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Unifying Method for Computing the Circumcircles of Three Circles

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Abstract – Given a set of three generator circles in a plane, we want to find a circumcircle of these generators. This problem is a part of well-known Apollonius' 10th Problem and is frequently encountered in various geometric computations such as the Voronoi diagram for circles. It turns out that this seemingly trivial problem is not at all easy to solve in a general setting. In addition, there can be several degenerate configurations of the generators. For example, there may not exist any circumcircle, or there could be one or two circumcircle(s) depending on the generator configuration. Sometimes, a circumcircle itself may degenerate to a line. We show that the problem can be reduced to a point location problem among the regions bounded by two lines and two transformed circles via Möbius transformations in a complex space. The presented algorithm is simple and the required computation is negligible. In addition, several degenerate cases are all incorporated into a unified framework.

***Keywords*: Apollonius' 10th Problem, circumcircle, inversion, Möbius transformation**

1. Introduction

Suppose that a set of three circles is given in a plane, where the radii of the circles are not necessarily equal and where the circles are possibly intersecting one another. A circle, however, is not allowed to entirely internal to another one. Given this circle set, we want to compute the circumcircle of these circles. A circumcircle is a circle tangent to the given circles and places them outside of itself. This problem is frequently encountered in various geometric computations such as the Voronoi diagram for circles. In particular, the computation of the Voronoi diagram of circles requires this problem solved [9, 11, 15, 17, 20].

The problem was first considered by Apollonius around 200 B.C., and has been known as Apollonius' 10th problem [2, 4, 5, 8, 13]. Ever since, there have been several efforts to solve the problem using various approaches [1, 3, 16, 19]. Recently, Rokne reported a noble approach based on Möbius transformation, which is in fact an inversion, in the complex plane [18]. Using the fact that Möbius transformation in a complex plane maps circles to lines and vice versa, he suggested to compute a tangent line of two circles in a mapped space in order to back-transform into a circumcircle. Most recently, Gavrilova reported an analytic solution which involves trigonometric functions [6].

Solutions to the problem can be approached in various ways. One straightforward approach could be computing the center of a circumcircle as an intersection between two bisectors, which are hyperbolic arcs, of pairs of circles. It turns out that this involves the solution process of a quartic equation that can be solved by either the Ferrari formula or a numerical process [10]. Since the formula involves square root operations, it is relatively expensive and inevitably contains truncation errors. In addition, this approach can be applied only after the number of circumcircles of the generators is determined. Otherwise, we may either not find all circumcircles or waste computing time. On the other hand, the solution may be symbolically generated via tools like Mathematica. It turns out, however, that the cost of such symbolic generation for the equation of the center of the circumcircle can be quite high. For example, we generated the equation using Mathematica, where the problem is formulated as an intersection between two hyperbolic arcs, and it took approximately 73,000 lines of C code and 2.99 MB of data in ASCII format.

Even though the problem is quite complicated in Euclidean space, it turns out that it can be rather easily solved in a complex system. Using Möbius transformation, based on Rokne's suggestion, we reduce the given problem into the problem of finding appropriate tangent lines of two circles in a mapped space.

We have found out that there are six possible configurations of circumcircles to a set of three circles. Proposed in this paper is an algorithm to identify what the case of three given circles is out of the six cases. By locating a region, based on a point location problem, among six mutually exclusive regions bounded by lines and/or circles, a possible configuration of circumcircle out of six possibilities is identified. Since the whole solution space is composed of those six mutually exclusive regions, the proposed algorithm always produces correct

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circumcircle. In addition, it turns out that all of the degenerate configurations of generators can be handled in a unified way within the framework of the proposed algorithm. The proposed algorithm is also easy to program, numerically robust, and computationally very efficient.

In Section 2, we provide more descriptive explanations of the problem. In Section 3, the properties of Möbius transformation in a complex plane are provided so that the problem can be transformed to an easier one. Based on the transformation, we present in Section 4 the core part of the algorithm that parses the cases of configuration of generators using a point location problem in the transformed space. Section 5 contains the conclusions of the research.

