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Limiting distributions and almost sure limit theorems for the normalized maxima of complete and incomplete samples from Gaussian sequence

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Abstract: Let $\{X_k, k \geqslant 1\}$ be a stationary Gaussian sequence with partial maximum $M_n = \max\{X_k, 1 \leqslant k \leqslant n\}$ and sample mean $\overline{X}_n = \sum_{k=1}^{n} X_k/n$. Suppose that some of the random variables $X_1, X_2, \ldots$ can be observed and the others not. Denote by $\widetilde{M}_n$ the maximum of the observed random variables from the set $\{X_1, X_2, \ldots, X_n\}$. Under some mild conditions, we prove the joint limiting distribution and the almost sure limit theorem for $(\widetilde{M}_n - \overline{X}_n, M_n - \overline{X}_n)$.

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Contents

1. Introduction

Let $\{X_k, k \geqslant 1\}$ be a standardized stationary Gaussian sequence. Let $r_n = \mathrm{E}\, X_1 X_{n+1}$ and $M_n = \max\{X_k, 1 \leqslant k \leqslant n\}$. The limiting distribution of M_n for weakly dependent stationary Gaussian sequences has been studied by Berman [2], i.e.,

$$\lim_{n\to\infty} \mathrm{P}\left\{a_n\left(M_n - b_n\right) \leqslant x\right\} = \exp\left\{-e^{-x}\right\} =: \Lambda(x)$$

as $r_n \log n \to 0$, where

$$a_n = (2\log n)^{1/2}, \quad b_n = a_n - \log(4\pi \log n)/2a_n. \tag{1.1}$$

For the limiting distribution of the partial maxima of strongly dependent stationary Gaussian sequences, see Lin [10] and Mittal and Ylvisaker [14] for the case of $r_n \log n \to \gamma \in (0, \infty)$ and McCormick and Mittal [13] for the case $r_n \log n \to \infty$ with some additional conditions.

The following more general condition for stationary Gaussian sequences was introduced by McCormick [12]:

$$\frac{\log n}{n} \sum_{k=1}^{n} |r_k - r_n| = o(1). \tag{1.2}$$

Under the condition (1.2), McCormick [12] proved the limiting distribution of the partial maximum centered at the sample mean, i.e.,

Theorem A. *Let $\{X_k, k \geqslant 1\}$ be a standardized stationary Gaussian sequence with correlations $\{r_n, n \geqslant 1\}$ such that $r_k < 1$ for some $k \geqslant 1$. Assume that (1.2) holds. Then*

$$\lim_{n\to\infty} \mathrm{P}\left\{a_n\left(\frac{M_n - \overline{X}_n}{(1-r_n)^{\frac{1}{2}}} - b_n\right) \leqslant x\right\} = \Lambda(x),$$

where a_n and b_n are defined by (1.1), and $\overline{X}_n = \sum_{k=1}^{n} X_k/n$.

In this note, we are interested in the joint limiting distribution and the almost sure limit theorem (ASLT) of maxima centered at sample mean for complete and incomplete samples from stationary Gaussian sequences. The joint limiting distribution of the maxima of complete and incomplete samples has been studied by Mladenović and Piterbarg [15]. See Theorems 3.1 and 3.2 in Mladenović and Piterbarg [15]. The ASLTs for the maximum of i.i.d. random variables have been studied by Fahrner and Stadtmüller [9], Cheng et al. [5] and Berkes and Csáki [1]. For weakly dependent stationary Gaussian sequences, Csáki and Gonchigdanzan [6] showed that

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k}\, \mathbb{I}(M_k \leqslant u_k) = e^{-\tau} \quad a.s. \tag{1.3}$$

for some sequence u_n satisfying $n(1-\Phi(u_n)) \to \tau$ and the covariance function r_n satisfying

$$r_n \log n (\log\log n)^{1+\varepsilon} = O(1)$$

for some $\varepsilon > 0$, where $\mathbb{I}$ denotes an indicator function. Chen and Lin [3, 4] extended (1.3) to the nonstationary Gaussian case. Dudziński [7, 8] and Peng et al. [16] studied the joint ASLT for the partial sum and maximum of stationary sequences. For applications of ASLTs, see Peng et al. [17].

Throughout this note, let $\{X_k, k \geqslant 1\}$ be a standardized stationary Gaussian sequence with the marginal distribution $\Phi(x) = (2\pi)^{-1/2} \int_{-\infty}^{x} \exp(-t^2/2)dt$ and correlation function $r_n = \mathrm{E}\, X_1 X_{n+1}$. Suppose that some of the random variables $X_1, X_2, \ldots$ can be observed and the others not. Let ε_i denote the indicator random variable that X_i is observed. Let $S_{k,n} = \varepsilon_{k+1} + \varepsilon_{k+2} + \cdots + \varepsilon_n$ denote the number of observed random variables from the set $\{X_{k+1}, X_{k+2}, \ldots, X_n\}$, and $S_n = S_{0,n}$. Define

$$\widetilde{M}_n = \begin{cases} \max\{X_j, 1 \leqslant j \leqslant n, \varepsilon_j = 1\} & \text{if } S_n \geqslant 1, \\ -\infty & \text{if } S_n = 0. \end{cases}$$

Assume that $\{\varepsilon_k, k \geqslant 1\}$ and $\{X_k, k \geqslant 1\}$ are mutually independent sequences. Further assume that

$$\frac{S_n}{n} \xrightarrow{\mathrm{P}} p \in (0, 1]. \tag{1.4}$$

2. The joint limiting distribution

In this section, we consider the joint limit distribution of the partial maxima (centered at sample means) for complete and incomplete samples from dependent Gaussian sequences.

