Brazilian Journal of Probability and Statistics
Volume 22, Number 1, Pages 35–59
ISSN 0103-0752

ABOUT TEST CRITERIA IN MULTIVARIATE ANALYSIS

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ABSTRACT. The exact distribution of a certain criterion announced as new by Olson (1974), but really defined by Wilks (1932), is studied. Tables of that criterion are computed for a sort of particular parameters. Some errors in two of the criteria obtained by Wilks (1932) are detected and corrected. The moments and the exact distribution of Dempster's test criterion are found. At the end, an example of the literature determines all the criteria and their tests.

1. Introduction

Consider the general multivariate linear model

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon} \tag{1.1}$$

where: $\mathbf{Y} \in \Re^{n\times p}$ is the matrix of the observed values; $\boldsymbol{\beta} \in \Re^{q\times p}$ is the parameter matrix; $\mathbf{X} \in \Re^{n\times q}$ is the design matrix or the regression matrix of rank $r \leq q$; $\boldsymbol{\epsilon} \in \Re^{n\times p}$ is the error matrix which has a matrix variate normal distribution, specifically $\boldsymbol{\epsilon} \sim \mathcal{N}_{n\times p}(\mathbf{0}, \mathbf{I}_n \otimes \boldsymbol{\Sigma})$, see Muirhead (1982, p.430); $\otimes$ denotes the Kronecker product; and $\boldsymbol{\Sigma} \in \Re^{p\times p}$, $\boldsymbol{\Sigma} > \mathbf{0}$. For this model, we want to test the hypothesis

$$H_0 : \mathbf{C}\boldsymbol{\beta}\mathbf{M} = \mathbf{0} \text{ versus } H_a : \mathbf{C}\boldsymbol{\beta}\mathbf{M} \neq \mathbf{0} \tag{1.2}$$

where $\mathbf{C} \in \Re^{\nu_H\times q}$ of rank $\nu_H \leq r$ and $\mathbf{M} \in \Re^{p\times g}$ of rank $g \leq p$. As in the univariate case, the matrix $\mathbf{C}$ concerns to the hypothesis among the elements of the parameter matrix columns, while the matrix $\mathbf{M}$ allows hypothesis among the different response parameters. The matrix $\mathbf{M}$ plays a role in profile analysis, for example; in ordinary hypothesis test it is taken to be the identity matrix, $\mathbf{M} = \mathbf{I}_p$.

Let $\mathbf{S}_H$ be the matrix of sums of squares and sums of products due to the hypothesis and let $\mathbf{S}_E$ be the matrix of sums of squares and sums of products due to the error. Both are defined like this

$$\begin{aligned} \mathbf{S}_H &= (\mathbf{C}\widetilde{\boldsymbol{\beta}}\mathbf{M})'(\mathbf{C}(\mathbf{X}'\mathbf{X})^{-}\mathbf{C}')^{-1}(\mathbf{C}\widetilde{\boldsymbol{\beta}}\mathbf{M}) \\ \mathbf{S}_E &= \mathbf{M}'\mathbf{Y}'(\mathbf{I}_n - \mathbf{X}\mathbf{X}^{-})\mathbf{Y}\mathbf{M}, \end{aligned} \tag{1.3}$$

Key words and phrases. Anderson's criterion, Dempster's criterion, Gnanadesikan's criterion, Lawley-Hotelling trace, likelihood ratio test, multivariate linear model, new Pillai's criteria, Olson's criterion, Pillai's criteria, Pillai's trace, Roy's criteria, Wilks' criteria.

Received April/2007, accepted June/2007.

respectively; where $\widetilde{\boldsymbol{\beta}} = \mathbf{X}^{-}\mathbf{Y}$ and $\mathbf{X}^{-}$ is any generalised inverse of $\mathbf{X}$ such that $\mathbf{X} = \mathbf{X}\mathbf{X}^{-}\mathbf{X}$. Besides, under the null hypothesis, $\mathbf{S}_H$ has a g-dimensional Wishart distribution with ν_H degrees of freedom and parameter matrix $\mathbf{M}'\mathbf{\Sigma}\mathbf{M}$, i.e. $\mathbf{S}_H \sim \mathcal{W}_g(\nu_H, \mathbf{M}'\mathbf{\Sigma}\mathbf{M})$; similarly $\mathbf{S}_E$ has a g-dimensional Wishart distribution with ν_E degrees of freedom and parameter matrix $\mathbf{M}'\mathbf{\Sigma}\mathbf{M}$, i.e. $\mathbf{S}_E \sim \mathcal{W}_g(\nu_E, \mathbf{M}'\mathbf{\Sigma}\mathbf{M})$; specifically, ν_H and ν_E denote the degrees of freedom of the hypothesis and the error, respectively. All the results given below are true for $\mathbf{M} \neq \mathbf{I}_p$, just compute $\mathbf{S}_H$ and $\mathbf{S}_E$ from (1.3) and replace p by g. Now, let $\lambda_1, \cdots, \lambda_s$ be the $s = \min(\nu_H, g)$ non null eigenvalues of the matrix $\mathbf{S}_H\mathbf{S}_E^{-1}$ such that $0 < \lambda_s < \cdots < \lambda_1 < \infty$ and let $\theta_1, \cdots, \theta_s$ be the s non null eigenvalues of the matrix $\mathbf{S}_H(\mathbf{S}_H + \mathbf{S}_E)^{-1}$ with $0 < \theta_s < \cdots < \theta_1 < 1$; here we note $\lambda_i = \theta_i/(1-\theta_i)$ and $\theta_i = \lambda_i/(1+\lambda_i)$, $i = 1, \cdots, s$. Various authors have proposed a number of different criteria for testing the hypothesis (1.2). But it is known, see for example Kres (1983), that all the tests can be expressed in terms of the eigenvalues $\lambda' s$ or $\theta' s$. In our experience, a reason for which many of these test statistics are not used is due to lack and/or inaccessibility of tables for the respective critical values.

In this work the three test statistics proposed by Wilks (1932) are studied after correcting some errors in the published density functions. We emphasize that two of those statistics were proposed as new by Roy *et al*. (1971) (U-statistic) and Olson (1974) (V-statistic). Besides we show how to obtain the critical values for the U-statistic by starting from the tables of Wilks' Λ statistic. The density of the V-statistic is derived by three different methods and the tables for the critical points are constructed for several values of the parameters. The exact distribution of another test criterion proposed by Pillai (1955) is found. The moments and the exact distribution for the Dempster statistic are given. At the end, this work solves a problem in the literature by computing all the published test statistics studied, also we propose a way for finding the critical values for the remaining test criteria; special emphasis is given around some practical considerations that should be taken into account when the approximations are used.

2. Wilks' criteria

Unfortunately, there is not homogeneity in the symbol of the test statistics; moreover, some of them were renamed creating more confusion. For example, the well known statistic of Wilks is often represented by W, but in the literature it is also defined as Wilks' Λ. However, Anderson (1982, p. 299) denoted it by U, but Wilks (1932) named another statistics with that symbol. In order to avoid any confusion in notation we return to the original notation of Wilks (1932) and we define the three criteria in this way:

$$\begin{aligned} \Lambda = W &= \frac{|\mathbf{S}_E|}{|\mathbf{S}_E + \mathbf{S}_H|} = \prod_{i=1}^{s}\frac{1}{1+\lambda_i} \\ &= \prod_{i=1}^{s}(1-\theta_i) \qquad \text{Wilks (1932, p. 485)} \end{aligned}$$

$$\begin{aligned} U &= \frac{|\mathbf{S}_H|}{|\mathbf{S}_E + \mathbf{S}_H|} = \prod_{i=1}^{s} \frac{\lambda_i}{1+\lambda_i} \\ &= \prod_{i=1}^{s} \theta_i \qquad \text{Wilks (1932, p. 482)} \\ V &= \frac{|\mathbf{S}_H|}{|\mathbf{S}_E|} = \prod_{i=1}^{s} \lambda_i \\ &= \prod_{i=1}^{s} \frac{\theta_i}{(1-\theta_i)} \qquad \text{Wilks (1932, p. 486).} \end{aligned}$$

Curiously the U-statistic was proposed as a new statistic in the literature by Roy *et al.* (1971, last paragraph p. 72) with the same original notation of Wilks (1932). Similarly, the third statistic was proposed as new by Olson (1974); here we used that notation.

Moreover, Wilks (1932) proposed integral expressions for the densities of the three statistics, but even when the general expression for the W and the U statistics are correct (Wilks (1932, eq. (5), p. 475)), the density for W (Wilks (1932, eq. (35), p. 486)) is wrong. Maybe this fact explains the inconsistences of some particular expressions for the densities of W published by Wilks (1935), such as it is corroborated by Consul (1966) when the results are compared with the results obtained by Anderson (1982, p. 308). The correct density function of W in Wilks (1932) is obtained by replacing $(p-2)/2$ by $(p-3)/2$ in the exponent of the term $(v_1 v_2 \cdots v_{n-1})$. Now, by using our notation

Wilks' notation	Our notation
N	$\nu_H + \nu_E + 1$
p	$\nu_H + 1$
n	p

where $\nu_E = N - p$, note the distribution of U can be found as a function of the distribution of W (and vice versa), just changing the rules of ν_H and ν_E. This is, by making the transformation

$$(\nu_H, \nu_E) \to (\nu_E, \nu_H).$$

Observe that in Wilks' notation the density of U can be obtained from the density of W by making the transformation

$$(N-p, p-1) \to (p-1, N-p);$$

here, the above-mentioned error in the density of W is detected again. This equivalency can be easily seen by replacing particular values of ν_H and ν_E in the densities of U and W (the densities were derived by Hsu (1940) from the joint density of the eigenvalues of the θ's). For proving the equivalency, let us denote the density of $\mathbf{\Theta} = (\theta_1, \ldots, \theta_s)'$ by $p(\mathbf{\Theta}; s, m, h)$, where

$$m = \frac{|\nu_H - p| - 1}{2} \text{ and } h = \frac{\nu_E - p - 1}{2},$$

see Díaz-García and Gutiérrez-Jáimez (1997), Nanda (1948), Pillai (1955) or Rencher (1995, p. 165). See also Srivastava and Khatri (1979, Theorem 3.6.2, p. 93), (but first note some minor errors appear there: the exponent of π must be $p^2/2$ instead of $p/2$ and the exponent of the l_i should be $(n_2 - p - 1)/2$ in place of $(n_2 - p - 1)$).

Now, it is known that $W \sim$ Wilks' Λ. If $\mathbf{\Theta}^* = ((1-\theta_1), \dots, (1-\theta_s))' = (\theta_1^*, \dots, \theta_s^*)'$, the distribution of $\mathbf{\Theta}^*$is the same as that of $\mathbf{\Theta}$, by interchanging m and h, see Nanda (1948, Section 5). Then,

$$\Lambda^* = \prod_{i=1}^{s} \theta_i^* \sim \text{Wilks' } \Lambda, \quad \text{with } m \text{ and } h \text{ interchanged},$$

but note $\Lambda^* = U$. Therefore,

$$U \sim \text{Wilks' } \Lambda, \quad \text{with } m \text{ and } h \text{ interchanged}.$$

But $\nu_E > p$ and $\nu_H \geq p$, then the interchange of m and h is equivalent to the interchange of ν_H and ν_E.

In summary,

Theorem 1. *The distribution of U-statistic, can be obtained from the distribution of the Λ-statistic, by interchanging ν_H and ν_E; this is*

$$U_{\nu_H,\nu_E} \stackrel{d}{=} \Lambda_{\nu_E,\nu_H},$$

where $\stackrel{d}{=}$ denotes identically distributed and the statistics U and Λ were denoted with subindexes to indicate the interchange between the parameters ν_E and ν_H.

Note that, in general, all the distributions of the test statistics and the tabulation of the correspondent critical values were derived by assuming that $p \leq \nu_H$; however, if $p > \nu_H$, the associated densities and their respective critical values can be obtained by making the following transformations in the parameters, see Muirhead (1982, eq. (7), p. 455), Srivastava and Khatri (1979, p. 96) or Rencher (1995, p. 167),

$$(p, \nu_H, \nu_E) \to (\nu_H, p, \nu_E + \nu_H - p). \tag{2.1}$$

3. Third Wilks' statistic, V-statistic

Wilks' V-statistic have been rarely used, maybe because its exact and asymptotic distributions have not been derived, and of course, no tables of its critical values have been constructed, except the Table H in Olson (1973) where the critical values were obtained via Monte Carlo. Recently that statistic have been used in the context of sensitivity analysis in regression, see Díaz-García *et al.* (2007). In fact, Wilks never found its particular distribution; but the k-moments were derived and a suggestion for determining its distribution was given starting from the general equations (13) and (16) of Wilks (1932); however the equation (16) contains two errors.

