.77
	1.00	156.70	287.85	128.33	103.83	889.18	222.88	150.31	130.41
1.25	1.25	152.40	240.67	126.28	103.61	496.70	201.48	144.95	127.77
	1.50	140.80	161.34	120.50	102.98	213.71	156.42	130.94	120.45
	1.75	124.95	104.13	111.97	101.95	109.62	113.95	112.78	109.96
	2.00	107.94	69.59	101.87	100.53	65.18	82.56	94.43	98.00
	Range of λ	(0.00, 2.12)	(0.23, 1.77)	(0.00, 2.04)	(0.00, 2.08)	(0.21, 1.79)	(0.15, 1.85)	(0.08, 1.92)	(0.04, 1.96)
	$C(p.q)$	0.6778	0.5450	0.8385	0.9445	0.2846	0.6194	0.7747	0.8428
	0.05	108.76	46.52	84.39	102.65	63.22	63.95	80.86	89.30
	0.25	135.41	71.75	106.31	109.38	98.82	93.02	103.90	106.21
	0.50	174.77	143.05	139.66	116.39	204.83	159.77	140.74	128.59
	0.75	211.68	354.22	172.03	121.04	574.82	280.56	178.78	147.20
	1.00	227.71	697.40	186.43	122.68	1444.64	375.09	196.49	154.66
1.75	1.25	211.68	354.22	172.03	121.04	574.82	280.56	178.78	147.20
	1.50	174.77	143.05	139.66	116.39	204.83	159.77	140.74	128.59
	1.75	135.41	71.75	106.31	109.38	98.82	93.02	103.90	106.21
	2.00	102.95	42.26	79.68	100.87	57.30	58.69	76.03	85.40
	Range of λ	(0.00, 2.03)	(0.38, 1.62)	(0.20, 1.80)	(0.00, 2.02)	(0.25, 1.75)	(0.29, 1.71)	(0.22, 1.78)	(0.18, 1.82)
	$C(p.q)$	0.5623	0.3501	0.6957	0.8689	0.2232	0.4774	0.6776	0.7739

Table 4 *PARBs of proposed estimator* $\hat{\sigma}^2_{(p,q)}$ *with respect to MLE* $\hat{\sigma}^2_{ml}$

$q \downarrow$	$p \rightarrow$	-2				2			
	$\lambda\downarrow$ $n\rightarrow$	5	10	15	20	5	10	15	20
1.00	0.05	84.49	31.54	68.77	189.65	32.97	35.89	40.34	43.11
	0.25	107.02	39.95	87.11	240.23	41.76	45.46	51.10	54.60
	0.50	160.53	59.92	130.67	360.34	62.63	68.19	76.65	81.90
	0.75	321.06	119.84	261.33	720.68	125.27	136.38	153.30	163.81
	1.00	u.b.	u.b.	u.b.	u.b.	u.b.	u.b.	u.b.	u.b.
	1.25	321.06	119.84	261.33	720.68	125.27	136.38	153.30	163.81
	1.50	160.53	59.92	130.67	360.34	62.63	68.19	76.65	81.90
	1.75	107.02	39.95	87.11	240.23	41.76	45.46	51.10	54.60
	2.00	80.26	29.96	65.33	180.17	31.32	34.09	38.33	40.95
	Range of λ	(0.20, 1.80)	(0.70, 1.30)	(0.35, 1.61)	(0.00, 2.80)	(0.69, 1.31)	(0.66, 1.34)	(0.62, 1.38)	(0.59, 1.41)
	$C(p.q)$	0.7508	0.6662	0.8980	0.9722	0.3614	0.7067	0.8261	0.8779
1.25	0.05	65.35	23.13	43.45	94.87	29.43	27.65	31.15	33.47
	0.25	82.77	29.30	55.04	120.17	37.27	35.03	39.46	42.40
	0.50	124.16	43.96	82.56	180.26	55.91	52.54	59.19	63.60
	0.75	248.32	87.91	165.11	360.52	111.82	105.09	118.38	127.19
	1.00	u.b.	u.b.	u.b.	u.b.	u.b.	u.b.	u.b.	u.b.
	1.25	248.32	87.91	165.11	360.52	111.82	105.09	118.38	127.19
	1.50	124.16	43.96	82.56	180.26	55.91	52.54	59.19	63.60
	1.75	82.77	29.30	55.04	120.17	37.27	35.03	39.46	42.40
	2.00	62.08	21.98	41.28	90.13	27.95	26.27	29.60	31.80
	Range of λ	(0.38, 1.62)	(0.78, 1.22)	(0.59, 1.41)	(0.10, 1.90)	(0.72, 1.28)	(0.74, 1.26)	(0.70, 1.30)	(0.68, 1.32)
	$C(p.q)$	0.6778	0.5450	0.8385	0.9445	0.2846	0.6194	0.7747	0.8428
1.75	0.05	48.10	16.20	23.06	40.16	27.10	20.14	21.77	23.28
	0.25	60.93	20.52	29.21	50.86	34.33	25.52	27.57	29.48
	0.50	91.39	30.78	43.81	76.29	51.50	38.27	41.36	44.22
	0.75	182.78	61.55	87.62	152.59	102.99	76.55	82.72	88.45
	1.00	u.b.	u.b.	u.b.	u.b.	u.b.	u.b.	u.b.	u.b.
	1.25	182.78	61.55	87.62	152.59	102.99	76.55	82.72	88.45
	1.50	91.39	30.78	43.81	76.29	51.50	38.27	41.36	44.22
	1.75	60.93	20.52	29.21	50.86	34.33	25.52	27.57	29.48
	2.00	45.69	15.39	21.91	38.15	25.75	19.14	20.68	22.11
	Range of λ	(0.54, 1.46)	(0.85, 1.15)	(0.78, 1.22)	(0.62, 1.38)	(0.74, 1.26)	(0.81, 1.19)	(0.79, 1.21)	(0.78, 1.22)
	$C(p.q)$	0.5623	0.3501	0.6957	0.8689	0.2232	0.4774	0.6776	0.7739