Theorem 1. *For the standardized stationary Gaussian sequence $\{X_k, k \geqslant 1\}$, assume that $r_k < 1$ for some $k \geqslant 1$ and the conditions (1.2) and (1.4) hold. Then*

$$\lim_{n\to\infty} \mathrm{P}\left\{ \frac{\widetilde{M}_n - \overline{X}_n}{(1-r_n)^{\frac{1}{2}}} \leqslant u_n(x), \frac{M_n - \overline{X}_n}{(1-r_n)^{\frac{1}{2}}} \leqslant u_n(y) \right\} = \Lambda^p(x)\Lambda^{1-p}(y)$$

for all $-\infty < x \leqslant y < \infty$, where $u_n(z) = z/a_n + b_n$ for $z \in R$ and a_n and b_n as defined by (1.1).

Proof. Let $\sigma_n^2(k) = \mathrm{E}(X_k - \overline{X}_n)^2$, $Y_{k,n} = (X_k - \overline{X}_n)/\sigma_n(k)$ and $\rho_n(i,j) = \mathrm{E}\, Y_{i,n} Y_{j,n}$. Then

$$\max_{1\leqslant k\leqslant n} \left|\sigma_n^2(k) - (1-r_n)\right| = o(\frac{1}{\log n}). \tag{2.1}$$

by (2.8) of McCormick [12].

Define $M_n^* = \max_{1\leqslant i\leqslant n}\{Y_{i,n}\}$. Also define $\widetilde{M}_n^* = \max_{1\leqslant i\leqslant n}\{Y_{i,n}, \varepsilon_i = 1\}$ if $S_n \geqslant 1$ and $\widetilde{M}_n^* = -\infty$ if $S_n = 0$. We prove that

$$\lim_{n\to\infty} \mathrm{P}\left\{\widetilde{M}_n^* \leqslant u_n(x), M_n^* \leqslant u_n(y)\right\} = \Lambda^p(x)\Lambda^{1-p}(y) \tag{2.2}$$

holds for all $x \leqslant y$.

Now suppose that the subset $\{X_{i_1}, \ldots, X_{i_m}\}$ has been observed from the set $\{X_1, \ldots, X_n\}$, where $0 \leqslant m \leqslant n$. Let $N = \{1, 2, \ldots, n\}$, $I_m = \{i_1, i_2, \ldots, i_m\}$ and $M_n^*(B) = \max\{Y_{k,n}, k \in B\}$. Then by the Normal Comparison Lemma (cf. Leadbetter et al. [11]) and by the arguments in the proof of Theorem 2.1 in McCormick [12],

$$\begin{aligned}
&\left|\mathrm{P}\left\{M_n^*(I_m) \leqslant u_n(x), M_n^*(N/I_m) \leqslant u_n(y)\right\} - \Phi^m(u_n(x))\Phi^{n-m}(u_n(y))\right| \\
\leqslant\ & \sum_{1\leqslant i<j\leqslant n} \frac{1}{2\pi}\left(1-\rho_n^2(i,j)\right)^{-\frac{1}{2}} |\rho_n(i,j)| \exp\left\{-\frac{u_n^2(x)}{1+|\rho_n(i,j)|}\right\} \\
=\ & o(1)
\end{aligned}$$

hold uniformly for all $I_m = \{i_1, i_2, \ldots, i_m\}, m \leqslant n$. So, by using the total probability formula and Theorem 3.1 in Mladenović and Piterbarg [15], we obtain

$$\begin{aligned}
& \lim_{n\to\infty} \mathrm{P}\left\{\widetilde{M}_n^* \leqslant u_n(x), M_n^* \leqslant u_n(y)\right\} \\
=\ & \lim_{n\to\infty} \sum_{m=0}^{n} \mathrm{P}\{S_n = m\}\Phi^m(u_n(x))\Phi^{n-m}(u_n(y)) \\
=\ & \Lambda^p(x)\Lambda^{1-p}(y),
\end{aligned}$$

which is (2.2). Clearly,

$$a_n\left(\frac{X_k - \overline{X}_n}{(1-r_n)^{\frac{1}{2}}} - b_n\right) = a_n(Y_{k,n} - b_n)\frac{\sigma_n(k)}{(1-r_n)^{\frac{1}{2}}} + a_n b_n\left(\frac{\sigma_n(k)}{(1-r_n)^{\frac{1}{2}}} - 1\right),$$

and noting that by (2.1) both

$$\sigma_n(k)/(1-r_n)^{\frac{1}{2}} = 1 + o(1)$$

and

$$a_n b_n\left(\frac{\sigma_n(k)}{(1-r_n)^{\frac{1}{2}}} - 1\right) = o(1)$$

hold uniformly for all $1 \leqslant k \leqslant n$, we have

$$a_n\left(\frac{\widetilde{M}_n - \overline{X}_n}{(1-r_n)^{\frac{1}{2}}} - b_n\right) = a_n(\widetilde{M}_n^* - b_n)(1+o(1)) + o(1)$$

and

$$a_n\left(\frac{M_n - \overline{X}_n}{(1-r_n)^{\frac{1}{2}}} - b_n\right) = a_n(M_n^* - b_n)(1+o(1)) + o(1).$$