2. Tangent circles

Apollonius considered a number of problems to construct circles simultaneously tangent to a set of three objects, each of which can be either a point, line, or circle in a plane. Among ten possible combinations, Apollonius' 10^{th} Problem is to construct the circles simultaneously tangent to three circles.

Given three circles called generators, there are at most eight circles simultaneously tangent to the generators, as shown in Fig. 1. In the figure, the black circles are the generators and the white ones are the tangent circles. Fig. 1(a),(b),(c), and (d) illustrate the cases of one circumcircle, three circles where one generator is internal to each circle while two generators are external, three circles where two generators are internal while one generator is external, and the case where all three generators are internal to a circle, respectively.

Among these tangent circles, we want to find the first one which is a circumcircle of three generator circles. Depending on the configuration of the three generators, however, there may be either no, one, or two circumcircles, as shown in Fig. 2. We want to determine which case a given generator set is and find such circumcircles with as little computation as possible if they exist.

3. Möbius transformation

Let a plane be complex. Then, a point (x, y) in the Euclidean plane is treated as a complex number $z=x+iy$. Also, let $\mathbf{c}_i=(z_i, r_i)$, $i=1$, 2, and 3, be the generator circles with a center (x_i, y_i) and a radius $r_1 \geq r_2 \geq r_3 \geq 0$ as shown in Fig. 3. Then, $\tilde{\mathbf{c}}_i=(z_i, r_i - r_3)$ transforms generator circles $\mathbf{c}_1$, $\mathbf{c}_2$ and $\mathbf{c}_3$ to shrunken circles $\tilde{\mathbf{c}}_1$, $\tilde{\mathbf{c}}_2$ and $\tilde{\mathbf{c}}_3$ respectively. Note that $\tilde{\mathbf{c}}_3$ degenerates to a point z_3. Then, if we can find a circle $\tilde{\mathbf{c}}$ passing through $z_3 \equiv \tilde{\mathbf{c}}_3$ and tangent to both $\tilde{\mathbf{c}}_1$ and $\tilde{\mathbf{c}}_2$, we can easily find a circle $\mathbf{c}$ which is simultaneously tangent to $\mathbf{c}_1$, $\mathbf{c}_2$ and $\mathbf{c}_3$ by simply subtracting r_3 from the radius of $\tilde{\mathbf{c}}$.

Let $W=W(z)=u(x, y)+iv(x, y)$ be an analytic function. Then, $W(z)$ is a conformal map except at critical points where the derivative $W'(z)$ is zero [14]. A conformal

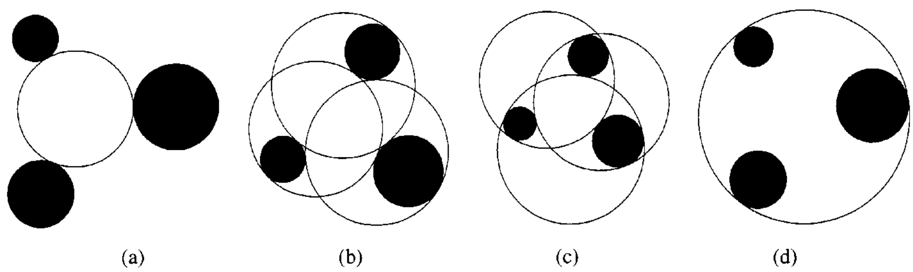


Fig. 1. Apollonius' 10^{th} Problem : the circles tangent to three circles.

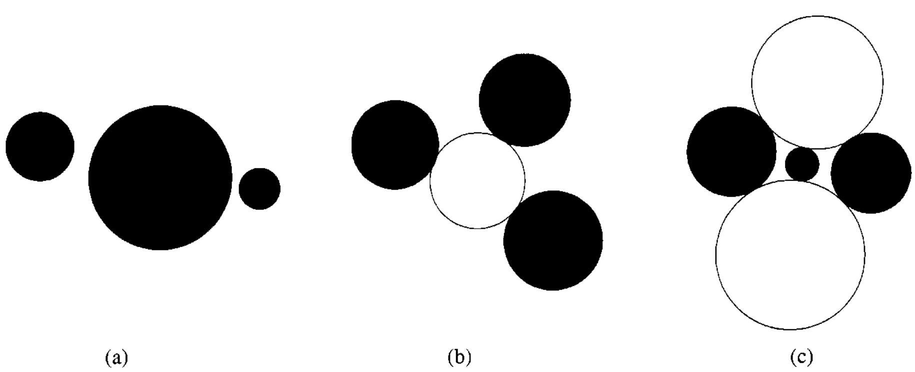


Fig. 2. Cases of circumcircles. (a) no circumcircle, (b) one circumcircle, and (c) two circumcircles.