Table 5 *PREs of proposed estimator $\hat{\sigma}^2_{(p,q)}$ with respect to Singh and Singh (1997) estimator $\hat{\sigma}^2_{(p)}$*

$q \downarrow$	$p \rightarrow$	-2				2			
	$\lambda\downarrow$ $n\rightarrow$	5	10	15	20	5	10	15	20
1.00	0.05	259.34	412.59	419.91	344.95	137.90	183.86	184.02	174.92
	0.50	81.65	180.24	128.29	108.02	107.62	93.43	89.70	89.58
	1.00	0.04	0.48	7.44	16.60	30.63	34.55	43.44	51.46
	1.50	81.65	180.24	128.29	108.02	107.62	93.43	89.70	89.58
	2.00	282.29	433.50	460.78	380.16	139.50	192.95	195.61	186.18
	2.50	518.19	586.35	931.68	826.82	148.54	265.10	307.61	305.33
	3.00	732.43	668.94	1459.63	1438.07	152.06	308.83	399.27	419.13
	3.50	905.76	715.60	1981.60	2200.76	153.77	335.23	467.35	515.39
	4.00	1039.38	743.79	2460.58	3099.20	154.72	351.82	516.51	592.31
	Ineffective range of λ	(0.44, 1.56)	(0.65, 1.35)	(0.56, 1.44)	(0.52, 1.48)	(0.56, 1.44)	(0.47, 1.53)	(0.44, 1.56)	(0.43, 1.57)
	$C(p.q)$	0.7508	0.6662	0.8980	0.9722	0.3614	0.7067	0.8261	0.8779
1.25	0.05	270.97	325.01	421.97	357.48	118.94	160.55	174.58	170.42
	0.50	94.97	193.59	141.28	113.74	106.86	101.91	95.73	93.91
	1.00	0.05	0.71	8.53	17.59	49.40	44.98	49.39	55.85
	1.50	94.97	193.59	141.28	113.74	106.86	101.91	95.73	93.91
	2.00	291.13	333.48	457.42	393.04	119.49	165.18	183.07	179.72
	2.50	471.56	385.11	811.59	830.12	122.47	196.32	254.75	268.51
	3.00	602.21	407.18	1117.94	1388.85	123.57	211.42	302.19	339.72
	3.50	690.80	418.28	1355.84	2029.05	124.09	219.47	332.46	391.78
	4.00	750.79	424.56	1533.47	2712.50	124.38	224.19	352.15	428.96
	Ineffective range of λ	(0.49, 1.51)	(0.69, 1.31)	(0.59, 1.41)	(0.54, 1.46)	(0.60, 1.40)	(0.51, 1.49)	(0.47, 1.53)	(0.46, 1.54)
	$C(p.q)$	0.6778	0.5450	0.8385	0.9445	0.2846	0.6194	0.7747	0.8428
1.75	0.05	264.75	201.23	342.54	363.89	104.95	116.73	144.07	151.52
	0.50	117.88	171.64	163.74	128.54	102.41	104.09	102.89	100.25
	1.00	0.07	1.73	12.39	20.78	80.26	75.70	64.56	66.23
	1.50	117.88	171.64	163.74	128.54	102.41	104.09	102.89	100.25
	2.00	277.68	202.53	357.79	394.37	105.05	117.42	147.40	156.61
	2.50	370.80	209.53	465.34	708.56	105.61	121.39	170.08	196.60
	3.00	420.12	212.09	520.74	995.45	105.81	122.97	181.27	220.49
	3.50	447.68	213.30	551.24	1228.65	105.91	123.73	187.29	234.79
	4.00	464.22	213.96	569.40	1409.09	105.96	124.16	190.83	243.72
	Ineffective range of λ	(0.55, 1.45)	(0.77, 1.23)	(0.64, 1.36)	(0.57, 1.43)	(0.63, 1.37)	(0.58, 1.42)	(0.53, 1.47)	(0.50, 1.50)
	$C(p.q)$	0.5623	0.3501	0.6957	0.8689	0.2232	0.4774	0.6776	0.7739