The right density function of the V-statistic is (in our notation)

$$\begin{aligned} f_V(v) &= \frac{\pi^{p(p-1)/2} \prod_{i=1}^{p}(\Gamma[(\nu_H+\nu_E)/2 - i + 1])}{\Gamma_p[\nu_H/2]\Gamma_p[\nu_E/2]} v^{(\nu_H-p-1)/2} (1+v)^{-(\frac{\nu_H+\nu_E}{2}-p+1)} \\ &\times \int_0^1 \cdots \int_0^1 \left\{ 1 - \frac{\prod_{i=1}^{p-1}(1-r_i)}{1+v} - \left[1 - \prod_{i=1}^{p-1}(1-r_i) - \frac{\prod_{i=1}^{p-1} r_i}{1+v} \right] \right\}^{-(\frac{\nu_H+\nu_E}{2}-p+1)} \\ &\times \prod_{i=1}^{p-1} [r_i(1-r_i)]^{(\nu_H+\nu_E)/2-(p+i)/2} dr_1 dr_2 \dots dr_{(p-1)}, \ v > 0. \end{aligned}$$

Also we have that

$$\frac{1-\prod_{i=1}^{p-1}(1-r_i)-\frac{\prod_{i=1}^{p-1}r_i}{1+v}}{1-\frac{\prod_{i=1}^{p-1}(1-r_i)}{1+v}}<1$$

and

$$\frac{\prod_{i=1}^{p-1}(1-r_i)}{1+v}<1$$

for $r_i \in [0,1]$ and $v > 0$. This allows us to expand in a double series of powers the term between the braces and later to integrate it term by term.

- For $p = 1$ we get

$$f_V(v)=\frac{\Gamma[\frac{\nu_H+\nu_E}{2}]}{\Gamma[\frac{\nu_H}{2}]\Gamma[\frac{\nu_E}{2}]}\,v^{\frac{\nu_H-2}{2}}\,(1+v)^{-\frac{\nu_H+\nu_E}{2}},\ v>0.$$

- For $p = 2$ we have

$$f_V(v)=k_2\,v^{\frac{\nu_H-3}{2}}\int_0^1[(1-r_1)v+r_1]^{-\frac{\nu_H+\nu_E-2}{2}}[r_1(1-r_1)]^{\frac{\nu_H+\nu_E-3}{2}}dr_1,\ v>0,$$

$$k_2=\frac{\Gamma[\frac{\nu_H+\nu_E}{2}]\Gamma[\frac{\nu_H+\nu_E-2}{2}]}{\Gamma[\frac{\nu_H}{2}]\Gamma[\frac{\nu_H-1)}{2}]\Gamma[\frac{\nu_E}{2}]\Gamma[\frac{\nu_E-1}{2}]}.$$

- For $p = 3$ we obtain

$$\begin{aligned} f_V(v) &= k_3\,v^{\frac{\nu_H-4}{2}}\int_0^1\int_0^1[(1-r_1)(1-r_2)v+r_1r_2]^{-\frac{\nu_H+\nu_E-4}{2}}[r_1(1-r_1)]^{\frac{\nu_H+\nu_E-4}{2}}\\ &\times\ [r_2(1-r_2)]^{\frac{\nu_H+\nu_E-5}{2}}dr_1dr_2,\ v>0,\end{aligned}$$

$$k_3=\frac{\Gamma[\frac{\nu_H+\nu_E}{2}]\Gamma[\frac{\nu_H+\nu_E-2}{2}]\Gamma[\frac{\nu_H+\nu_E-4}{2}]}{\Gamma[\frac{\nu_H}{2}]\Gamma[\frac{\nu_H-1}{2}]\Gamma[\frac{\nu_H-2}{2}]\Gamma[\frac{\nu_E}{2}]\Gamma[\frac{\nu_E-1}{2}]\Gamma[\frac{\nu_E-2}{2}]}.$$

- For $p = 4$ we get

$$\begin{aligned} f_V(v) &= k_4\,v^{\frac{\nu_H-5}{2}}\int_0^1\int_0^1\int_0^1[(1-r_1)(1-r_2)(1-r_3)v+r_1r_2r_3]^{-\frac{\nu_H+\nu_E-6}{2}}\\ &\times[r_1(1-r_1)]^{\frac{\nu_H+\nu_E-5}{2}}[r_2(1-r_2)]^{\frac{\nu_H+\nu_E-6}{2}}[r_3(1-r_3)]^{\frac{\nu_H+\nu_E-7}{2}}dr_1dr_2,\ v>0,\end{aligned}$$

$$k_4=\frac{\Gamma[\frac{\nu_H+\nu_E}{2}]\Gamma[\frac{\nu_H+\nu_E-2}{2}]\Gamma[\frac{\nu_H+\nu_E-4}{2}]\Gamma[\frac{\nu_H+\nu_E-6}{2}]}{\Gamma[\frac{\nu_H}{2}]\Gamma[\frac{\nu_H-1}{2}]\Gamma[\frac{\nu_H-2}{2}]\Gamma[\frac{\nu_H-3}{2}]\Gamma[\frac{\nu_E}{2}]\Gamma[\frac{\nu_E-1}{2}]\Gamma[\frac{\nu_E-2}{2}]\Gamma[\frac{\nu_E-3}{2}]}.$$

Observe that for $p = 1$, the distribution of V is a constant times an F-distribution; so, we can use the tables of F in order to find the critical values of V. Tables for $p = 2, 3$ and several values of ν_H and ν_E are tabulated in the Appendix A.

Alternatively, the exact distribution of the V-statistic can be determined via the approach of Hsu (1940), i.e. by the joint distribution of the λ's. In particular a simplified expression for $p = 2$ can be obtained as follows: from Muirhead (1982, pp. 451 and 454-455) we have that

$$f_{\lambda_1,\lambda_2}(\lambda_1, \lambda_2) = k\,(\lambda_1\lambda_2)^{(\nu_H-3)/2}[(1+\lambda_1)(1+\lambda_2)]^{-(\nu_H+\nu_E)/2}(\lambda_1-\lambda_2),$$

where

$$k = \frac{\pi\Gamma_2[(\nu_H+\nu_E)/2]}{\Gamma_2[\nu_H/2]\Gamma_2[\nu_E/2]};$$

if we define $V = \lambda_1\lambda_2$ and $R = (1+\lambda_1)(1+\lambda_2)$, then $dvdr = (\lambda_1-\lambda_2)d\lambda_1 d\lambda_2$. Thus

$$f_{V,R}(v, r) = k\,v^{(\nu_H-3)/2}r^{-(\nu_H+\nu_E)/2},$$

by integrating with respect to R and ranging from $(1+\sqrt{v})^2$ to ∞ we get the density function

$$f_V(v) = k_2^*\,v^{(\nu_H-3)/2}\left(1+\sqrt{v}\right)^{-(\nu_H+\nu_E-2)},\ v > 0,$$

with

$$k_2^* = \frac{2k}{(\nu_H+\nu_E-2)} = \frac{2\sqrt{\pi}\ \Gamma[(\nu_H+\nu_E)/2]\Gamma[(\nu_H+\nu_E-1)/2]}{(\nu_H+\nu_E-2)\Gamma[\nu_H/2]\Gamma[(\nu_H-1)/2]\Gamma[\nu_E/2]\Gamma[(\nu_E-1)/2]}.$$

A third approach for deriving the distribution of the V-statistic is proposed by Consul (1966). It is based on the Mellin transform which we define as follows:

If $M(s)$ is analytic in the strip $\sigma_0 < \Re e(s) < \sigma_1$, and if it tends to zero uniformly with increasing $\mathcal{I}m(s)$ for any real value c between a and b, with its integral along such a line converging absolutely, then if

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} M(s)x^{-s}ds.$$

we have that

$$M(s) = \int_0^\infty f(x)x^{s-1}dx$$

Conversely, suppose $f(x)$ is piecewise continuous on the positive real numbers, taking a value halfway between the limit values at any jump discontinuities, and suppose the integral

$$M(s) = \int_0^\infty f(x)x^{s-1}dx$$

is absolutely convergent when $\sigma_0 < \Re e(s) < \sigma_1$. Then f is recoverable via the inverse Mellin transform from its Mellin transform M.

From Wilks (1932, p. 486) (in our notation),

$$E(V^h) = \frac{\Gamma_p[\nu_H/2+h]\Gamma_p[\nu_E/2+h]}{\Gamma_p[\nu_H/2]\Gamma_p[\nu_E/2]}.$$

Then the exact density function of V is

$$f_V(v) = \frac{1}{\Gamma_p[\nu_H/2]\Gamma_p[\nu_E/2]}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} v^{-h-1}\Gamma_p[\nu_H/2+h]\Gamma_p[\nu_E/2+h]dh.$$

Putting $h + (\nu_E + 1 - p)/2 = t$, we obtain

$$f_V(v) = \frac{v^{(\nu_E-1-p)/2}}{\Gamma_p[\nu_H/2]\Gamma_p[\nu_E/2]}\frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty} v^{-t}\Gamma_p[t-(\nu_E-\nu_H+1-p)/2]\Gamma_p[t+(1-p)/2]dt,$$

where $c' = c + (\nu_E + 1 - p)/2$.

4. Exact distribution of new Pillai criterion, $W^{(s)}$

Pillai (1955) proposed another test criterion and its approximated distribution, apart from the other three criteria exposed in that work. The new criterion is defined like this

$$W^{(s)} = 1 - V^{(s)}/s = \frac{s - \sum_{i=1}^{s}\theta_i}{s} = \frac{\sum_{i=1}^{s}(1-\theta_i)}{s}$$

where

$$V^{(s)} = \operatorname{tr}((\mathbf{S}_E + \mathbf{S}_H)^{-1}\mathbf{S}_H) = \sum_{i=1}^{s}\frac{\lambda_i}{(1+\lambda_i)} = \sum_{i=1}^{s}\theta_i;$$

is the Pillai's statistic, see Muirhead (1982, p. 466), Rencher (1995, 168), Kres (1983, p. 6) and Seber (1984, p. 414), among many others.

Then

$$sW^{(s)} = \operatorname{tr}(\mathbf{S}_E(\mathbf{S}_E + \mathbf{S}_H)^{-1}) = \sum_{i=1}^{s}(1-\theta_i).$$

Now, as in Section 2, a similar result for the exact distribution of the $W^{(s)}$- statistic can be derived:

Exactly as before, if $\mathbf{\Theta}^* = ((1-\theta_1),\dots,(1-\theta_s))' = (\theta_1^*,\dots,\theta_s^*)'$, the distribution of $\mathbf{\Theta}^*$ and $\mathbf{\Theta}$ are the same, so, by interchanging m and h

$$sV^{(s)^*} = \sum_{i=1}^{s}\theta_i^* \sim \text{Pillai's } V^{(s)}, \quad \text{with } m \text{ and } h \text{ interchanged}$$

and by noting that $sV^{(s)^*} = sW^{(s)}$, we have

$$sW^{(s)} \sim \text{Pillai's } V^{(s)}, \quad \text{with } m \text{ and } h \text{ interchanged}$$

In summary,

Theorem 2. *The distribution of the $W^{(s)}$-statistic can be obtained from the distribution of $V^{(s)}$-statistic, by interchanging ν_H and ν_E. This is*

$$W^{(s)}_{\nu_H,\nu_E} \overset{d}{=} \frac{1}{s}V^{(s)}_{\nu_E,\nu_H}$$

where the statistics $V^{(s)}$ and $W^{(s)}$ were denoted with subindexes to indicate the interchange between the parameters ν_E and ν_H.

Note that this behavior of the parameters can be seen in the approximated distributions of both statistics given in Pillai (1955, eqs. (5) and (6), respectively).

5. Exact distribution of the Dempster criterion

For the case of one or two samples, Dempster (1958) and Dempster (1960) propose a non exact proof for testing the hypothesis (1.2). For the general case ($p > 2$), Fujikoshi *et al.* (2004) propose the following statistic

$$T_D = (\mathrm{tr}\mathbf{S}_H)/(\mathrm{tr}\mathbf{S}_E);$$

which is termed the Dempster trace criterion.

Dempster's criterion is rarely used. This is because its exact and asymptotic distribution are given in terms of the matrix of parameters $\mathbf{\Sigma}$.

Fujikoshi *et al.* (2004) derive asymptotic null and nonnull distributions of Dempster trace criterion when $n \to \infty$ and $p \to \infty$. They prove that

$$\frac{\widetilde{T}_D}{\sigma_D} \xrightarrow{d} \mathcal{N}(0,1), \tag{5.1}$$

where $\xrightarrow{d}$ denotes convergence in distribution, and

$$\widetilde{T}_D = \sqrt{p}\left\{n\frac{\mathrm{tr}\mathbf{S}_H}{\mathrm{tr}\mathbf{S}_E} - \nu_H\right\},$$

and

$$\sigma_D = \frac{\sqrt{2\nu_H\left(\mathrm{tr}\mathbf{\Sigma}^2\right)/p}}{(\mathrm{tr}\mathbf{\Sigma})/p}.$$

For a practical situation, a (n, p)-consistent estimator is given by

$$\widehat{\sigma}_D = \frac{\sqrt{2\nu_H\{(\mathrm{tr}\mathbf{S}_E^2)/n^2 - (\mathrm{tr}\mathbf{S}_E)^2/n^3\}/p}}{(\mathrm{tr}\mathbf{S}_E)/(np)}.$$

Next we derive the exact null distribution and the moments of the Dempster trace criterion.