The result follows by (2.2). □

Corollary 1. *Under the conditions of Theorem 1, for $x \in R$, we have*

$$\lim_{n\to\infty} \mathrm{P}\left\{\frac{\widetilde{M}_n - \overline{X}_n}{(1-r_n)^{\frac{1}{2}}} \leqslant u_n(x)\right\} = \Lambda^p(x).$$

Proof. Clearly, both

$$\begin{aligned}\mathrm{P}\left\{\frac{\widetilde{M}_n - \overline{X}_n}{(1-r_n)^{\frac{1}{2}}} \leqslant u_n(x)\right\} \quad \leqslant \quad & \mathrm{P}\left\{\frac{\widetilde{M}_n - \overline{X}_n}{(1-r_n)^{\frac{1}{2}}} \leqslant u_n(x), \frac{M_n - \overline{X}_n}{(1-r_n)^{\frac{1}{2}}} \leqslant u_n(y)\right\} \\ & + \mathrm{P}\left\{\frac{M_n - \overline{X}_n}{(1-r_n)^{\frac{1}{2}}} > u_n(y)\right\}\end{aligned}$$

and

$$\mathrm{P}\left\{\frac{\widetilde{M}_n - \overline{X}_n}{(1-r_n)^{\frac{1}{2}}} \leqslant u_n(x)\right\} \geqslant \mathrm{P}\left\{\frac{\widetilde{M}_n - \overline{X}_n}{(1-r_n)^{\frac{1}{2}}} \leqslant u_n(x), \frac{M_n - \overline{X}_n}{(1-r_n)^{\frac{1}{2}}} \leqslant u_n(y)\right\}$$

hold. So, by Theorem A and Theorem 1,

$$\limsup_{n\to\infty} \mathrm{P}\left\{\frac{\widetilde{M}_n - \overline{X}_n}{(1-r_n)^{\frac{1}{2}}} \leqslant u_n(x)\right\} \leqslant \Lambda^p(x)\Lambda^{1-p}(y) + 1 - \Lambda(y)$$

and

$$\liminf_{n\to\infty} \mathrm{P}\left\{\frac{\widetilde{M}_n - \overline{X}_n}{(1-r_n)^{\frac{1}{2}}} \leqslant u_n(x)\right\} \geqslant \Lambda^p(x)\Lambda^{1-p}(y).$$

Letting $y \uparrow \infty$, we obtain the desired result. □

As a direct consequence of Theorem 1, we have

Corollary 2. *Under the conditions of Theorem 1, we have*

$$\lim_{n\to\infty} \mathrm{P}\left\{\frac{\widetilde{M}_n - \overline{X}_n}{s_n} \leqslant u_n(x), \frac{M_n - \overline{X}_n}{s_n} \leqslant u_n(y)\right\} = \Lambda^p(x)\Lambda^{1-p}(y)$$

for $-\infty < x \leqslant y < \infty$, where $s_n^2 = \sum_{k=1}^n (X_k - \overline{X}_n)^2/n$.

Proof. Since

$$a_n\left(\frac{\widetilde{M}_n - \overline{X}_n}{s_n} - b_n\right) = a_n\left(\frac{\widetilde{M}_n - \overline{X}_n}{(1-r_n)^{\frac{1}{2}}} - b_n\right)\lambda_n + a_n b_n\left(\lambda_n - 1\right)$$

and

$$a_n\left(\frac{M_n - \overline{X}_n}{s_n} - b_n\right) = a_n\left(\frac{M_n - \overline{X}_n}{(1-r_n)^{\frac{1}{2}}} - b_n\right)\lambda_n + a_n b_n\left(\lambda_n - 1\right),$$

where $\lambda_n = (1-r_n)^{\frac{1}{2}}/s_n$, by using Lemma 2.2 in McCormick [12], we have $\lambda_n \xrightarrow{\mathrm{P}} 1$ and $a_n b_n\left(\lambda_n - 1\right) \xrightarrow{\mathrm{P}} 0$. So, the result follows by Theorem 1. □

3. The almost sure limit theorem

In this section, we present and prove the ASLT for the maxima of standardized stationary Gaussian sequences $\{X_k, k \geqslant 1\}$. For simplicity, denote:

$$\widetilde{M}_{k,n}^* = \begin{cases} \max\{Y_{j,n}, k+1 \leqslant j \leqslant n, \varepsilon_j = 1\} & \text{if } S_{k,n} \geqslant 1, \\ -\infty & \text{if } S_{k,n} = 0, \end{cases} \qquad \widetilde{M}_n^* = \widetilde{M}_{0,n}^*,$$

$$M_{k,n}^* = \max_{k+1 \leqslant j \leqslant n}\{Y_{j,n}\}, \quad M_n^* = M_{0,n}^*,$$

where $Y_{k,n} = (X_k - \overline{X}_n)/\sigma_n(k)$ and $\sigma_n^2(k) = \mathrm{E}(X_k - \overline{X}_n)^2$ for $1 \leqslant k \leqslant n$. Set $\overline{r}_n = \max_{1 \leqslant k \leqslant n} |r_k - r_n|$ and assume that

$$\overline{r}_n (\log n)(\log\log n)^{1+\varepsilon} = O(1) \tag{3.1}$$

for some $\varepsilon > 0$.