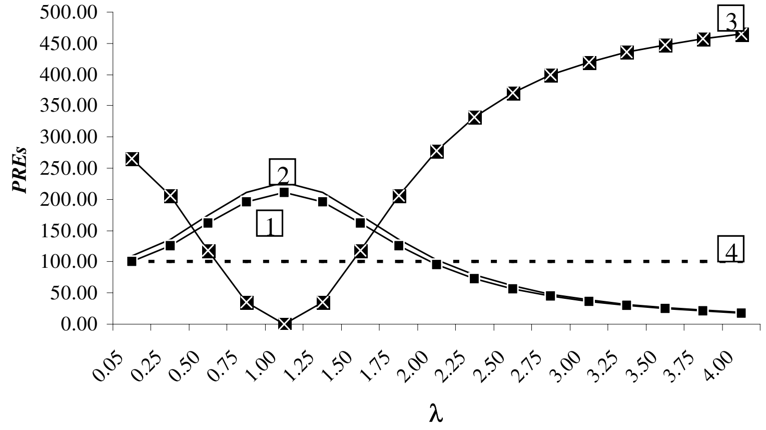
Table 6 *PARBs of proposed estimator $\hat{\sigma}^2_{(p,q)}$ with respect to Singh and Singh (1997) estimator $\hat{\sigma}^2_{(p)}$*

$p \rightarrow$	-2				2			
$q \downarrow$ $n \rightarrow$	5	10	15	20	5	10	15	20
1.00	1563.64	816.91	5475.97	47351.42	156.93	397.30	685.86	919.35
1.25	935.43	439.58	2185.89	11849.31	125.03	235.90	409.00	554.31
1.75	506.78	215.49	615.60	2122.72	106.08	125.16	199.69	268.04

It has been observed from Tables 1 to 4 that if n, p, q are fixed, the PREs as well as PARBs of proposed class of shrinkage estimators with respect to the MMSE estimator and the MLE increases up to $\lambda = 1$ (i.e. when the guessed value exactly coincide with the true value), procures its maximum at this point and then decreases symmetrically in magnitude, as λ increases in its range of dominance for all n, p and q. This at once leads to conclude that the range of dominance of λ cannot exceed $0 < \lambda < 2$. In other words, even if the guessed value σ_0^2 under estimates or over estimates almost two times or less (but not exactly two times) the true value σ^2, the proposed class of estimators yields better estimators than the MMSE estimator and the MLE in the effective interval of λ. It is interesting to note that at $\lambda = 1$, the proposed class of estimators is unbiased with largest efficiency and hence in the vicinity of $\lambda = 1$ also, the proposed class of estimators not only renders massive gain in efficiency but it is less biased in comparison of MMSE estimator and MLE. Moreover, it is observed that to get better estimators in the class, the value of $C(p, q)$ should be as small as possible in the interval of (0,1).