Theorem 3. *When $\nu_H > p-1$ and $\nu_E > p-1$, the exact null distribution of T_D is*

$$f_{T_D}(t) = |\delta^{-1}\mathbf{\Sigma}|^{-(\nu_H+\nu_E)/2}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\beta^{II}(t;(p\nu_H+2k)/2,(p\nu_E+2l)/2)}{k!\,l!}$$
$$\times\sum_{\kappa}\sum_{\mu}\left(\frac{1}{2}\nu_H\right)_{\kappa}\left(\frac{1}{2}\nu_E\right)_{\mu}C_{\kappa}\left(\mathbf{I}_p-\delta\mathbf{\Sigma}^{-1}\right)C_{\mu}\left(\mathbf{I}_p-\delta\mathbf{\Sigma}^{-1}\right),\quad t>0$$

where $\beta^{II}(t;b,c)$ denotes the density function of a univariate Type II Beta distribution of parameters b and c, see Gupta and Nagar (2000, p. 165); $\sum_{\kappa}$ denotes summation over all the partitions $\kappa = (k_1,\dots,k_p)$, $k_1 \geq \cdots \geq k_p \geq 0$, of k, $C_{\kappa}(X)$ is the zonal polynomial of X corresponding to κ and the generalised hypergeometric coefficient $(a)_{\kappa}$ is given by

$$(a)_{\kappa} = \prod_{i=1}^{p}(a-(i-1)/2)_{k_i}$$

$(r)_k = r(r+1)\cdots(r+k-1)$, $(a)_0 = 1$ *(see Muirhead (1982, p. 258))and* $\delta \in (0,\infty)$ *is an arbitrary parameter. Muirhead (1982, p. 341) proposes* $\delta = 2\delta_1\delta_p/(\delta_1+\delta_p)$ *as near value to the optimal one, where* δ_1, δ_p *are the largest and smallest eigenvalue of* $\mathbf{\Sigma}$ *respectively.*

Proof. Remember that $S_H \sim \mathcal{W}_p(\nu_H, \mathbf{\Sigma})$ and $S_E \sim \mathcal{W}_p(\nu_E, \mathbf{\Sigma})$ are independent. Let $X = \operatorname{tr} S_H$ and $Y = \operatorname{tr} S_E$, then X and Y are independent too. Using Theorem 8.3.4 in Muirhead (1982, p. 339), the joint density function of X and Y is

$$f_{X,Y}(x,y) = |\delta^{-1}\mathbf{\Sigma}|^{-(\nu_H+\nu_E)/2} \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{1}{k!\ l!} g(x; p\nu_H/2+k, 2\delta) g(y; p\nu_E/2+k, 2\delta)$$
$$\times \sum_{\kappa}\sum_{\mu} \left(\frac{1}{2}\nu_H\right)_{\kappa} \left(\frac{1}{2}\nu_E\right)_{\mu} C_{\kappa}\left(\mathbf{I} - \delta\mathbf{\Sigma}^{-1}\right) C_{\mu}\left(\mathbf{I}_p - \delta\mathbf{\Sigma}^{-1}\right),$$

where

$$g(x; r, 2\delta) = \frac{\exp(-x/(2\delta))x^{r-1}}{(2\delta^r)\Gamma[r]}.$$

Making the change of variables

$$T_D = X/Y, \quad Z = Y \quad (T_D > 0,\ Z > 0),$$

with $dxdy = zdzdt$, the joint density function of T_D and Z is

$$|\delta^{-1}\mathbf{\Sigma}|^{-(\nu_H+\nu_E)/2} \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{z}{k!\ l!} g(tz; p\nu_H/2+k, 2\delta) g(z; p\nu_E/2+k, 2\delta)$$
$$\times \sum_{\kappa}\sum_{\mu} \left(\frac{1}{2}\nu_H\right)_{\kappa} \left(\frac{1}{2}\nu_E\right)_{\mu} C_{\kappa}\left(\mathbf{I}_p - \delta\mathbf{\Sigma}^{-1}\right) C_{\mu}\left(\mathbf{I}_p - \delta\mathbf{\Sigma}^{-1}\right).$$

Now integrating with respect to z over $z \in (0,\infty)$ it gives the desired marginal density function of T_D. □

Corollary 1. *Observe that if in Theorem 3,* $\mathbf{\Sigma} = \delta\mathbf{I}_p$*, then*

$$f_{T_D}(t) = \beta^{II}(t; p\nu_H/2, p\nu_E)/2),$$

or alternatively

$$\frac{\nu_E}{\nu_H} T_D \sim \mathcal{F}(p\nu_H, p\nu_E),$$

where $\mathcal{F}(b,c)$ *is a central* F*-distribution with* b *and* c *degrees of freedom.*

Corollary 2. *Under the condition of Theorem 3 the moments of* T_D *are given by*

$$E(T_D^h) = |\delta^{-1}\mathbf{\Sigma}|^{-(\nu_H+\nu_E)/2} \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{\Gamma[(p\nu_H+2k)/2+h]\Gamma[(p\nu_E+2l)/2-h]}{k!\ l!\Gamma[(p\nu_H+2k)/2]\Gamma[(p\nu_E+2l)/2]}$$
$$\times \sum_{\kappa}\sum_{\mu} \left(\frac{1}{2}\nu_H\right)_{\kappa} \left(\frac{1}{2}\nu_E\right)_{\mu} C_{\kappa}\left(\mathbf{I}_p - \delta\mathbf{\Sigma}^{-1}\right) C_{\mu}\left(\mathbf{I}_p - \delta\mathbf{\Sigma}^{-1}\right).$$

Similarly, if $\mathbf{\Sigma} = \delta\mathbf{I}_p$

$$E(T_D^h) = \frac{\Gamma[p\nu_H/2+h]\Gamma[p\nu_E/2-h]}{\Gamma[p\nu_H/2]\Gamma[p\nu_E/2]}.$$

Proof. The proof follows easily from the moments of univariate Type II Beta distribution. □

Remark 1. *Alternative expressions of the density function and the moments of T_D given in Theorem 3 and Corollary 2 can be derived in terms of the* invariant polynomials, *Davis (1980); specifically, by the eq. (5.1) and (5.10) in Davis (1980), the following results are obtained (or see also Chikuse (1980)):*

$$f_{T_D}(t) = |\delta^{-1}\mathbf{\Sigma}|^{-(\nu_H+\nu_E)/2} \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{\beta^{II}(t;(p\nu_H+2k)/2,(p\nu_E+2l)/2)}{k!\ l!}$$

$$\times \sum_{\kappa}\sum_{\mu}\sum_{\phi\in\kappa\cdot\mu} \left(\frac{1}{2}\nu_H\right)_{\kappa}\left(\frac{1}{2}\nu_E\right)_{\mu}\left(\theta_{\phi}^{\kappa,\mu}\right)^2 C_{\phi}\left(\mathbf{I}_p-\delta\mathbf{\Sigma}^{-1}\right), \quad t>0,$$

and

$$E(T_D^h) = |\delta^{-1}\mathbf{\Sigma}|^{-(\nu_H+\nu_E)/2} \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{\Gamma[(p\nu_H+2k)/2+h]\Gamma[(p\nu_E+2l)/2-h]}{k!\ l!\Gamma[(p\nu_H+2k)/2]\Gamma[(p\nu_E+2l)/2]}$$

$$\times \sum_{\kappa}\sum_{\mu}\sum_{\phi\in\kappa\cdot\mu} \left(\frac{1}{2}\nu_H\right)_{\kappa}\left(\frac{1}{2}\nu_E\right)_{\mu}\left(\theta_{\phi}^{\kappa,\mu}\right)^2 C_{\phi}\left(\mathbf{I}_p-\delta\mathbf{\Sigma}^{-1}\right),$$

where

$$\theta_{\phi}^{\kappa,\mu} = \frac{C_{\phi}^{\kappa,\mu}(\mathbf{I}_p,\mathbf{I}_p)}{C_{\phi}(\mathbf{I}_p)}.$$

and $C_{\phi}^{\kappa,\mu}(\mathbf{I}_p,\mathbf{I}_p)$ is an invariant polynomial evaluated in the identity matrix, $\mathbf{I}_p$.

6. Example

The present example describes and spreads the computation of the different statistics for testing the multivariate linear hypothesis and proposes practical ways for finding the corresponding critical values by using: the published tables, the integration of the exact distribution or approximations. Also, we emphasize some directions about the use of approximations for computing the critical values; unfortunately, such considerations are not described in the texts where those approximations are established, see Kres (1983), Rencher (1995), among many others; however, in most of the original sources we can find some important directions for approximations, see for example Pillai (1955).

The following application is a modification of the example 9.4.3 of Srivastava (2002, p. 294).

Example. The original observation matrix consists of a 32 vector corresponding to the 12 responses of each rat; for our exposition, the first two dependent variables Y_1 and Y_2 are considered (i.e. the first two days). In this case we propose the following multivariate linear model:

$$\underset{(32\times2)}{\mathbf{Y}} = \underset{(32\times5)}{\mathbf{X}}\ \underset{(5\times2)}{\beta} + \underset{(32\times2)}{\mathbf{E}},$$

where the design matrix is also provided in Srivastava (2002, p. 294) and

$$\beta = \begin{pmatrix} \beta_{01} & \beta_{02} \\ \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \\ \beta_{31} & \beta_{32} \\ \beta_{41} & \beta_{42} \end{pmatrix}.$$

We want to test the hypothesis

$$H : \mathbf{C}\beta\mathbf{M} = \mathbf{0} \quad \text{versus} \quad A : \mathbf{C}\beta\mathbf{M} \neq \mathbf{0},$$

where $\mathbf{M} = \mathbf{I}_2$ and

$$\mathbf{C} = \begin{pmatrix} 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

The matrices of sums of squares and sums of products due to the error and the hypothesis are, respectively,

$$\mathbf{S}_E = \begin{pmatrix} 255.80 & 112.62 \\ 112.62 & 415.25 \end{pmatrix} \text{ and } \mathbf{S}_H = \begin{pmatrix} 10.05 & 27.55 \\ 27.55 & 81.30 \end{pmatrix}.$$

The test statistics for all the known criteria are tabulated in Table 1.

TABLE 1. Criteria to test the null hypothesis

Criteria	Statistics	$\alpha(= 0.05)$ Critical value
Wilks' Λ	0.832	0.626
Wilks' U	5.190E-4	0.025
Wilks' V	6.235E-4	0.038
Lawley-Hotelling's $U^{(s)}$	0.200	0.548
Pillai's $V^{(s)}$	0.168	0.415
Pillai's $W^{(s)}$	0.915	0.792
Pillai's $H^{(s)}$	0.908	0.969
Pillai's $R^{(s)}$	0.006	0.969
Pillai's $T^{(s)}$	0.167	3.168
Roy's $\lambda_{\max}$	0.197	0.489
Roy's $\theta_{\max}$	0.164	0.328
Anderson's $\lambda_{\min}$	0.041	0.117
Roy's $\theta_{\min}$	0.003	0.105
Dempster's T_D	0.136	0.182

Some definitions and comments about the results in Table 1:

(1) General remarks:

(a) The decision rule for all the criteria is:

reject H_0 if the statistic $\geq$ critical value

However, for Wilks' Λ and Pillai's $W^{(s)}$ criteria, the decision rule is (this class of test is known in statistical literature as **inverse test**, see Rencher (1995, p. 162)):

reject H_0 if the statistic $\leq$ critical value.

(b) The tables for critical values of all the criteria are tabulated in terms of the parameters (p, ν_H, ν_E) or in terms of the parameters (s, m, h), where remember that

$$s = \min(p, \nu_H), \quad m = (|\nu_H - p| - 1)/2 \text{ and } h = (\nu_E - p - 1)/2.$$

Besides, the tables (in general) have been computed by assuming that $p \leq \nu_H$ and $p \leq \nu_E$. If $p > \nu_H$ then use the combination of parameters $(\nu_H, p, \nu_E + \nu_H - p)$ in place of (p, ν_H, ν_E), see Muirhead (1982, eq. (7), p. 455), Srivastava and Khatri (1979, p. 96) or Rencher (1995, p. 167).

(c) Observe that the null hypothesis is not rejected under any criteria. This is remarkable, because usually, when several statistics are applied, in the same test, some contradictory conclusions can appear, i.e. a hypothesis can be rejected by some statistics and accepted by the remaining ones. This is due to the multidimensional nature of the space in which the vectors involved in the hypothesis lie, see Rencher (1995, p. 169). Some directions for choosing one of these tests are given by the comparison of power functions, see for example Morrison (1978, pp. 223-224), Anderson (1982, Section 8.6.5), Olson (1974) and Rencher (1995, Section 6.2), among many others.

(2) Wilks' Λ statistic:

$$\Lambda = \frac{|\mathbf{S}_E|}{|\mathbf{S}_E + \mathbf{S}_H|} = \prod_{i=1}^{s} \frac{1}{1+\lambda_i} = \prod_{i=1}^{s} (1 - \theta_i);$$

see Wilks (1932), Rencher (1995, p. 161) and Kres (1983, p. 5) among many others. The critical value was taken from Table 1 in Kres (1983, pp. 14-51), besides, it was computed with the correct expression by using Mathematica.