The main results are:

Theorem 2. *Let $\{X_k, k \geqslant 1\}$ be a standardized stationary Gaussian sequence satisfying (3.1) and $r_k < 1$ for some $k \geqslant 1$. Suppose that the sequence $\{\varepsilon_k, k \geqslant 1\}$ is independent and independent of $\{X_k, k \geqslant 1\}$ and $S_n/n \xrightarrow{\mathrm{P}} p \in (0,1]$. Then*

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \mathbb{I}\left\{\widetilde{M}_k^* \leqslant u_k(x), M_k^* \leqslant u_k(y)\right\} = \Lambda^p(x)\Lambda^{1-p}(y) \quad a.s. \tag{3.2}$$

for all $x \leq y$.

The following result shows that $\sigma_n(k)$ in (3.2) can be replaced by $(1-r_k)^{1/2}$.

Theorem 3. *Under the conditions of Theorem 2, for all $x \leqslant y$, we have*

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \mathbb{I}\left\{\frac{\widetilde{M}_k - \overline{X}_k}{(1-r_k)^{\frac{1}{2}}} \leqslant u_k(x), \frac{M_k - \overline{X}_k}{(1-r_k)^{\frac{1}{2}}} \leqslant u_k(y)\right\} = \Lambda^p(x)\Lambda^{1-p}(y) \quad a.s.$$

The following two results prove the ASLTs of the maxima for complete and incomplete samples.

Corollary 3. *Under the conditions of Theorem 2, we have*

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \mathbb{I}\left\{\frac{M_k - \overline{X}_k}{(1-r_k)^{\frac{1}{2}}} \leqslant u_k(y)\right\} = \Lambda(y) \quad a.s.$$

Corollary 4. *Under the conditions of Theorem 2, we have*

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \mathbb{I}\left\{\frac{\widetilde{M}_k - \overline{X}_k}{(1-r_k)^{\frac{1}{2}}} \leqslant u_k(x)\right\} = \Lambda^p(x) \quad a.s.$$

To prove the main results, we need some auxiliary Lemmas. For convenience, let $\mathcal{C}$ denote an absolute positive constant taken to vary from line to line.

Lemma 1. *For the standardized stationary Gaussian sequence $\{X_k, k \geqslant 1\}$, assume that $r_k < 1$ for some $k \geqslant 1$ and that (3.1) holds. Then*

$$\overline{\rho}_n(\log n)(\log\log n)^{1+\varepsilon} = O(1),$$

where $\overline{\rho}_n = \max\{|\rho_n(i,j)|, 1 \leqslant i < j \leqslant n\}$ and $\rho_n(i,j) = \mathrm{E}\, Y_{i,n}Y_{j,n}$.

Proof. By arguments similar to those of (2.7) and (2.8) in McCormick [12] and (3.1), we have $\sup_{n\geqslant 1}|r_n| = \delta < 1$ for some $\delta > 0$ and

$$\max_{1\leqslant k\leqslant n}\left|\sigma_n^2(k) - (1-r_n)\right| \leqslant \frac{\mathcal{C}}{\log n(\log\log n)^{1+\varepsilon}}. \tag{3.3}$$

So,

$$\begin{aligned}
|\rho_n(i,j)| &= \frac{1}{\sigma_n(i)\sigma_n(j)}\left|\frac{1}{n^2}\sum_{k=1}^{n}\sum_{l=1}^{n} r_{k-l} - \frac{1}{n}\sum_{k=1}^{n}(r_{k-i}+r_{k-j}) + r_{i-j}\right| \\
&\leqslant \frac{1}{1-\delta}\Bigg(\frac{1}{n^2}\sum_{k=1}^{n}\sum_{l=1}^{n}|r_{k-l}-r_n| + \frac{1}{n}\sum_{k=1}^{n}|r_{k-i}-r_n| \\
&\qquad + \frac{1}{n}\sum_{k=1}^{n}|r_{k-j}-r_n| + |r_{i-j}-r_n|\Bigg) \\
&\leqslant \frac{\mathcal{C}}{\log n(\log\log n)^{1+\varepsilon}}
\end{aligned}$$

for sufficiently large n. The desired result follows. □

Lemma 2. *Under the conditions of Lemma 1, both*

$$\sum_{1\leqslant i<j\leqslant n}|\rho_n(i,j)|\exp\left(-\frac{u_n^2(x)}{1+|\rho_n(i,j)|}\right) \leqslant \frac{\mathcal{C}}{(\log\log n)^{1+\varepsilon}} \tag{3.4}$$

and

$$\sum_{i=1}^{k}\sum_{j=k+1}^{n}|\rho_n(i,j)|\exp\left(-\frac{u_k^2(x)+u_n^2(x)}{2(1+|\rho_n(i,j)|)}\right) \leqslant \frac{\mathcal{C}}{(\log\log n)^{1+\varepsilon}} \tag{3.5}$$

hold for $k < n$.