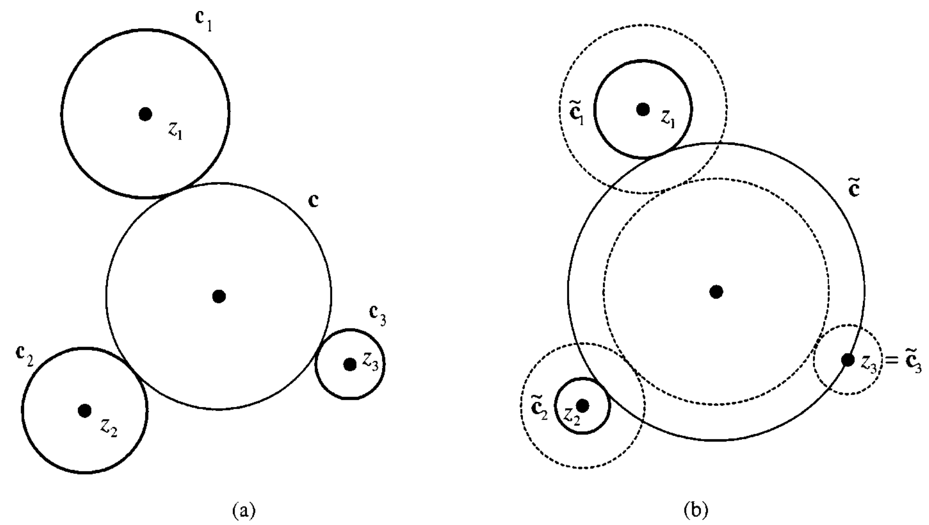


Fig. 3. Circumcircle and the inflated circumcircle. (a) generators and the desired circumcircle, (b) shrunken generators and a circumcircle passing through z_3.

mapping is known to preserve angles between any oriented curves both in magnitudes and in orientations. Among various conformal mappings, consider Möbius transformation defined as $W(z)=(az+b)/(cz+d)$, where $ad-bc\neq 0$, and a, b, c and d are either complex or real numbers. Note that $W(z)$ is analytic so that the mapping $W(z)$ is everywhere conformal and maps circles and straight lines in the Z-plane onto circles and straight lines in the W-plane. Among others, we note a particular linear mapping $W(z)=1/(z-z_0)$ as was suggested by [7,18]. The following lemma is provided without a proof. For the details of the properties of $W(z)$, the readers are recommended to refer to material on the subject such as [14].

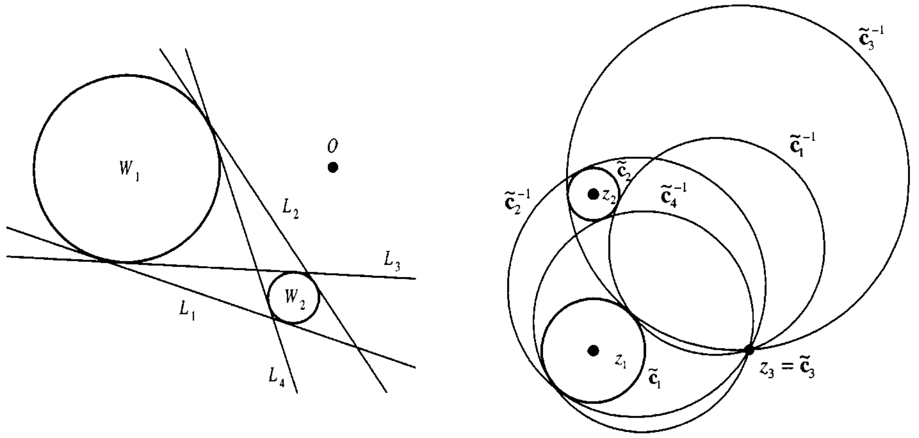
Lemma 1. Möbius transformation $W(z)=1/(z-z_0)$ has the following properties.