(3) Wilks' U statistic:

$$U = \frac{|\mathbf{S}_H|}{|\mathbf{S}_E + \mathbf{S}_H|} = \prod_{i=1}^{s} \frac{\lambda_i}{1+\lambda_i} = \prod_{i=1}^{s} \theta_i;$$

see Wilks (1932), Roy *et al*. (1971, p. 72), Seber (1984, p. 413) and Kres (1983, p. 6) among many others. This criterion is also known as Gnanadesikan's U statistic. The critical value was computed with the expression (31) in Wilks (1932) by the use of Mathematica. Observe, also, that this statistic is wrongly defined as a function of the eigenvalues λ's and θ's in Kres (1983, p. 6).

(4) Wilks' V statistic:

$$V = \frac{|\mathbf{S}_H|}{|\mathbf{S}_E|} = \prod_{i=1}^{s} \lambda_i = \prod_{i=1}^{s} \frac{\theta_i}{(1 - \theta_i)};$$

see Wilks (1932), Olson (1974) and Kres (1983, p. 8). This statistic is also known as Olson's V statistic. The critical value given in the Table 1 was taken from the tables in the Appendix A.

(5) Lawley-Hotelling's $U^{(s)}$ statistic:

$$U^{(s)} = \operatorname{tr}(\mathbf{S}_E^{-1}\mathbf{S}_H) = \sum_{i=1}^{s} \lambda_i = \sum_{i=1}^{s} \frac{\theta_i}{(1-\theta_i)};$$

see Muirhead (1982, p. 466), Rencher (1995, 167) and Kres (1983, p. 6) among many others. Unfortunately, the tables for the critical values do not include the minimum required possible combinations between the parameters s, m and h; see Table 6 in Kres (1983, pp.118-135). In the example of the Table 1 we have used an F-approximation, see equation (6.30) in Rencher (1995, p. 167), see also Pillai (1955, eq. (7)). Observe that this approximation is useful for $h + s \geq 30$ when $s = 2$; when s increases by 1, $h + s$ must increase by 10 to give satisfactory results, see Pillai (1955).

(6) Pillai's $V^{(s)}$ statistic:

$$V^{(s)} = \operatorname{tr}((\mathbf{S}_E + \mathbf{S}_H)^{-1}\mathbf{S}_H) = \sum_{i=1}^{s} \frac{\lambda_i}{(1+\lambda_i)} = \sum_{i=1}^{s} \theta_i;$$

see Muirhead (1982, p. 466), Rencher (1995, 168) and Kres (1983, p. 6) among many others. The corresponding critical value in the Table 1 was taken from Table 7 in Kres (1983, pp. 136-153). However, note that for the critical value, a Type I Beta approximation can be used (see Gupta and Nagar (2000, p.165)), see equation (5) in Pillai (1955), see also Rencher (1995, Section 6.1.5). This approximation is useful for $m + h \geq 30$ when $s = 2$; but, if s increases by 1, then, $m + h$ must be increased by 10, for getting satisfactory results, see Pillai (1955).

(7) Pillai's $W^{(s)}$ statistic:

$$W^{(s)} = \operatorname{tr}((\mathbf{S}_E + \mathbf{S}_H)^{-1}\mathbf{S}_E) = \sum_{i=1}^{s} \frac{1}{(1+\lambda_i)} = \sum_{i=1}^{s}(1-\theta_i) = (1 - V^{(s)}/s);$$

see Pillai (1955). For the critical values we can use a Type I Beta approximation, see equation (6) in Pillai (1955). For practical use, this approach is satisfactory for $m + h \geq 30$ when $s = 2$; but, if s increases by 1, then, $m + h$ must be increased by 10, for getting satisfactory results, see Pillai (1955). However, note the exact critical value can be obtained from $V^{(s)}$-statistic and the expression $W^{(s)} = (1 - V^{(s)}/s)$. In fact, Table 1 contains the exact value.

(8) Pillai's $H^{(s)}$ statistic:

$$H^{(s)} = \frac{s}{\sum_{i=1}^{s}(1+\lambda_i)} = s\left\{\sum_{i=1}^{s}(1-\theta_i)^{-1}\right\}^{-1} = (1 + U^{(s)}/s)^{-1};$$

see Pillai (1955) and Kres (1983, p. 8). The corresponding critical value in the Table 1 was obtained by using a Type I Beta approximation, see equation (9) in

Pillai (1955). Here we have to apply the above mentioned practical conditions for the correct use of the approximation of the $U^{(s)}$ statistic.

(9) Pillai's $R^{(s)}$ statistic:

$$R^{(s)} = \frac{s}{\sum_{i=1}^{s} \frac{1+\lambda_i}{\lambda_i}} = s\left\{\sum_{i=1}^{s} \theta_i^{-1}\right\}^{-1} = (1 + U'^{(s)}/s)^{-1};$$

see Pillai (1955) and Kres (1983, p. 8). Here $U'^{(s)}$ is the same $U^{(s)}$ but with m and h interchanged. For this criterion, the corresponding critical value in the Table 1 was computed by using a Type I beta approximation, see equation (11) in Pillai (1955). Again, the same conditions explained before for the $U^{(s)}$ statistic have to be applied for a satisfactory result in the approximations. Besides, we need the conditions $m \geq 0$, or $|\nu_H - p| \geq 1$ for getting satisfactory approximations. The last condition was not considered by Pillai (1955), but it is required, because for a Beta distribution $\beta(a, b)$ it is known that $a > 0$, which is guaranteed when $m \geq 0$ in the approximation.

(10) Pillai's $T^{(s)}$ statistic:

$$T^{(s)} = s\left\{\sum_{i=1}^{s} \lambda_i^{-1}\right\}^{-1} = \frac{s}{\sum_{i=1}^{s} \frac{1-\theta_i}{\theta_i}} = \frac{R^{(s)}}{1 - R^{(s)}};$$

see Pillai (1955) and Kres (1983, p. 8). In the Table 1, the critical value was obtained by using a Type II Beta (see Gupta and Nagar (2000, 165)) approximation, see equation (13) in Pillai (1955). Again, for a satisfactory approximation, including the restriction over m, we use the same rules applied to the $R^{(s)}$ statistic.

(11) Roy's $\lambda_{\max}$:

$$\lambda_{\max} = \frac{\theta_{\max}}{1 - \theta_{\max}};$$

see Roy (1957) and Kres (1983, p. 7). The corresponding critical value in the Table 1 was obtained from table 3 in Kres (1983, pp. 62-86). Besides, we got the critical value by integrating the joint distribution of the λ's via Mathematica.

(12) Roy's $\theta_{\max}$:

$$\theta_{\max} = \frac{\lambda_{\max}}{1 + \lambda_{\max}};$$

see Roy (1957), Muirhead (1982, p. 481), Rencher (1995, p. 164) and Kres (1983, p. 7) among many others. For this criterion the corresponding critical value in the Table 1 can be obtained from table 2, 4 or 5 in Kres (1983, pp. 52-61, 87-104 and 105-117, respectively). Again, Mathematica was used for finding the critical value of that criterion by integrating the joint distribution of the eigenvalues θ's.

(13) Anderson's $\lambda_{\min}$:

$$\lambda_{\min} = \frac{\theta_{\min}}{1 - \theta_{\min}};$$

see Roy (1957), Anderson (1982) and Kres (1983, p. 7) among many others. As above, the critical value in the Table 1 was computed via Mathematica by integrating the joint distribution of the eigenvalues λ's. However, note that the critical value can be determined as a function of the critical value for $\theta_{\min}$.

(14) Roy's $\theta_{\min}$:

$$\theta_{\min} = \frac{\lambda_{\min}}{1 + \lambda_{\min}};$$

see Pillai (1955) and Roy (1957). Similarly to Anderson's criterion, the corresponding critical value in the Table 1 was obtained by integration of the joint distribution of the eigenvalues λ's via Mathematica. However those values can be obtained from the distribution of $\theta_{\max}$ by using

$$\theta_{\min}(\alpha, s, \nu_H, \nu_E) = 1 - \theta_{\max}(\alpha, s, \nu_E, \nu_H),$$

see Nanda (1948); but, again, the published tables do not allow us to read the values because, they do not incorporate such combinations of the parameters; in fact there are many similar particular cases for which the critical value can not be found from those tables.

(15) Dempster's T_D:

$$T_D = (\mathrm{tr}\mathbf{S}_H)/(\mathrm{tr}\mathbf{S}_E);$$

see Dempster (1958), Dempster (1960) and Fujikoshi *et al.* (2004). In this criterion, the critical value in the Table 1 was obtained by using the normal approximation (5.1).

7. Conclusions

A sort of important problems appears when we try to test a multivariate hypothesis: 1. from a practical point of view, we know that a test of multivariate hypothesis can be performed by several criteria (for example, here we present 14 statistics for the same multivariate linear model), then the choice of the "best" statistics is an important question to solve; 2. finding the corresponding critical values is also problematic, because a few combinations of the parameters are provided in the published tables; 3. an additional problem is related with the approximations of the critical values given in the books and softwares, because most of them do not indicate the conditions for the use of those approximations; 4. only six of the 14 statistics have been studied considerably (around 70's), but even for the analyzed cases, the theoretical recommendations for their use are not clear today. In this work, we tried to answer those questions, first, by correcting some wrong published results about the probability functions associated to these criteria, to explain the historical discrepancies. Also, the exact distributions of many statistics were found, these provide the initial step in future works for obtaining theoretical comparison among the criteria. In fact, we proposed three alternative ways for determining the density of Wilks' V statistic and provided tables for a number of parameter combinations, improving the existing tables and giving the exact formula for generating any combination if necessary. With the exact densities we avoid the classical problems of using approximations without clarifying their right use. Even with the above theoretical results, which add bases for an advanced discussion or comparison among the statistics and their critical values, an important question remains about the best choice of the statistic. In particular, there are some works about the power function, as it was explained in the general remark point (c) of the example, which recommend some statistics under certain conditions. However, mainly they study the following criteria: Wilks' Λ, Lawley-Hotelling's $U^{(s)}$, Pillai's $V^{(s)}$ and Roy's $\lambda_{\max}$, and relationships with the remaining ten statistics are not clear yet.

8. Appendix A. Tables of the third criterion V of S. S. Wilks (Olson's criterion)

(1) Contents of the tables and definition of the test statistic: The tables contain the upper percentage points of the test statistic

$$V = \frac{|\mathbf{S}_H|}{|\mathbf{S}_E|} = \prod_{i=1}^{s} \lambda_i = \prod_{i=1}^{s} \frac{\theta_i}{(1-\theta_i)} \quad \text{Wilks (1932, p. 486).}$$

Here, $\mathbf{S}_H$ is the matrix of sums of squares and sums of products due to the hypothesis and $\mathbf{S}_E$ is the matrix of sums of squares and sums of products due to the error. Also, $\lambda_1, \cdots, \lambda_s$ are the $s = \min(\nu_H, g)$ non null eigenvalues of the matrix $\mathbf{S}_H\mathbf{S}_E^{-1}$ such that $0 < \lambda_s < \cdots < \lambda_1 < \infty$ and $\theta_1, \cdots, \theta_s$ are the s non null eigenvalues of the matrix $\mathbf{S}_H(\mathbf{S}_H + \mathbf{S}_E)^{-1}$ with $0 < \theta_s < \cdots < \theta_1 < 1$; recall that $\lambda_i = \theta_i/(1-\theta_i)$ and $\theta_i = \lambda_i/(1+\lambda_i)$, $i = 1, \cdots, s$.

(2) Dimensions of the tables and definition of the parameters:

(a) The parameter α:
α = error probability
α = 5% and 1%.

(b) The parameter p:
p = dimension of the variates
for $p = 2, 3$

(c) The parameter ν_H:
ν_H = degree of freedom of the hypothesis

$$\text{for } \nu_H = \begin{cases} 1(1)30, 35(5)100, 120, & \text{p = 2;} \\ 3, 5, 10, 15, 20, 30, 50, 80, 100, 120, & \text{p = 3.} \end{cases}$$

(d) The parameter ν_E:
ν_E = degree of freedom of the hypothesis

$$\text{for } \nu_E = \begin{cases} 2(1)30, 40(20)140, 170, 200, 240, 320, 440, 600, 800, 1000 \\ \text{(p = 2);} \\ 10(10)50, 80, 100, 200, 400, 600 \\ \text{(p = 3).} \end{cases}$$

These tables are valid for $p \le \nu_H$; if $p > \nu_H$, the respective critical values can be obtained by making the following transformations in the parameters,

$$(p, \nu_H, \nu_E) \to (\nu_H, p, \nu_E + \nu_H - p).$$

TABLE 2. Upper percentage points of the test statistic $\mathbf{V}_{p,\nu_H,\nu_E}$ ($p = 2$ and $\alpha = 0.05$).