Proof. Note that

$$\exp\left(-\frac{u_n^2(x)}{2}\right) \sim \frac{\mathcal{C}u_n(x)}{n}, \quad u_n(x) \sim (2\log n)^{1/2}$$

for large n. So,

$$\begin{aligned}
& \sum_{1\leqslant i<j\leqslant n} |\rho_n(i,j)| \exp\left(-\frac{u_n^2(x)}{1+|\rho_n(i,j)|}\right) \\
\leqslant\ & n^2\overline{\rho}_n \exp\left(-\frac{u_n^2(x)}{1+\overline{\rho}_n}\right) \\
\leqslant\ & n^2\overline{\rho}_n \exp\left(-(1-\overline{\rho}_n)u_n^2(x)\right) \\
\leqslant\ & \mathcal{C}\overline{\rho}_n u_n^2(x) \exp\left(\overline{\rho}_n u_n^2(x)\right) \\
\leqslant\ & \frac{\mathcal{C}}{(\log\log n)^{1+\varepsilon}}
\end{aligned}$$

since $\overline{\rho}_n u_n^2(x) \sim 2\overline{\rho}_n \log n \leqslant \frac{\mathcal{C}}{(\log\log n)^{1+\varepsilon}}$ for large n by Lemma 1. Similarly,

$$\begin{aligned}
& \sum_{i=1}^{k}\sum_{j=k+1}^{n} |\rho_n(i,j)| \exp\left(-\frac{u_k^2(x)+u_n^2(x)}{2(1+|\rho_n(i,j)|)}\right) \\
\leqslant\ & kn\overline{\rho}_n \exp\left(-\frac{u_k^2(x)+u_n^2(x)}{2(1+\overline{\rho}_n)}\right) \\
\leqslant\ & kn\overline{\rho}_n \exp\left(-\frac{u_k^2(x)+u_n^2(x)}{2}\right) \exp\left(\frac{\overline{\rho}_n\left(u_k^2(x)+u_n^2(x)\right)}{2}\right) \\
\leqslant\ & \mathcal{C}kn\overline{\rho}_n \left(\frac{u_k(x)u_n(x)}{kn}\right) \exp\left(\overline{\rho}_n u_n^2(x)\right) \\
\leqslant\ & \mathcal{C}\overline{\rho}_n u_n^2(x) \exp\left(\overline{\rho}_n u_n^2(x)\right) \\
\leqslant\ & \frac{\mathcal{C}}{(\log\log n)^{1+\varepsilon}}.
\end{aligned}$$

The proof is complete. □

Lemma 3. *Under the conditions of Lemma 1, we have*

$$\begin{aligned}
& \left|Cov\left(\mathbb{I}\left\{\widetilde{M}_k^* \leqslant u_k(x), M_k^* \leqslant u_k(y)\right\}, \mathbb{I}\left\{\widetilde{M}_{k,n}^* \leqslant u_n(x), M_{k,n}^* \leqslant u_n(y)\right\}\right)\right| \\
\leqslant\ & \frac{\mathcal{C}}{(\log\log n)^{1+\varepsilon}}
\end{aligned}$$

for $k<n$, $x \leqslant y$.

Proof. Observe that

$$\begin{aligned}
& \mathrm{Cov}\left(\mathbb{I}\left\{\widetilde{M}_k^* \leqslant u_k(x), M_k^* \leqslant u_k(y)\right\}, \mathbb{I}\left\{\widetilde{M}_{k,n}^* \leqslant u_n(x), M_{k,n}^* \leqslant u_n(y)\right\}\right) \\
=\ & \mathrm{P}\left\{\widetilde{M}_k^* \leqslant u_k(x), M_k^* \leqslant u_k(y); \widetilde{M}_{k,n}^* \leqslant u_n(x), M_{k,n}^* \leqslant u_n(y)\right\} \\
& -\mathrm{P}\left\{\widetilde{M}_k^* \leqslant u_k(x), M_k^* \leqslant u_k(y)\right\}\mathrm{P}\left\{\widetilde{M}_{k,n}^* \leqslant u_n(x), M_{k,n}^* \leqslant u_n(y)\right\}.
\end{aligned}$$

Suppose that subsets $\{X_{i_1}, \dots, X_{i_s}\}$ and $\{X_{j_1}, \dots, X_{j_t}\}$ have been observed, where $\{X_{i_1}, \cdots, X_{i_s}\} \subset \{X_1, \cdots, X_k\}$ and $\{X_{j_1}, \dots, X_{j_t}\} \subset \{X_{k+1}, \dots, X_n\}$ for $s = 0, 1, 2, \dots, k$ and $t = 0, 1, 2, \dots, n-k$. Set $I = \{1, 2, \dots, k\}$, $J = \{k+1, k+2, \dots, n\}$, $I_s = \{i_1, i_2, \dots, i_s\}$ and $J_t = \{j_1, j_2, \dots, j_t\}$. By Lemma 1, there exists $0 < \eta < 1$ such that $|\rho_n(i,j)| \leqslant \eta$ for sufficiently large n. Now define

$$\begin{aligned} A_1 \quad = \quad & \mathrm{P}\left\{M^*(I_s) \leqslant u_k(x), M^*(I/I_s) \leqslant u_k(y);\right. \\ & \left. M^*(J_t) \leqslant u_n(x), M^*(J/J_t) \leqslant u_n(y)\right\} \end{aligned}$$

and

$$\begin{aligned} A_2 \quad = \quad & \mathrm{P}\left\{M^*(I_s) \leqslant u_k(x), M^*(I/I_s) \leqslant u_k(y)\right\} \\ & \times P\left\{M^*(J_t) \leqslant u_n(x), M^*(J/J_t) \leq u_n(y)\right\}, \end{aligned}$$