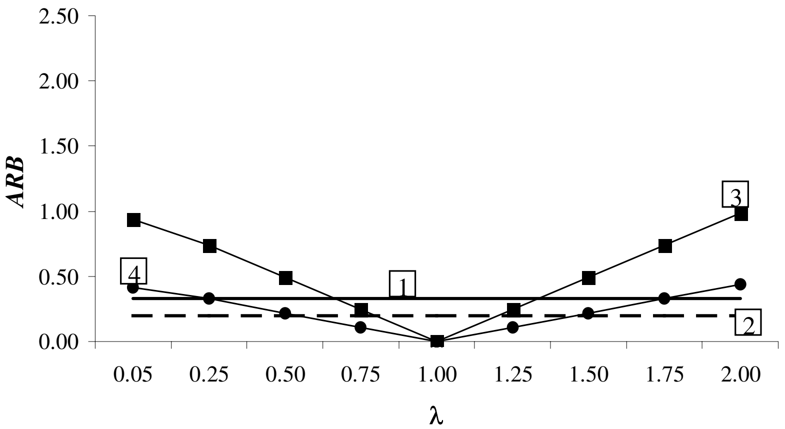
If the proposed class of estimators is compared in terms of efficiency with Singh and Singh (1997) class of estimators, Table 5 depicts that for fixed n, p, q the PREs decreases up to $\lambda = 1$, attains its minimum at this point and then increases as λ increases. This implies that the suggested class of estimators is more efficient than Singh and Singh (1997) class of estimators in the region lies either sides of the closed proximity of $\lambda = 1$. Thus if the experimenter has less confidence in the guessed value, the suggested class of estimators performs better than Singh and Singh (1997) class of estimators. The PARBs on the other hand are independent of λ and Table 6 shows results in support of the proposed class of estimators. It is worth mentioning that if $w_{(p)} < C(p, q)$, the suggested class of estimators is always less biased than Singh and Singh (1997) class of estimators, no matter what is the value of λ.

Figures 1 and 2 clearly show the above discussion. Figure 1 represents the efficiency performance and Figure 2 represents the bias performance.



$\boxed{1}$ PRE$\left(\hat{\sigma}^2_{(p,q)}, \hat{\sigma}^2_m\right)$, $\boxed{2}$ PRE$\left(\hat{\sigma}^2_{(p,q)}, \hat{\sigma}^2_{ml}\right)$, $\boxed{3}$ PRE$\left(\hat{\sigma}^2_{(p,q)}, \hat{\sigma}^2_{(p)}\right)$, $\boxed{4}$ Cut-Off Point

Figure 1 *Efficiency performance*



$\boxed{1}$ ARB$\left(\hat{\sigma}^2_m\right)$, $\boxed{2}$ ARB$\left(\hat{\sigma}^2_{ml}\right)$, $\boxed{3}$ ARB$\left(\hat{\sigma}^2_{(p)}\right)$, $\boxed{4}$ ARB$\left(\hat{\sigma}^2_{(p,q)}\right)$

Figure 2 *Bias performance*

5 Conclusion

A modification in the class of estimators reported by Singh and Singh (1997) has been suggested. The class of estimators thus obtained seems to be an improved version of Singh and Singh (1997) class of estimators subject to certain conditions. The proposed class leads to formulate many interesting estimators of shrinkage type. It is identified that when the guessed value σ_0^2 coincides exactly with the true value σ^2 and also when σ_0^2 is moderately far away from σ^2, we get larger gain in efficiency over the MMSE estimator and the MLE in the effective interval of λ. The suggested class of shrinkage estimators has substantial gain in efficiency for a number of choices of p and q, when the sample size is small. Even for large sample sizes, so far as the proper selection of scalars is concerned, some of the estimators from the suggested class are found more efficient than the MMSE estimator and the MLE. Thus, even if the experimenter has less confidence in the guessed value, the efficiency of the proposed class can be increased considerably by suitably choosing the scalars p and q. The superiority of the suggested class of estimators over Singh and Singh (1997) class of estimators has also been recognized. The suggested class of shrinkage estimators are therefore recommended for its use in practice.

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