ν_E	ν_H															ν_E
	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	
2	361.0000	1481.756	3362.677	6003.765	9405.020	13566.44	18488.03	24169.78	30611.70	37813.79	45776.04	54498.47	63981.05	74223.81	85226.73	2
3	12.05573	40.80952	85.46445	145.9769	222.3357	314.5369	422.5784	546.4593	686.1791	841.7373	1013.134	1200.368	1403.441	1622.351	1857.099	3
4	2.939228	9.135212	18.35151	30.57064	45.78694	63.99793	85.20236	109.3995	136.5889	166.7704	199.9436	236.1086	275.2652	317.4133	362.5529	4
5	1.242651	3.682280	7.211562	11.82054	17.50547	24.26458	32.09689	41.00185	50.97909	62.02837	74.14954	87.34247	101.6071	116.9433	133.3511	5
6	0.673326	1.935493	3.726076	6.038455	8.869896	12.21902	16.08506	20.46754	25.36614	30.78067	36.71097	43.15694	50.11851	57.59561	65.58819	6
7	0.419320	1.180241	2.244184	3.606366	5.264680	7.218019	9.465739	12.00744	14.84285	17.97178	21.39413	25.10975	29.11862	33.42066	38.01583	7
8	0.285292	0.790702	1.489508	2.378068	3.454699	4.718490	6.168894	7.805563	9.628263	11.63683	13.83115	16.21113	18.77672	21.52784	24.46448	8
9	0.206312	0.565097	1.056767	1.678445	2.428753	3.306924	4.312490	5.445146	6.704686	8.090964	9.603873	11.24333	13.00929	14.90168	16.92049	9
10	0.155986	0.423293	0.786949	1.244621	1.795156	2.437901	3.172448	3.998530	4.915963	5.924615	7.024390	8.215214	9.497031	10.86979	12.33348	10
11	0.121999	0.328577	0.607922	0.958099	1.378132	1.867454	2.425710	3.052661	3.748142	4.512035	5.344251	6.244723	7.213400	8.250240	9.355210	11
12	0.097989	0.262276	0.483301	0.759435	1.089837	1.474013	1.911649	2.402532	2.946513	3.543483	4.193363	4.896090	5.651617	6.459905	7.320923	12
13	0.080411	0.214105	0.393188	0.616268	0.882612	1.191787	1.543512	1.937596	2.373904	2.852338	3.372823	3.935302	4.539730	5.186073	5.874301	13
14	0.067161	0.178031	0.325982	0.509808	0.728868	0.982776	1.271280	1.594210	1.951440	2.342882	2.768466	3.228140	3.721862	4.249598	4.811323	14
15	0.056929	0.150331	0.274559	0.428563	0.611772	0.823844	1.064550	1.333736	1.631288	1.957123	2.311179	2.693405	3.103763	3.542223	4.008758	15
16	0.048865	0.128606	0.234357	0.365191	0.520601	0.700277	0.904013	1.131669	1.383140	1.658349	1.957238	2.279762	2.625883	2.995573	3.388809	16
17	0.042398	0.111259	0.202345	0.314832	0.448267	0.602369	0.776951	0.971883	1.187070	1.422441	1.677941	1.953528	2.249168	2.564833	2.900501	17
18	0.037133	0.097189	0.176446	0.274167	0.389941	0.523515	0.674718	0.843429	1.029562	1.233049	1.453841	1.691898	1.947188	2.219684	2.509366	18
19	0.032790	0.085623	0.155203	0.240867	0.342243	0.459098	0.591277	0.738668	0.901191	1.078785	1.271401	1.479002	1.701559	1.939047	2.191446	19
20	0.029166	0.076001	0.137566	0.213261	0.302748	0.405813	0.522313	0.652145	0.795235	0.951524	1.120968	1.303533	1.499189	1.707915	1.929691	20
21	0.026110	0.067911	0.122765	0.190126	0.269685	0.361246	0.464677	0.579882	0.706791	0.845351	0.995520	1.157264	1.330558	1.515380	1.711712	21
22	0.023510	0.061045	0.110223	0.170548	0.241735	0.323603	0.416029	0.518925	0.632225	0.755879	0.889848	1.034101	1.188613	1.353364	1.528335	22
23	0.021280	0.055168	0.099506	0.153836	0.217898	0.291524	0.374600	0.467043	0.568791	0.679799	0.800029	0.929451	1.068043	1.215784	1.372658	23
24	0.019352	0.050099	0.090275	0.139458	0.197409	0.263970	0.339035	0.422528	0.514391	0.614580	0.723060	0.839803	0.964787	1.097994	1.239407	24
25	0.017675	0.045698	0.082269	0.127001	0.179669	0.240130	0.308283	0.384056	0.467395	0.558259	0.656615	0.762437	0.875703	0.996396	1.124500	25
26	0.016206	0.041851	0.075280	0.116136	0.164210	0.219367	0.281514	0.350583	0.426524	0.509296	0.598870	0.695219	0.798325	0.908170	1.024739	26
27	0.014913	0.038469	0.069144	0.106605	0.150658	0.201176	0.258073	0.321284	0.390762	0.466469	0.548376	0.636459	0.730700	0.831081	0.937591	27
28	0.013769	0.035481	0.063728	0.098199	0.138712	0.185151	0.237431	0.295494	0.359295	0.428797	0.503973	0.584801	0.671263	0.763342	0.861027	28
29	0.012752	0.032828	0.058922	0.090746	0.128129	0.170960	0.219162	0.272677	0.331464	0.395489	0.464725	0.539151	0.618750	0.703507	0.793410	29
30	0.011843	0.030461	0.054640	0.084110	0.118710	0.158336	0.202915	0.252395	0.306733	0.365898	0.429866	0.498616	0.572132	0.650398	0.733405	30
40	0.006375	0.016291	0.029089	0.044617	0.062781	0.083519	0.106788	0.132554	0.160793	0.191484	0.224611	0.260161	0.298122	0.338488	0.381246	40
60	0.002713	0.006889	0.012246	0.018714	0.026250	0.034827	0.044422	0.055021	0.066609	0.079179	0.092720	0.107226	0.122692	0.139113	0.156484	60
80	0.001494	0.003781	0.006706	0.010229	0.014325	0.018979	0.024178	0.029912	0.036174	0.042959	0.050260	0.058074	0.066398	0.075228	0.084562	80
100	0.000944	0.002385	0.004223	0.006435	0.009004	0.011918	0.015171	0.018757	0.022669	0.026904	0.031459	0.036331	0.041518	0.047017	0.052826	100
120	0.000650	0.001640	0.002902	0.004418	0.006178	0.008173	0.010398	0.012849	0.015523	0.018415	0.021525	0.024849	0.028387	0.032136	0.036096	120
140	0.000475	0.001197	0.002116	0.003220	0.004500	0.005951	0.007568	0.009349	0.011290	0.013390	0.015647	0.018059	0.020625	0.023343	0.026214	140
170	0.000320	0.000806	0.001423	0.002165	0.003024	0.003998	0.005082	0.006276	0.007576	0.008983	0.010493	0.012107	0.013824	0.015642	0.017561	170
200	0.000230	0.000579	0.001023	0.001555	0.002171	0.002869	0.003646	0.004502	0.005433	0.006440	0.007522	0.008677	0.009906	0.011206	0.012579	200
240	0.000159	0.000400	0.000706	0.001074	0.001499	0.001980	0.002516	0.003105	0.003747	0.004440	0.005185	0.005980	0.006825	0.007720	0.008664	240
320	0.000089	0.000224	0.000395	0.000599	0.000837	0.001105	0.001404	0.001732	0.002089	0.002475	0.002889	0.003332	0.003802	0.004299	0.004824	320
440	0.000047	0.000118	0.000208	0.000315	0.000440	0.000581	0.000737	0.000910	0.001097	0.001299	0.001517	0.001749	0.001995	0.002255	0.002530	440
600	0.000025	0.000063	0.000111	0.000169	0.000236	0.000311	0.000395	0.000487	0.000587	0.000695	0.000811	0.000935	0.001067	0.001206	0.001353	600
800	0.000014	0.000035	0.000062	0.000095	0.000132	0.000174	0.000221	0.000273	0.000329	0.000390	0.000455	0.000524	0.000598	0.000678	0.000758	800
1000	0.000009	0.000023	0.000040	0.000061	0.000084	0.000111	0.000141	0.000174	0.000210	0.000249	0.000290	0.000335	0.000382	0.000431	0.000484	1000

TABLE 3. Upper percentage points of the test statistic $\mathbf{V}_{p,\nu_H,\nu_E}$ ($p = 2$ and $\alpha = 0.05$).

ν_E	ν_H = 17	18	19	20	21	22	23	24	25	26	27	28	29	30	ν_E
2	96989.82	109513.1	122796.5	136840.1	151643.8	167207.8	183531.8	200616.1	218460.5	237065.1	256429.9	276554.8	297439.9	319085.1	2
3	2107.686	2374.109	2656.371	2954.470	3268.407	3598.182	3943.794	4305.244	4682.533	5075.657	5484.619	5909.420	6350.058	6806.534	3
4	410.6839	461.8063	515.9200	573.0252	633.1216	696.2094	762.2885	831.3589	903.4207	978.4737	1056.518	1137.554	1221.580	1308.598	4
5	150.8304	169.3812	189.00351	209.6972	231.4624	254.2990	278.2070	303.1864	329.2373	356.3595	384.5531	413.8181	444.1544	475.5621	5
6	74.09623	83.11969	92.65855	102.7128	113.2824	124.3674	135.9677	148.0833	160.7143	173.8606	187.5222	201.6991	216.3913	231.5988	6
7	42.90410	48.08545	53.55984	59.32727	65.38771	71.74116	78.38761	85.32704	92.55945	100.0848	107.9032	116.0145	124.4188	133.1159	7
8	27.58659	30.89415	34.38714	38.06554	41.92934	45.97853	50.21309	54.63303	59.23832	64.02896	69.00496	74.16629	79.51297	85.04499	8
9	19.06567	21.33719	23.73507	26.25925	28.90974	31.68653	34.58958	37.61891	40.77451	44.05638	47.46449	50.99887	54.65948	58.44634	9
10	13.88805	15.53348	17.26976	19.09687	21.01479	23.02353	25.12306	27.31338	29.59448	31.96635	34.42899	36.98241	39.62659	42.36152	10
11	10.52829	11.76945	13.07867	14.45595	15.90126	17.41460	18.99596	20.64533	22.36271	24.14808	26.00145	27.92281	29.91215	31.96948	11
12	8.234648	9.201058	10.22014	11.29187	12.41625	13.59326	14.82289	16.10515	17.44001	18.82748	20.26755	21.76021	23.30547	24.90331	12
13	6.604391	7.376324	8.190084	9.045657	9.943033	10.88220	11.86316	12.88589	13.95039	15.05665	16.20468	17.39447	18.62600	19.89929	13
14	5.407014	6.036653	6.700225	7.397717	8.129118	8.894420	9.693613	10.52669	11.39365	12.29448	13.22918	14.19775	15.20018	16.23647	14
15	4.503349	5.025977	5.576629	6.155292	6.761956	7.396613	8.059254	8.749873	9.468464	10.21502	10.98954	11.79202	12.62246	13.48085	15
16	3.805569	4.245839	4.709604	5.196853	5.707577	6.241765	6.799412	7.380511	7.985056	8.613043	9.264467	9.939324	10.63761	11.35933	16
17	3.256154	3.631775	4.027353	4.442875	4.878333	5.333718	5.809023	6.304243	6.819371	7.354404	7.909337	8.484166	9.078889	9.693501	17
18	2.816216	3.140220	3.481364	3.839640	4.215036	4.607547	5.017164	5.443883	5.887698	6.348604	6.826598	7.321675	7.833833	8.363070	18
19	2.458739	2.740913	3.037955	3.349857	3.676608	4.018201	4.374631	4.745891	5.131977	5.532884	5.948608	6.379145	6.824494	7.284650	19
20	2.164501	2.412332	2.673174	2.947015	3.233848	3.533666	3.846462	4.172232	4.510970	4.862672	5.227334	5.604954	5.995527	6.399052	20
21	1.919538	2.138846	2.369625	2.611867	2.865562	3.130705	3.407290	3.695310	3.994762	4.305641	4.627944	4.961668	5.306810	5.663366	21
22	1.713513	1.908886	2.114444	2.330178	2.556079	2.792142	3.038362	3.294731	3.561248	3.837906	4.124704	4.421637	4.728704	5.045901	22
23	1.538651	1.713753	1.897953	2.091243	2.293616	2.505065	2.725585	2.955170	3.193818	3.441523	3.698282	3.964093	4.238952	4.522858	23
24	1.389014	1.546804	1.712767	1.886896	2.069183	2.259623	2.458209	2.664938	2.879805	3.102807	3.333940	3.573201	3.820588	4.076099	24
25	1.260004	1.402897	1.553170	1.710816	1.875827	2.048198	2.227924	2.414999	2.609421	2.811186	3.020289	3.236730	3.460504	3.691610	25
26	1.148022	1.278009	1.414690	1.558058	1.708106	1.864830	2.028224	2.198284	2.375006	2.558386	2.748421	2.945110	3.148448	3.358434	26
27	1.050217	1.168950	1.293782	1.424705	1.561713	1.704801	1.853964	2.009198	2.170499	2.337864	2.511290	2.690773	2.876313	3.067906	27
28	0.964306	1.073171	1.187613	1.307625	1.433202	1.564338	1.701029	1.843271	1.991059	2.144391	2.303264	2.467676	2.637623	2.813105	28
29	0.888448	0.988613	1.093896	1.204292	1.319795	1.440399	1.566101	1.696895	1.832780	1.973750	2.119805	2.270941	2.427156	2.588449	29
30	0.821141	0.913598	1.010769	1.112648	1.219228	1.330506	1.446477	1.567137	1.692482	1.822510	1.957219	2.096604	2.240665	2.389400	30
40	0.426395	0.473925	0.523830	0.576110	0.630756	0.687770	0.747145	0.808880	0.872970	0.939416	1.008211	1.079360	1.152860	1.228705	40
60	0.174801	0.194062	0.214263	0.235402	0.257477	0.280485	0.304425	0.329295	0.355093	0.381819	0.409471	0.438048	0.467548	0.497972	60
80	0.094397	0.104731	0.115562	0.126889	0.138711	0.151025	0.163831	0.177128	0.190914	0.205189	0.219952	0.235203	0.250939	0.267162	80
100	0.058945	0.065371	0.072104	0.079142	0.086484	0.094129	0.102077	0.110326	0.118876	0.127727	0.136877	0.146326	0.156074	0.166120	100
120	0.040265	0.044642	0.049227	0.054017	0.059014	0.064215	0.069621	0.075230	0.081043	0.087058	0.093276	0.099696	0.106317	0.113139	120
140	0.029235	0.032407	0.035728	0.039197	0.042815	0.046580	0.050493	0.054552	0.058757	0.063108	0.067605	0.072248	0.077035	0.081966	140
170	0.019581	0.021699	0.023918	0.026235	0.028651	0.031164	0.033776	0.036485	0.039290	0.042193	0.045192	0.048287	0.051478	0.054765	170
200	0.014023	0.015538	0.017124	0.018780	0.020506	0.022302	0.024168	0.026103	0.028107	0.030179	0.032320	0.034530	0.036808	0.039154	200
240	0.009658	0.010699	0.011790	0.012928	0.014114	0.015349	0.016630	0.017959	0.019335	0.020759	0.022229	0.023746	0.025309	0.026920	240
320	0.005376	0.005955	0.006560	0.007193	0.007851	0.008536	0.009247	0.009985	0.010748	0.011538	0.012353	0.013194	0.014061	0.014953	320
440	0.002819	0.003122	0.003439	0.003770	0.004115	0.004473	0.004845	0.005231	0.005630	0.006043	0.006469	0.006908	0.007361	0.007828	440
600	0.001507	0.001669	0.001838	0.002015	0.002198	0.002390	0.002588	0.002794	0.003007	0.003227	0.003454	0.003688	0.003930	0.004178	600
800	0.000844	0.000936	0.001029	0.001128	0.001231	0.001338	0.001449	0.001564	0.001683	0.001806	0.001933	0.002064	0.002199	0.002338	800
1000	0.000539	0.000597	0.000657	0.000720	0.000786	0.000854	0.000925	0.000998	0.001074	0.001153	0.001234	0.001317	0.001403	0.001492	1000