- It transforms lines and circles passing through z_0 in the Z-plane to straight lines in the W-plane.
- It transforms lines and circles not passing through z_0 in the Z-plane to circles in the W-plane.
- It transforms a point at infinity in the Z-plane to the origin of the W-plane.

Therefore, the mapping $W(z)=1/(z-z_3)$ transforms $\tilde{\mathbf{c}}_1$ and $\tilde{\mathbf{c}}_2$ in the Z-plane to circles W_1 and W_2 in the W-plane, if z_3 is not on $\tilde{\mathbf{c}}_1$ and $\tilde{\mathbf{c}}_2$. Then, the desired circle $\tilde{\mathbf{c}}$, tangent to circles $\tilde{\mathbf{c}}_1$ and $\tilde{\mathbf{c}}_2$ in the Z-plane, will be mapped to a line L, tangent to W_1 and W_2 in the W-plane by $W(z)$. It can be shown that $W(z)$ maps circles $\tilde{\mathbf{c}}_i=(z_i, r_i-r_3)$ into circles $W_i=(\omega_i, R_i)$ defined as $\omega_i=((x_i-x_3)/D_i,-(y_i-y_3)/D_i)$ and $R_i=(r_i-r_3)/D_i$ where $D_i=(x_i-x_3)^2+(y_i-y_3)^2-(r_i-r_3)^2$, $i=1$ and 2. Similarly, it can be also shown that the inverse transformation $W^{-1}(z)=Z(w)=1/w+z_3$ is also another conformal mapping, and hence, maps lines not passing through the origin of the W-plane to circles in the Z-plane. Suppose that a line is given as $au+bv+1=0$ in the W-plane. Then, its inverse in the Z-plane is a circle $\tilde{\mathbf{c}}=(z_0,r_0)$, where $z_0=(-a/2+x_3, b/2+y_3)$ and $r_0=\sqrt{a^2+b^2}/2$. Now we can get the circumcircle $\mathbf{c}=(z_0, r_0-r_3)$. Note that if the circumcircle has negative radius, then circle $\mathbf{c}$ does not exist in the original plane. In this case, the three circles are intersect each other.

4. Point location problem in the *W*-plane

Let W_1 and W_2 be two circles with radii R_1 and R_2 in the W-plane, respectively. Suppose that $R_1>R_2>0$, as shown in Fig. 4(a). Then, there could be at most four distinct lines simultaneously tangent to both W_1 and W_2. Suppose that the black dot in Fig. 4(a) is the origin O of the coordinate system in the W-plane. Then, the line L_1 maps to the circumcircle $\tilde{\mathbf{c}}_1^{-1}$ in the Z-plane, as shown in Fig. 4(b), by the inverse-mapping $Z(w)$ because the circles W_1 and W_2 as well as the origin O are located on the same side with respect to L_1. Note that the origin O of the W-plane corresponds to infinity in the Z-plane, and $Z(w)$ is conformal. Similarly, L_2 maps to the inscribing circle $\tilde{\mathbf{c}}_2^{-1}$ since the circles W_1 and W_2 are on the opposite side of O which corresponds to infinity in the Z-plane. The cases of L_3 and L_4 correspond to $\tilde{\mathbf{c}}_3^{-1}$ and $\tilde{\mathbf{c}}_4^{-1}$, respectively. Therefore, the line L which corresponds to a circumcircle in the Z-plane is either one or both of the *exterior tangent lines*, L_1 and/or L_2. Between L_1 and L_2, the one containing W_1, W_2 and the origin O on the same side of the line will map to the desired circumcircle(s). Remember that zero, one or both exterior tangent lines may be the correct result depending on the configuration of the initially given generator circles. From now on, we will drop the word *exterior* from the term for the convenience of presentation,



(a) (b)

Fig. 4. Inverse-mapping $W^{-1}(z)=Z(w)=\{1/w\}+z_3$ which maps from the W-plane to the Z-plane. (a) the W-plane, (b) the Z-plane.

unless otherwise needed.