where $M^*(B) = \max\{Y_{k,n}, k \in B\}$. By using the Normal Comparison Lemma and (3.5),

$$\begin{aligned} & |A_1 - A_2| \\ \leqslant \quad & \mathcal{C}\left(\sum_{i\in I_s}\sum_{j\in J_t}|\rho_n(i,j)|\exp\left(-\frac{u_k^2(x)+u_n^2(x)}{2(1+|\rho_n(i,j)|)}\right)\right. \\ & +\sum_{i\in I_s}\sum_{j\in J/J_t}|\rho_n(i,j)|\exp\left(-\frac{u_k^2(x)+u_n^2(y)}{2(1+|\rho_n(i,j)|)}\right) \\ & +\sum_{i\in I/I_s}\sum_{j\in J_t}|\rho_n(i,j)|\exp\left(-\frac{u_k^2(y)+u_n^2(x)}{2(1+|\rho_n(i,j)|)}\right) \\ & \left.+\sum_{i\in I/I_s}\sum_{j\in J/J_t}|\rho_n(i,j)|\exp\left(-\frac{u_k^2(y)+u_n^2(y)}{2(1+|\rho_n(i,j)|)}\right)\right) \\ \leqslant \quad & \mathcal{C}\sum_{i\in I}\sum_{j\in J}|\rho_n(i,j)|\exp\left(-\frac{u_k^2(x)+u_n^2(x)}{2(1+|\rho_n(i,j)|)}\right) \\ \leqslant \quad & \frac{\mathcal{C}}{(\log\log n)^{1+\varepsilon}} \end{aligned}$$

uniformly for all $I_s = \{i_1, i_2, \dots, i_s\}$, $s \leqslant k$ and $J_t = \{j_1, j_2, \dots, j_t\}$, $t \leqslant n-k$. So, the result follows by using the total probability formula. □

Lemma 4. *Under the conditions of Theorem 2, we have*

$$\begin{aligned} & \mathrm{E}\left|\mathbb{I}\left\{\widetilde{M}_n^* \leq u_n(x), M_n^* \leq u_n(y)\right\} - \mathbb{I}\left\{\widetilde{M}_{k,n}^* \leq u_n(x), M_{k,n} \leq u_n(y)\right\}\right| \\ \leqslant \quad & \mathcal{C}\left(\frac{k}{n} + (\log\log n)^{-(1+\varepsilon)}\right) \end{aligned}$$

for $k < n$.

Proof. Let $\{Z_k,\ k \geqslant 1\}$ be the independent sequence associated with $\{Y_{k,n},\ k \geqslant 1\}$ and let $\{Z_k,\ k \geqslant 1\}$ be independent of $\{\varepsilon_k,\ k \geqslant 1\}$. Define $M'_{k,n} = \max_{k+1 \leqslant j \leqslant n}\{Z_j\}$. Also define $\widetilde{M}'_{k,n} = \max_{k+1 \leqslant j \leqslant n}\{Z_j, \varepsilon_j = 1\}$ if $S_{k,n} \geqslant 1$ and $\widetilde{M}'_{k,n} = -\infty$ if $S_{k,n} = 0$. Then

$$\begin{aligned} & \mathrm{E}\left|\mathbb{I}\left\{\widetilde{M}^*_n \leq u_n(x), M^*_n \leqslant u_n(y)\right\} - \mathbb{I}\left\{\widetilde{M}^*_{k,n} \leqslant u_n(x), M^*_{k,n} \leqslant u_n(y)\right\}\right| \\ \leq\ & \left|\mathrm{P}\left(\widetilde{M}^*_{k,n} \leqslant u_n(x), M^*_{k,n} \leqslant u_n(y)\right) - \mathrm{P}\left(\widetilde{M}'_{k,n} \leqslant u_n(x), M'_{k,n} \leqslant u_n(y)\right)\right| \\ & + \left|\mathrm{P}\left(\widetilde{M}^*_n \leqslant u_n(x), M^*_n \leqslant u_n(y)\right) - \mathrm{P}\left(\widetilde{M}'_n \leqslant u_n(x), M'_n \leqslant u_n(y)\right)\right| \\ & + \left|\mathrm{P}\left(\widetilde{M}'_{k,n} \leqslant u_n(x), M'_{k,n} \leqslant u_n(y)\right) - \mathrm{P}\left(\widetilde{M}'_n \leqslant u_n(x), M'_n \leqslant u_n(y)\right)\right| \\ =:\ & D_1 + D_2 + D_3. \end{aligned}$$

By the Normal Comparison Lemma and (3.4), $D_i \leqslant \mathcal{C}(\log\log n)^{-(1+\varepsilon)}$ for $i = 1, 2$. It remains to prove that $D_3 \leqslant \mathcal{C}\frac{k}{n}$. Clearly,

$$\begin{aligned} D_3 \ \leqslant\ & \left|\mathrm{P}\left(\widetilde{M}'_{k,n} \leqslant u_n(x)\right) - \mathrm{P}\left(\widetilde{M}'_n \leqslant u_n(x)\right)\right| \\ & + \left|\mathrm{P}\left(M'_{k,n} \leqslant u_n(y)\right) - \mathrm{P}\left(M'_n \leqslant u_n(y)\right)\right| \\ \leqslant\ & \mathrm{P}\left(\widetilde{M}'_k > \widetilde{M}'_{k,n}\right) + \mathrm{P}\left(M'_k > M'_{k,n}\right). \end{aligned}$$