TABLE 4. Upper percentage points of the test statistic $\mathbf{V}_{p,\nu_H,\nu_E}$ ($p = 2$ and $\alpha = 0.05$).

ν_E	ν_H 35	40	45	50	55	60	65	70	75	80	85	90	95	100	120	ν_E
2	438713.8	577346.8	734983.9	911625.1	1107270	1321920	1555573	1808231	2079894	2370560	2680230	3008905	3356584	3723267	5378759	2
3	9326.477	12242.36	15554.18	19261.95	23365.65	27865.30	32760.88	38052.40	43739.87	49823.27	56302.61	63177.90	70449.12	78116.28	112744.3	3
4	1788.558	2343.299	2972.821	3677.125	4456.210	5310.075	6238.722	7242.149	8320.358	9473.348	10701.11	12003.67	13381.00	14833.12	21389.38	4
5	648.6712	848.5642	1075.241	1328.701	1608.945	1915.973	2249.784	2610.379	2997.757	3411.919	3852.864	4320.592	4815.104	5336.399	7689.416	5
6	315.3658	412.0150	521.5462	643.9593	779.2543	927.4310	1088.489	1262.361	1449.252	1648.955	1861.541	2087.008	2325.357	2576.587	3710.325	6
7	180.9963	236.2001	298.7272	368.5776	445.7510	530.2476	622.0673	721.2100	827.6757	941.4644	1062.576	1191.011	1326.769	1469.849	2115.402	7
8	115.4850	150.5581	190.2640	234.6027	283.5741	337.1782	395.4149	458.2842	525.7861	597.9205	674.6875	756.0870	842.1191	932.7837	1341.767	8
9	79.27420	103.2578	130.3969	160.6915	194.1415	230.7468	270.5075	313.4234	359.4945	408.7209	461.1026	516.6394	575.3315	637.1788	916.1201	9
10	57.39748	74.70206	94.27510	116.1165	140.2262	166.6040	195.2501	226.1644	259.3468	294.7974	332.5160	372.5028	414.7577	459.2807	660.0537	10
11	43.27574	56.28122	70.98573	87.38919	105.4915	125.2926	146.7926	169.9912	194.8887	221.4848	249.7797	279.7732	311.4655	344.8564	495.4070	11
12	33.68127	43.77362	55.18018	67.90086	81.93558	97.28429	113.9470	131.9236	151.2141	171.8185	193.7368	216.9690	241.5151	267.3750	383.9533	12
13	26.89189	34.92790	44.00717	54.12959	65.29509	77.50363	90.75516	105.0497	120.3871	136.7675	154.1908	172.6571	192.1662	212.7183	305.3554	13
14	21.92570	28.46108	35.84244	44.06969	53.14276	63.06159	73.82615	85.43642	97.89238	111.1940	125.3414	140.3342	156.1728	172.8570	248.0501	14
15	18.19199	23.60162	29.70959	36.51580	44.02017	52.22267	61.12324	70.72188	81.01855	92.01324	103.7060	116.0967	129.1854	142.9721	205.0986	15
16	15.31922	19.86449	24.99497	30.71057	37.01122	43.89687	51.36748	59.42303	68.06350	77.28888	87.09915	97.49430	108.4743	120.0392	172.1474	16
17	13.06484	16.93313	21.29823	26.16004	31.51848	37.37352	43.72511	50.57324	57.91788	65.75901	74.09663	82.93073	92.26129	102.0883	146.3609	17
18	11.26534	14.59427	18.34971	22.53156	27.13975	32.17425	37.63500	43.52200	49.83521	56.57462	63.74023	71.33201	79.34998	87.79411	125.8322	18
19	9.807474	12.70021	15.96273	19.59491	23.59670	27.96806	32.70894	37.81933	43.29919	49.14853	55.36732	61.95555	68.91323	76.24034	109.2430	19
20	8.610872	11.14620	14.00490	17.18688	20.69204	24.52038	28.67184	33.14641	37.94406	43.06477	48.50854	54.27536	60.36521	66.77810	95.65989	20
21	7.617302	9.856347	12.38036	15.18925	18.28296	21.66143	25.32463	29.27254	33.50514	38.02241	42.82435	47.91094	53.28217	58.93804	84.40783	21
22	6.783772	8.774644	11.01838	13.51489	16.26411	19.26599	22.52050	26.02762	29.78733	33.79961	38.06445	42.58184	47.35178	52.37425	74.98943	22
23	6.078011	7.859066	9.865887	12.09839	14.55650	17.24019	20.14941	23.28415	26.64438	30.23009	34.04127	38.07791	42.34000	46.82754	67.03203	23
24	5.475433	7.077602	8.882476	10.88997	13.10001	15.51256	18.12760	20.94509	23.96501	27.18735	30.61210	34.23926	38.06880	42.10073	60.25221	24
25	4.957051	6.405542	8.036957	9.851207	11.84823	14.02799	16.39046	18.93559	21.66339	24.57383	27.66690	30.94259	34.40090	38.04182	54.43147	25
26	4.508022	5.823572	7.304961	8.952103	10.76494	12.74343	14.88753	17.19724	19.67252	22.31336	25.11975	28.09169	31.22915	34.53215	49.39933	26
27	4.116614	5.316432	6.667237	8.168947	9.821503	11.62486	13.57900	15.68388	17.93949	20.34582	22.90285	25.61058	28.46899	31.47809	45.02121	27
28	3.773459	4.871936	6.108417	7.482820	8.995088	10.64518	12.43306	14.35871	16.42211	18.62325	20.96211	23.43869	26.05298	28.80497	41.18984	28
29	3.471005	4.480269	5.616122	6.878485	8.267302	9.782532	11.42414	13.19211	15.08642	17.10706	19.25401	21.52727	23.92683	26.45268	37.81891	29
30	3.203114	4.133450	5.180291	6.343561	7.623204	9.019179	10.53146	12.16001	13.90483	15.76589	17.74320	19.83672	22.04647	24.37243	34.83831	30
40	1.643097	2.116011	2.647352	3.237058	3.885081	4.591387	5.355949	6.178746	7.059764	7.998989	8.996410	10.05202	11.16581	12.33778	17.60731	40
60	0.663903	0.852798	1.064595	1.299247	1.556723	1.836996	2.140046	2.465857	2.814418	3.185716	3.579745	3.996497	4.435966	4.898147	6.973917	60
80	0.355541	0.455990	0.568463	0.692929	0.829362	0.977742	1.138055	1.310287	1.494430	1.690473	1.898412	2.118239	2.349950	2.593540	3.686634	80
100	0.220808	0.282894	0.352345	0.429137	0.513249	0.604667	0.703378	0.809373	0.922643	1.043181	1.170983	1.306043	1.448357	1.597921	2.268626	100
120	0.150256	0.192359	0.239422	0.291425	0.348354	0.410197	0.476942	0.548583	0.625113	0.706526	0.792817	0.883982	0.980018	1.080921	1.533164	120
140	0.108786	0.139189	0.173155	0.210668	0.251715	0.296286	0.344374	0.395972	0.451074	0.509676	0.571773	0.637362	0.706440	0.779004	1.104083	140
170	0.072633	0.092873	0.115470	0.140412	0.167691	0.197297	0.229226	0.263471	0.300030	0.338896	0.380069	0.423544	0.469320	0.517394	0.732640	170
200	0.051901	0.066333	0.082437	0.100205	0.119630	0.140705	0.163426	0.187787	0.213787	0.241421	0.270688	0.301585	0.334109	0.368260	0.521096	200
240	0.035666	0.045562	0.056600	0.068774	0.082076	0.096504	0.112053	0.128719	0.146502	0.165397	0.185403	0.206518	0.228741	0.252070	0.356426	240
320	0.019798	0.025277	0.031383	0.038114	0.045465	0.053434	0.062017	0.071215	0.081023	0.091442	0.102469	0.114104	0.126345	0.139192	0.196619	320
440	0.010358	0.013218	0.016403	0.019912	0.023743	0.027894	0.032364	0.037151	0.042254	0.047673	0.053406	0.059454	0.065814	0.072488	0.102300	440
600	0.005527	0.007050	0.008746	0.010614	0.012652	0.014860	0.017236	0.019781	0.022492	0.025371	0.028416	0.031627	0.035003	0.038545	0.054358	600
800	0.003092	0.003943	0.004891	0.005934	0.007072	0.008304	0.009630	0.011050	0.012562	0.014168	0.015865	0.017656	0.019538	0.021511	0.030320	800
1000	0.001973	0.002515	0.003119	0.003784	0.004509	0.005294	0.006138	0.007042	0.008005	0.009027	0.010108	0.011248	0.012445	0.013702	0.019306	1000

TABLE 5. Upper percentage points of the test statistic $\mathbf{V}_{p,\nu_H,\nu_E}$ ($p = 2$ and $\alpha = 0.01$).

ν_E	ν_H															ν_E
	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	
2	9801.000	39401.75	88802.67	158003.8	247005.0	355806.4	484408.0	632809.8	801011.7	989013.8	1196816	1424418	1671821	1939024	2226027	2
3	81.00000	255.2653	520.3097	876.0284	1322.395	1859.400	2487.038	3205.308	4014.207	4913.736	5903.892	6984.677	8156.090	9418.130	10770.80	3
4	13.26117	37.19618	71.67528	116.6876	172.2271	238.2906	314.8765	401.9837	499.6114	607.7592	726.4269	855.6142	995.3209	1145.547	1306.292	4
5	4.675445	12.27128	22.82939	36.34728	52.82190	72.25134	94.63444	119.9704	148.2589	179.4994	213.6917	250.8357	290.9312	333.9782	379.9765	5
6	2.285801	5.749135	10.44251	16.36489	23.51423	31.88910	41.48862	52.31220	64.35942	77.63001	92.12377	107.8405	124.7802	142.9427	162.3279	6
7	1.332719	3.254358	5.809482	8.997461	12.81673	17.26617	22.34506	28.05290	34.38936	41.35418	48.94720	57.16828	66.01732	75.49424	85.59899	7
8	0.866198	2.069798	3.646714	5.596458	7.917777	10.60977	13.67181	17.10349	20.90451	25.07466	29.61378	34.52175	39.79848	45.44390	51.45796	8
9	0.605719	1.423594	2.482555	3.782193	5.321475	7.099640	9.116168	11.37069	13.86295	16.59276	19.55998	22.76450	26.20624	29.88513	33.80113	9
10	0.446358	1.035436	1.790812	2.712129	3.798518	5.049332	6.464118	8.042557	9.784419	11.68954	13.75778	15.98905	18.38328	20.94041	23.66038	10
11	0.342100	0.785241	1.348915	2.032805	2.836170	3.758449	4.799246	5.958279	7.235344	8.630289	10.14300	11.77338	13.52137	15.38691	17.36995	11
12	0.270308	0.615064	1.050607	1.576653	2.192561	2.897840	3.692143	4.575216	5.546877	6.606987	7.755442	8.992160	10.31708	11.73014	13.23131	12
13	0.218836	0.494313	0.840301	1.256542	1.742476	2.297672	2.921819	3.614693	4.376125	5.205992	6.104197	7.070664	8.105335	9.208161	10.37910	13
14	0.180712	0.405663	0.686760	1.023769	1.416198	1.863664	2.365887	2.922665	3.533846	4.199317	4.918988	5.692790	6.520669	7.402578	8.338483	14
15	0.151707	0.338729	0.571393	0.849488	1.172573	1.540308	1.952442	2.408789	2.909213	3.453608	4.041893	4.674002	5.349886	6.069502	6.832816	15
16	0.129137	0.286993	0.482602	0.715772	0.986111	1.293310	1.637144	2.017442	2.434081	2.886962	3.376010	3.901165	4.462380	5.059616	5.692840	16
17	0.111237	0.246199	0.412857	0.611034	0.840377	1.100607	1.391518	1.712956	2.064806	2.446977	2.859400	3.302019	3.774789	4.277674	4.810644	17
18	0.096804	0.213480	0.357107	0.527523	0.724408	0.947510	1.196640	1.471657	1.772454	2.098947	2.451072	2.828778	3.232021	3.660767	4.114989	18
19	0.085001	0.186846	0.311862	0.459903	0.630676	0.823952	1.039557	1.277361	1.537265	1.819192	2.123082	2.448885	2.796563	3.166082	3.557417	19
20	0.075227	0.164881	0.274653	0.404407	0.553877	0.722852	0.911172	1.118716	1.345394	1.591133	1.855876	2.139578	2.442203	2.763718	3.104100	20
21	0.067042	0.146558	0.243692	0.358317	0.490191	0.639117	0.804950	0.987578	1.186915	1.402894	1.635462	1.884576	2.150202	2.432310	2.730877	21
22	0.060122	0.131118	0.217661	0.319633	0.436811	0.569014	0.716107	0.877986	1.054572	1.245800	1.451622	1.671998	1.906894	2.156282	2.420141	22
23	0.054218	0.117987	0.195569	0.286856	0.391640	0.509755	0.641075	0.785504	0.942965	1.113401	1.296765	1.493019	1.702131	1.924076	2.158832	23
24	0.049141	0.106728	0.176664	0.258847	0.353086	0.459227	0.577152	0.706770	0.848012	1.000822	1.165156	1.340978	1.528259	1.726974	1.937103	24
25	0.044744	0.097002	0.160362	0.234728	0.319924	0.415805	0.522262	0.639209	0.766581	0.904326	1.052402	1.210776	1.379419	1.558309	1.747425	25
26	0.040911	0.088544	0.146208	0.213815	0.291198	0.378224	0.474789	0.580815	0.696239	0.821012	0.955095	1.098457	1.251069	1.412911	1.583963	26
27	0.037549	0.081143	0.133843	0.195565	0.266154	0.345486	0.433464	0.530012	0.635074	0.748602	0.870559	1.000915	1.139645	1.286727	1.442144	27
28	0.034585	0.074631	0.122977	0.179545	0.244192	0.316798	0.397273	0.485548	0.581566	0.685285	0.796668	0.915688	1.042320	1.176543	1.318341	28
29	0.031958	0.068870	0.113379	0.165409	0.224827	0.291521	0.365406	0.446415	0.534497	0.629610	0.731721	0.840802	0.956830	1.079786	1.209654	29
30	0.029619	0.063751	0.104859	0.152873	0.207668	0.269137	0.337202	0.411799	0.492879	0.580403	0.674339	0.774660	0.881346	0.994376	1.113737	30
40	0.015709	0.033510	0.054770	0.079445	0.107463	0.138761	0.173290	0.211013	0.251899	0.295923	0.343064	0.393306	0.446635	0.503038	0.562504	40
60	0.006590	0.013935	0.022630	0.032658	0.043983	0.056576	0.070415	0.085481	0.101758	0.119235	0.137900	0.157746	0.178765	0.200949	0.224294	60
80	0.003603	0.007585	0.012279	0.017674	0.023750	0.030491	0.037882	0.045914	0.054576	0.063862	0.073765	0.084281	0.095403	0.107129	0.119455	80
100	0.002267	0.004760	0.007692	0.011054	0.014834	0.019021	0.023607	0.028584	0.033945	0.039687	0.045806	0.052296	0.059156	0.066383	0.073974	100
120	0.001557	0.003263	0.005266	0.007560	0.010136	0.012986	0.016105	0.019488	0.023129	0.027026	0.031175	0.035574	0.040221	0.045114	0.050251	120
140	0.001135	0.002375	0.003830	0.005494	0.007361	0.009426	0.011684	0.014130	0.016763	0.019579	0.022577	0.025753	0.029107	0.032637	0.036342	140
170	0.000763	0.001595	0.002570	0.003683	0.004932	0.006311	0.007818	0.009451	0.011207	0.013083	0.015080	0.017195	0.019426	0.021775	0.024238	170
200	0.000548	0.001145	0.001843	0.002640	0.003533	0.004519	0.005596	0.006762	0.008016	0.009355	0.010779	0.012288	0.013879	0.015552	0.017307	200
240	0.000379	0.000790	0.001271	0.001819	0.002434	0.003112	0.003852	0.004653	0.005514	0.006433	0.007410	0.008445	0.009536	0.010683	0.011886	240
320	0.000211	0.000441	0.000708	0.001014	0.001355	0.001732	0.002143	0.002587	0.003065	0.003575	0.004116	0.004689	0.005293	0.005928	0.006594	320
440	0.000111	0.000232	0.000372	0.000532	0.000711	0.000908	0.001123	0.001356	0.001605	0.001872	0.002155	0.002454	0.002769	0.003101	0.003448	440
600	0.000060	0.000124	0.000199	0.000285	0.000380	0.000485	0.000600	0.000724	0.000857	0.000999	0.001150	0.001310	0.001478	0.001654	0.001839	600
800	0.000033	0.000070	0.000112	0.000159	0.000213	0.000272	0.000336	0.000406	0.000480	0.000559	0.000644	0.000733	0.000827	0.000926	0.001029	800
1000	0.000021	0.000044	0.000071	0.000102	0.000136	0.000174	0.000215	0.000259	0.000306	0.000357	0.000411	0.000468	0.000528	0.000590	0.000656	1000

TABLE 6. Upper percentage points of the test statistic $\mathbf{V}_{p,\nu_H,\nu_E}$ ($p = 2$ and $\alpha = 0.01$).

	ν_H														
ν_E	17	18	19	20	21	22	23	24	25	26	27	28	29	30	ν_E
2	2532830	2859433	3205836	3572040	3958044	4363848	4789452	5234856	5700060	6185065	6689870	7214475	7758880	8323085	2
3	12214.09	13748.01	15372.56	17087.74	18893.54	20789.97	22777.03	24854.72	27023.03	29281.97	31631.53	34071.73	36602.55	39223.99	3
4	1477.557	1659.340	1851.643	2054.465	2267.805	2491.665	2726.044	2970.941	3226.358	3492.293	3768.747	4055.721	4353.213	4661.224	4
5	428.9261	480.8270	535.6791	593.4823	654.2367	717.9423	784.5990	854.2067	926.7656	1002.276	1080.737	1162.149	1246.512	1333.826	5
6	182.9359	204.7665	227.8197	252.0955	277.5938	304.3148	332.2582	361.4242	391.8126	423.4236	456.2570	490.3129	525.5913	562.0922	6
7	96.33150	107.6917	119.6797	132.2953	145.5386	159.4095	173.9080	189.0342	204.7879	221.1692	238.1781	255.8146	274.0786	292.9702	7
8	57.84060	64.59180	71.71152	79.19974	87.05643	95.28159	103.8752	112.8372	122.1677	131.8665	141.9338	152.3695	163.1735	174.3460	8
9	37.95418	42.34426	46.97134	51.83539	56.93640	62.27434	67.84921	73.66099	79.70968	85.99526	92.51772	99.27706	106.2733	113.5064	9
10	26.54315	29.58870	32.79701	36.16804	39.70177	43.39821	47.25732	51.27910	55.46354	59.81062	64.32036	68.99272	73.82772	78.82534	10
11	19.47046	21.68842	24.02378	26.47654	29.04668	31.73418	34.53903	37.46121	40.50072	43.65755	46.93170	50.32315	53.83190	57.45794	11
12	14.82055	16.49783	18.26314	20.11644	22.05773	24.08699	26.20420	28.40937	30.70247	33.08350	35.55245	38.10932	40.75410	43.48679	12
13	11.61813	12.92522	14.30034	15.74348	17.25462	18.83375	20.48086	22.19593	23.97896	25.82994	27.74887	29.73573	31.79052	33.91325	13
14	9.328353	10.37216	11.46989	12.62152	13.82704	15.08643	16.39969	17.76679	19.18774	20.66253	22.19115	23.77359	25.40986	27.09994	14
15	7.639800	8.490430	9.384688	10.32256	11.30402	12.32906	13.39768	14.50986	15.66560	16.86489	18.10771	19.39408	20.72397	22.09739	15
16	6.362026	7.067152	7.808199	8.585151	9.397994	10.24672	11.13131	12.05176	13.00807	14.00022	15.02821	16.09204	17.19169	18.32717	16
17	5.373673	5.966741	6.589829	7.242923	7.926009	8.639076	9.382115	10.15512	10.95808	11.79098	12.65383	13.54662	14.46934	15.42199	17
18	4.594661	5.099766	5.630285	6.186203	6.767510	7.374193	8.006244	8.663654	9.346417	10.05453	10.78797	11.54676	12.33087	13.14031	18
19	3.970544	4.405446	4.862104	5.340507	5.840642	6.362498	6.906067	7.471342	8.058314	8.666978	9.297328	9.949361	10.62307	11.31845	19
20	3.463327	3.841380	4.238244	4.653907	5.088357	5.541584	6.013580	6.504336	7.013847	7.542106	8.089108	8.654849	9.239325	9.842531	20
21	3.045882	3.377309	3.725142	4.089369	4.469979	4.866963	5.280312	5.710020	6.156080	6.618486	7.097232	7.592316	8.103732	8.631478	21
22	2.698452	2.991197	3.298362	3.619937	3.955910	4.306273	4.671017	5.050135	5.443622	5.851471	6.273679	6.710241	7.161152	7.626409	22
23	2.406381	2.666708	2.939798	3.225641	3.524227	3.835547	4.159592	4.496358	4.845837	5.208025	5.582916	5.970506	6.370793	6.783771	23
24	2.158629	2.391535	2.635811	2.891444	3.158426	3.436747	3.726401	4.027381	4.339681	4.663297	4.998223	5.344456	5.701992	6.070828	24
25	1.946751	2.156274	2.375980	2.605859	2.845902	3.096101	3.356449	3.626940	3.907568	4.198329	4.499217	4.810230	5.131362	5.462612	25
26	1.764210	1.953639	2.152237	2.359994	2.576902	2.802953	3.038140	3.282457	3.535899	3.798462	4.070139	4.350929	4.640828	4.939832	26
27	1.605880	1.777924	1.958262	2.146887	2.343788	2.548960	2.762395	2.984088	3.214033	3.452226	3.698662	3.953339	4.216252	4.487399	27
28	1.467700	1.624607	1.789051	1.961023	2.140515	2.327519	2.522031	2.724042	2.933550	3.150549	3.375036	3.607006	3.846457	4.093386	28
29	1.346419	1.490071	1.640599	1.797994	1.962247	2.133353	2.311304	2.496097	2.687725	2.886184	3.091472	3.303583	3.522516	3.748267	29
30	1.239414	1.371397	1.509675	1.654240	1.805084	1.962201	2.125585	2.295230	2.471132	2.653287	2.841691	3.036340	3.237232	3.444364	30
40	0.625025	0.690594	0.759202	0.830844	0.905515	0.983210	1.063925	1.147655	1.234398	1.324151	1.416910	1.512673	1.611438	1.713203	40
60	0.248795	0.274447	0.301247	0.329190	0.358275	0.388498	0.419856	0.452348	0.485972	0.520725	0.556606	0.593614	0.631746	0.671002	60
80	0.132378	0.145895	0.160003	0.174701	0.189987	0.205858	0.222312	0.239350	0.256969	0.275168	0.293945	0.313300	0.333232	0.353740	80
100	0.081927	0.090240	0.098912	0.107941	0.117326	0.127065	0.137157	0.147602	0.158398	0.169544	0.181040	0.192885	0.205077	0.217618	100
120	0.055631	0.061251	0.067112	0.073211	0.079548	0.086122	0.092932	0.099977	0.107256	0.114769	0.122516	0.130495	0.138706	0.147149	120
140	0.040220	0.044270	0.048492	0.052885	0.057448	0.062179	0.067080	0.072148	0.077383	0.082786	0.088355	0.094089	0.099989	0.106055	140
170	0.026815	0.029507	0.032311	0.035227	0.038256	0.041395	0.044646	0.048007	0.051477	0.055058	0.058748	0.062546	0.066454	0.070469	170
200	0.019143	0.021060	0.023056	0.025132	0.027287	0.029521	0.031833	0.034223	0.036690	0.039235	0.041858	0.044557	0.047333	0.050185	200
240	0.013144	0.014457	0.015824	0.017245	0.018720	0.020249	0.021831	0.023466	0.025153	0.026893	0.028686	0.030531	0.032428	0.034377	240
320	0.007289	0.008015	0.008771	0.009556	0.010371	0.011215	0.012088	0.012990	0.013922	0.014881	0.015870	0.016887	0.017933	0.019007	320
440	0.003811	0.004189	0.004583	0.004993	0.005417	0.005857	0.006311	0.006781	0.007266	0.007766	0.008280	0.008809	0.009353	0.009912	440
600	0.002033	0.002234	0.002444	0.002661	0.002887	0.003121	0.003363	0.003613	0.003871	0.004136	0.004410	0.004691	0.004980	0.005276	600
800	0.001137	0.001249	0.001366	0.001488	0.001614	0.001745	0.001880	0.002019	0.002163	0.002311	0.002464	0.002621	0.002782	0.002947	800
1000	0.000725	0.000797	0.000871	0.000949	0.001029	0.001112	0.001198	0.001287	0.001379	0.001473	0.001570	0.001670	0.001773	0.001878	1000

TABLE 7. Upper percentage points of the test statistic $\mathbf{V}_{p,\nu_H,\nu_E}$ ($p = 2$ and $\alpha = 0.01$).