By (5.6) in Berkes and Csáki [1], for $1 \leq k < n$, we have

$$\mathrm{P}\left(M'_k > M'_{k,n}\right) \leq \frac{k}{n}.$$

So, we need to prove $\mathrm{P}\big(\widetilde{M}'_k > \widetilde{M}'_{k,n}\big) \leq k/n$. Noting that $\{Z_k, k \geqslant 1\}$ is independent of $\{\varepsilon_k, k \geqslant 1\}$ and $\mathrm{P}\big(\widetilde{M}'_k > \widetilde{M}'_{k,n}, S_k = 0\big) = 0$, we obtain

$$\begin{aligned} \mathrm{P}\left(\widetilde{M}'_k > \widetilde{M}'_{k,n}\right) &= \sum_{t=1}^{k}\sum_{s=0}^{n-k} \mathrm{P}\left(\widetilde{M}'_k > \widetilde{M}'_{k,n}, S_k = t, S_{k,n} = s\right) \\ &= \sum_{t=1}^{k}\sum_{s=0}^{n-k} \mathrm{P}(S_k = t, S_{k,n} = s) \int_{-\infty}^{\infty} \Phi^s(x) d\Phi^t(x) \\ &= \sum_{t=1}^{k}\sum_{s=0}^{n-k} \mathrm{P}(S_k = t, S_{k,n} = s)\frac{t}{t+s} \\ &= \mathrm{E}\left(\frac{S_k}{S_n}\right). \end{aligned}$$

Here, we use the convention $0/0 =: 0$.

Now let $p_n = \mathrm{E}(S_n)/n$. We have $p_n \to p$ as $n \to \infty$ from the dominated convergence theorem. Note that $\mathrm{Var}(S_n) = \sum_{i=1}^{n} \mathrm{Var}(\varepsilon_i) \leqslant n/4$ by the independence of $\{\varepsilon_k, k \geqslant 1\}$ and that $x(1-x) \leqslant 1/4$ for $0 \leqslant x \leqslant 1$. So, for

$0 < \epsilon < p/2$ and sufficiently large n, we have

$$\begin{aligned}
\mathrm{E}\left(\frac{S_k}{S_n}\right) &= \mathrm{E}\left(\frac{S_k}{S_n}\left(\mathbb{I}\left\{\left|\frac{S_n}{n}-p_n\right| \geqslant \epsilon\right\}+\mathbb{I}\left\{\left|\frac{S_n}{n}-p_n\right| < \epsilon\right\}\right)\right) \\
&\leqslant \mathrm{P}\left(\left|\frac{S_n}{n}-p_n\right| \geqslant \epsilon\right)+\mathrm{E}\left(\frac{S_k}{n(p_n-\epsilon)}\right) \\
&\leqslant \frac{\mathrm{Var}(S_n)}{n^2\epsilon^2}+\frac{k}{n(p_n-\epsilon)} \\
&\leqslant \mathcal{C}\frac{k}{n}.
\end{aligned}$$

The desired result follows. □

Proof of Theorem 2. Let

$$\xi_n = \mathbb{I}\left\{\widetilde{M}_n^* \leqslant u_n(x), M_n^* \leqslant u_n(y)\right\} - \mathrm{P}\left(\widetilde{M}_n^* \leqslant u_n(x), M_n^* \leqslant u_n(y)\right).$$

Then

$$\begin{aligned}
&\mathrm{Var}\left(\sum_{k=1}^{n}\frac{1}{k}\mathbb{I}\left\{\widetilde{M}_k^* \leqslant u_k(x), M_k^* \leqslant u_k(y)\right\}\right) \\
=\;& \mathrm{E}\left(\sum_{k=1}^{n}\frac{1}{k}\xi_k\right)^2 \\
=\;& \sum_{k=1}^{n}\frac{1}{k^2}\mathrm{E}\,|\xi_k|^2 + 2\sum_{1\leqslant k<l\leqslant n}\frac{|\,\mathrm{E}(\xi_k\xi_l)|}{kl} \\
\leqslant\;& \sum_{k=1}^{n}\frac{1}{k^2} + 2\sum_{1\leqslant k<l\leqslant n}\frac{|\,\mathrm{E}(\xi_k\xi_l)|}{kl}.
\end{aligned}$$

It is clear that $\sum_{k=1}^{n} 1/k^2 < \infty$. Now consider $\mathrm{E}(\xi_k\xi_l)$. By Lemma 3 and Lemma 4, for $1 \leq k < n$, we have

$$\begin{aligned}
& |\mathrm{E}(\xi_k\xi_l)| \\
=\;& \left|\mathrm{Cov}\left(\mathbb{I}\{\widetilde{M}_k^* \leqslant u_k(x), M_k \leqslant u_k(y)\}, \mathbb{I}\{\widetilde{M}_l^* \leqslant u_l(x), M_l^* \leqslant u_l(y)\}\right)\right| \\
\leqslant\;& \mathrm{E}\left|\mathbb{I}\{\widetilde{M}_k^* \leqslant u_k(x), M_k^* \leqslant u_k(y)\} - \mathbb{I}\{\widetilde{M}_{k,l}^* \leqslant u_l(x), M_{k,l}^* \leqslant u_l(y)\}\right| \\
&+ \left|\mathrm{Cov}\left(\mathbb{I}\{\widetilde{M}_k^* \leqslant u_k(x), M_k^* \leqslant u_k(y)\}, \mathbb{I}\{\widetilde{M}_{k,l}^* \leqslant u_l(x), M_{k,l}^* \leqslant u_l(x)\}\right)\right| \\
\leqslant\;& \mathcal{C}\left(\frac{k}{l} + (\log\log l)^{-(1+\varepsilon)}\right)
\end{aligned}$$

for sufficient large l. By arguments similar to those in the proof of Theorem 1.1 in Csáki and Gonchigdanzan [6], we obtain

$$\mathrm{Var}\left(\sum_{k=1}^{n}\frac{1}{k}\mathbb{I}\left\{\widetilde{M}_k^* \leqslant u_k(x), M_k^* \leqslant u_k(y)\right\}\right) = O\left((\log n)^2(\log\log n)^{-(1+\varepsilon)}\right).$$

So, by Lemma 3.1 in Csáki and Gonchigdanzan [6], we have

$$\lim_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\xi_k=0.$$

So, the desired result follows by (2.2). □

Proof of Theorem 3. For every fixed $\theta>0$, there exists an integer $k_0=k_0(\theta)$ such that

$$\frac{(z+a_kb_k)}{\log k(\log\log k)^{1+\varepsilon}}<\theta$$

for $k>k_0$ and fixed $z\in R$. According to (3.3),

$$\left|\frac{(1-r_k)^{\frac{1}{2}}}{\sigma_k(i)}-1\right|\leqslant\frac{\mathcal{C}}{\log k(\log\log k)^{1+\varepsilon}}$$

holds uniformly for all $i\leqslant k$. So, for $k>k_0$

$$\{Y_{i,k}\leqslant u_k(z-\theta)\}\subset\left\{\frac{X_i-\overline{X}_k}{(1-r_k)^{\frac{1}{2}}}\leqslant u_k(z)\right\}\subset\{Y_{i,k}\leqslant u_k(z+\theta)\}.$$

Combining with Theorem 2, we obtain

$$\begin{aligned}&\limsup_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\mathbb{I}\left\{\frac{\widetilde{M}_k-\overline{X}_k}{(1-r_k)^{\frac{1}{2}}}\leqslant u_k(x),\frac{M_k-\overline{X}_k}{(1-r_k)^{\frac{1}{2}}}\leqslant u_k(y)\right\}\\ \leqslant\ &\Lambda^p(x+\theta)\Lambda^{1-p}(y+\varepsilon)\end{aligned}$$

and

$$\begin{aligned}&\liminf_{n\to\infty}\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\mathbb{I}\left\{\frac{\widetilde{M}_k-\overline{X}_k}{(1-r_k)^{\frac{1}{2}}}\leqslant u_k(x),\frac{M_k-\overline{X}_k}{(1-r_k)^{\frac{1}{2}}}\leqslant u_k(y)\right\}\\ \geqslant\ &\Lambda^p(x-\theta)\Lambda^{1-p}(y-\varepsilon).\end{aligned}$$

The result follows by letting $\theta\downarrow 0$. □

Proof of Corollary 3. It is clear that

$$\mathbb{I}\left\{\frac{M_k-\overline{X}_k}{(1-r_k)^{\frac{1}{2}}}\leqslant u_k(y)\right\}=\mathbb{I}\left\{\frac{\widetilde{M}_k-\overline{X}_k}{(1-r_k)^{\frac{1}{2}}}\leqslant u_k(y),\frac{M_k-\overline{X}_k}{(1-r_k)^{\frac{1}{2}}}\leqslant u_k(y)\right\}.$$

Furthermore, for arbitrary $\theta>0$

$$\begin{aligned}&\mathbb{I}\left\{\frac{\widetilde{M}_k-\overline{X}_k}{(1-r_k)^{\frac{1}{2}}}\leqslant u_k(y),\frac{M_k-\overline{X}_k}{(1-r_k)^{\frac{1}{2}}}\leqslant u_k(y)\right\}\\ \leqslant\ &\mathbb{I}\left\{\frac{\widetilde{M}_k-\overline{X}_k}{(1-r_k)^{\frac{1}{2}}}\leqslant u_k(y),\frac{M_k-\overline{X}_k}{(1-r_k)^{\frac{1}{2}}}\leqslant u_k(y+\theta)\right\}\end{aligned}$$

and

$$
\begin{aligned}
& \mathbb{I}\left\{\frac{\widetilde{M}_k-\overline{X}_k}{(1-r_k)^{\frac{1}{2}}} \leqslant u_k(y-\theta), \frac{M_k-\overline{X}_k}{(1-r_k)^{\frac{1}{2}}} \leqslant u_k(y)\right\} \\
\leqslant \quad & \mathbb{I}\left\{\frac{\widetilde{M}_k-\overline{X}_k}{(1-r_k)^{\frac{1}{2}}} \leqslant u_k(y), \frac{M_k-\overline{X}_k}{(1-r_k)^{\frac{1}{2}}} \leqslant u_k(y)\right\}.
\end{aligned}
$$

So, the desired result follows by Theorem 3 and continuity of $\Lambda(z)$. □

Proof of Corollary 4. The result follows by using Theorem 3 and Corollary 3. □

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