ν_E	ν_H															ν_E
	35	40	45	50	55	60	65	70	75	80	85	90	95	100	120	
2	11441114	15054147	19162184	23765225	28863270	34456320	40544374	47127432	54205494	61778560	69846630	78409705	87467784	97020867	140183240	2
3	53690.63	70422.94	89420.93	110684.6	134213.9	160008.9	188069.6	218395.9	250987.9	285845.6	322969.0	362358.0	404012.7	447933.1	646271.3	3
4	6359.061	8319.871	10543.65	13030.40	15780.13	18792.82	22068.48	25607.12	29408.72	33473.30	37800.84	42391.36	47244.85	52361.30	75456.84	4
5	1814.662	2369.275	2997.662	3699.825	4475.762	5325.475	6248.963	7246.226	8317.263	9462.075	10680.66	11973.02	13339.16	14779.07	21276.47	5
6	762.9334	994.3357	1256.299	1548.822	1871.907	2225.551	2609.757	3024.522	3469.848	3945.734	4452.181	4989.188	5556.755	6154.882	8852.995	6
7	396.8413	516.4008	651.6484	802.5840	969.2074	1151.519	1349.518	1563.204	1792.579	2037.640	2298.390	2574.827	2866.952	3174.764	4562.890	7
8	235.7339	306.3309	386.1367	475.1513	573.3744	680.8061	797.4463	923.2948	1058.352	1202.617	1356.091	1518.773	1690.663	1871.762	2688.241	8
9	153.2246	198.8641	250.4245	307.9056	371.3074	440.6297	515.8726	597.0360	684.1199	777.1242	876.0497	980.8940	1091.660	1208.345	1734.293	9
10	106.2527	137.7452	173.3024	212.9245	256.6112	304.3624	356.1781	412.0582	472.0027	536.0117	604.0850	676.2227	752.4248	832.6912	1194.400	10
11	77.34748	100.1689	125.9220	154.6066	186.2225	220.7698	258.2484	298.6582	341.9992	388.2714	437.4747	489.6092	544.6749	602.6716	863.9700	11
12	58.46869	75.64778	95.02381	116.5966	140.3662	166.3323	194.4950	224.8542	257.4099	292.1620	329.1107	368.2558	409.5973	453.1353	649.2512	12
13	45.54564	58.87576	73.90337	90.62830	109.0505	129.1698	150.9863	174.4998	199.7103	226.6179	255.2225	285.5241	317.5227	351.2182	502.9700	13
14	36.35744	46.95986	58.90698	72.19866	86.83477	102.8153	120.1401	138.8092	158.8225	180.1801	202.8820	226.9280	252.3182	279.0526	399.4320	14
15	29.61729	38.22492	47.92006	58.70256	70.57232	83.52927	97.57337	112.7046	128.9228	146.2282	164.6205	184.0999	204.6663	226.3197	323.8032	15
16	24.54185	31.65173	39.65659	48.55630	58.35075	69.03987	80.62362	93.10196	106.4748	120.7423	135.9042	151.9607	168.9116	186.7571	267.0836	16
17	20.63406	26.59395	33.30142	40.75635	48.95864	57.90821	67.60501	78.04902	89.24020	101.1785	113.8640	127.2966	141.4763	156.4030	223.5810	17
18	17.56731	22.62707	28.31938	34.64409	41.60113	49.19040	57.41188	66.26552	75.75128	85.86916	96.61914	108.0012	120.0153	132.6615	189.5667	18
19	15.12036	19.46372	24.34830	29.77398	35.74066	42.24828	49.29678	56.88613	65.01630	73.68728	82.89903	92.65156	102.9448	113.7789	162.5224	19
20	13.13941	16.90418	21.13662	25.83660	31.00402	36.63884	42.74098	49.31043	56.34714	63.85111	71.82230	80.26072	89.16634	98.53916	140.7023	20
21	11.51504	14.80644	18.50550	22.61207	27.12607	32.04743	37.37611	43.11206	49.25526	55.80570	62.76334	70.12818	77.90021	86.07942	122.8679	21
22	10.16779	13.06746	16.32522	19.94095	23.91456	28.24598	32.93516	37.98208	43.38669	49.14899	55.26895	61.74656	68.58180	75.77468	108.1224	22
23	9.038952	11.61108	14.49997	17.70550	21.22757	25.06612	29.22111	33.69249	38.48026	43.58437	49.00482	54.74159	60.79467	67.16405	95.80447	23
24	8.084407	10.38013	12.95781	15.81733	18.95860	22.38155	26.08614	30.07234	34.34011	38.88945	43.72032	48.83272	54.22664	59.90206	85.41869	24
25	7.270528	9.331036	11.64396	14.20916	17.02658	20.09613	23.41779	26.99150	30.81725	34.89502	39.22478	43.80653	48.64026	53.72595	76.58823	25
26	6.571344	8.430163	10.51611	12.82907	15.36896	18.13571	21.12929	24.34965	27.79677	31.47064	35.37122	39.49852	43.85252	48.43321	69.02273	26
27	5.966552	7.651226	9.541248	11.63650	13.93691	16.44240	19.15294	22.06848	25.18902	28.51452	32.04496	35.78034	39.72065	43.86587	62.49578	27
28	5.440110	6.973463	8.693278	10.59944	12.69187	14.97050	17.43530	20.08622	22.92325	25.94636	29.15554	32.55077	36.13205	39.89936	56.82881	28
29	4.979212	6.380308	7.951391	9.692349	11.60310	13.68359	15.93377	18.35362	20.94309	23.70219	26.63088	29.72915	32.99701	36.43442	51.87963	29
30	4.573542	5.858419	7.298834	8.894679	10.64587	12.55236	14.61410	16.83106	19.20321	21.73053	24.41301	27.25063	30.24339	33.39127	47.53390	30
40	2.266961	2.895482	3.598637	4.376336	5.228512	6.155118	7.156116	8.231477	9.381179	10.60520	11.90354	13.27617	14.72308	16.24427	23.07170	40
60	0.884097	1.125134	1.394025	1.690708	2.015136	2.367274	2.747093	3.154571	3.589690	4.052436	4.542796	5.060762	5.606324	6.179476	8.747872	60
80	0.464892	0.590337	0.730016	0.883880	1.051897	1.234038	1.430282	1.640612	1.865014	2.103476	2.355987	2.622541	2.903130	3.197748	4.516422	80
100	0.285511	0.362015	0.447084	0.540683	0.642785	0.753369	0.872419	0.999919	1.135859	1.280230	1.433023	1.594231	1.763849	1.941871	2.737929	100
120	0.192823	0.244229	0.301333	0.364106	0.432528	0.506582	0.586253	0.671532	0.762407	0.858872	0.960920	1.068545	1.181742	1.300506	1.831177	120
140	0.138846	0.175721	0.216649	0.261610	0.310585	0.363562	0.420530	0.481477	0.546398	0.615284	0.688131	0.764934	0.845689	0.930391	1.308619	140
170	0.092165	0.116537	0.143564	0.173229	0.205521	0.240427	0.277940	0.318052	0.360758	0.406051	0.453928	0.504384	0.557417	0.613022	0.861129	170
200	0.065588	0.082877	0.102036	0.123053	0.145918	0.170622	0.197159	0.225522	0.255707	0.287710	0.321526	0.357153	0.394588	0.433828	0.608805	200
240	0.044895	0.056693	0.069758	0.084082	0.099655	0.116473	0.134530	0.153821	0.174343	0.196092	0.219066	0.243262	0.268678	0.295312	0.413995	240
320	0.024799	0.031290	0.038471	0.046337	0.054883	0.064106	0.074002	0.084567	0.095801	0.107700	0.120262	0.133487	0.147372	0.161917	0.226667	320
440	0.012922	0.016292	0.020018	0.024096	0.028524	0.033300	0.038420	0.043885	0.049692	0.055839	0.062327	0.069154	0.076319	0.083821	0.117187	440
600	0.006875	0.008664	0.010640	0.012801	0.015147	0.017676	0.020386	0.023278	0.026348	0.029599	0.033027	0.036634	0.040417	0.044378	0.061981	600
800	0.003839	0.004836	0.005936	0.007140	0.008446	0.009853	0.011361	0.012969	0.014676	0.016483	0.018388	0.020391	0.022492	0.024691	0.034460	800
1000	0.002446	0.003080	0.003781	0.004547	0.005377	0.006272	0.007230	0.008252	0.009337	0.010485	0.011695	0.012967	0.014302	0.015698	0.021898	1000

TABLE 8. Upper percentage points of the test statistic $\mathbf{V}_{p,\nu_H,\nu_E}$.

$p = 3$ $\alpha = 0.05$

	ν_H										
ν_E	3	5	10	15	20	30	50	80	100	120	ν_E
10	0.073322	0.700454	7.520021	27.12518	66.10158	228.4902	1075.229	4439.711	8692.978	15045.67	10
20	0.005379	0.047264	0.465367	1.611285	3.830858	12.86758	58.95089	239.3027	465.7294	802.7410	20
30	0.001358	0.011617	0.110825	0.377156	0.886524	2.935041	13.24701	53.21887	103.1730	177.3476	30
40	0.000530	0.004477	0.042014	0.141614	0.330660	1.085038	4.849324	19.34044	37.38835	64.13839	40
50	0.000259	0.002173	0.020188	0.067632	0.157224	0.512805	2.275836	9.027131	17.41264	29.82274	50
80	0.000059	0.000490	0.004485	0.014879	0.034340	0.110847	0.485705	1.906212	3.660518	6.248559	80
100	0.000030	0.000245	0.002226	0.007359	0.016939	0.054466	0.237457	0.927768	1.778092	3.030608	100
200	0.000004	0.000029	0.000262	0.000858	0.001965	0.006263	0.026990	0.104288	0.198837	0.337505	200
400	0.000000	0.000004	0.000032	0.000104	0.000237	0.000750	0.003212	0.012322	0.023410	0.039631	400
600	0.000000	0.000001	0.000009	0.000030	0.000069	0.000219	0.000936	0.003580	0.006795	0.011484	600

$p = 3$ $\alpha = 0.01$

	ν_H										
ν_E	3	5	10	15	20	30	50	80	100	120	ν_E
10	0.264631	1.931939	17.81384	61.22305	145.6237	491.2244	2266.356	9253.390	18049.99	31162.11	10
20	0.017034	0.110726	0.903894	2.932160	6.735621	21.81983	96.93550	386.3916	747.3104	1282.661	20
30	0.004143	0.025957	0.202688	0.641679	1.450483	4.603821	20.02558	78.66108	151.3051	258.6989	30
40	0.001590	0.009780	0.074625	0.233072	0.521909	1.635863	7.017457	27.28280	52.26909	89.11286	40
50	0.000770	0.004685	0.035243	0.109124	0.242851	0.754698	3.205514	12.36654	23.61863	40.17610	50
80	0.000173	0.001037	0.007632	0.023306	0.051338	0.157219	0.655844	2.492506	4.730317	8.008860	80
100	0,000086	0,000514	0,003755	0,011413	0,025048	0,076287	0.315991	1,193271	2,258323	3,815224	100
200	0.000010	0.000060	0.000434	0.001305	0.002842	0.008551	0.034839	0.129519	0.243347	0.408692	200
400	0.000001	0.000007	0.000052	0.000156	0.000338	0.001011	0.004080	0.015018	0.028073	0.046955	400
600	0.000000	0.000002	0.000015	0.000046	0.000099	0.000294	0.001183	0.004336	0.008089	0.013507	600

ACKNOWLEDGEMENTS

The authors wish to express their gratitude to the editor and the referees for their helpful comments and suggestions. This work was supported partially by the research project 45974-F of CONACYT-México.

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