Lemma 2. If $z_3 \notin \tilde{\mathbf{c}}_1$ and $z_3 \notin \tilde{\mathbf{c}}_2$, then $O \notin (W_1 \cup W_2)$.

Proof. The origin O of the W-plane corresponds to infinity in the Z-plane. Since the generator circles in the Z-plane are assumed to be finite, the shrunken circles are also finite. Therefore, O cannot lie on W_1 or W_2.

$O \notin (W_1 \cup W_2)$ means that the origin O of the W-plane cannot lie on or interior to the circles W_1 and W_2.

4.1. $R_1>R_2>0$: General case

Suppose W_1 and W_2 are given as shown in Fig. 5(a). Let L_1 and L_2 be the tangent lines to both circles. In this general case, $R_1>R_2>0$ and therefore an intersection δ between L_1 and L_2 exists. Let us define "+" operator as follows: L_i^+ is the half-space, defined by L_i, containing W_1 as well as W_2. Similarly, L_i^- means the opposite side of L_i^+. Note that the boundary of the half-space, which is the line L_i itself, is excluded from both half-spaces.

Definition 1. W-plane consists of six mutually exclusive regions as follows.

$$\alpha=(L_1^+ \cap L_2^-) \cup (L_1^- \cap L_2^+)$$

$$\beta=L_1^- \cap L_2^-$$

$$\gamma=(L_1^+ \cap L_2^+)-(W_1 \cup W_2)$$

$$\delta=L_1 \cap L_2$$

$$\varepsilon=(L_1 \cap L_2^-) \cup (L_1^- \cap L_2)$$

$$\zeta=(L_1 \cap L_2^+) \cup (L_1^+ \cap L_2)$$

As shown in the figure, the region α consists of two subregions and the region γ consists of three subregions (or four, if W_1 and W_2 intersect each other. This case occurs when $\tilde{\mathbf{c}}_1$ and $\tilde{\mathbf{c}}_2$ intersect each other. However, this case does not make the theory any different.). Once the W-plane is decomposed into such regions, the problem of computing a circumcircle(s) now further reduces to a point location problem among the regions. Since this particular point location problem is obvious, the details are not elaborated here. Transforming the problem into a point location problem yields the following theorem. Note that, in Fig. 5, the shaded circles are shrunken circles, and black dots are the shrunken circles with zero radii and thus degenerate to a point in the Z-plane. In addition, a circumcircle is shown in a solid curve while an inscribing circle is shown in a broken curve.

Theorem 3. If $R_1>R_2>0$, there are six cases as follows.

- Case α: If $O \in \alpha$, one tangent line maps to a circumcircle while the other tangent line maps to an inscribing circle. (Fig. 5(b)-α)
- Case β: If $O \in \beta$, both tangent lines map to inscribing circles. (Fig. 5(b)-β)
- Case γ: If $O \in \gamma$, both tangent lines map to circumcircles. (Fig. 5(b)-γ)
- Case δ: If $O \equiv \delta$, both tangent lines map to lines intersecting at a point. (Fig. 5(b)-δ)
- Case ε: If $O \in \varepsilon$, a tangent line on which O lies maps to a line, while the other tangent line maps to an inscribing circle. (Fig. 5(b)-ε)
- Case ζ: If $O \in \zeta$, the tangent line on which O lies maps to a line, while the other tangent line maps to a circumcircle. (Fig. 5(b)-ζ)

Proof.

- Case α: Suppose that $\alpha_1=(L_1^- \cap L_2^+)$ and $\alpha_2=(L_1^+ \cap L_2^-)$. Without loss of generality we can assume that $O \in \alpha_1$. Then, L_1 in the W-plane is inverse-mapped to a circle $\tilde{\mathbf{c}}_1^{-1}$ inscribing $\tilde{\mathbf{c}}_1$ and $\tilde{\mathbf{c}}_2$ in the Z-plane, as illustrated by a dotted curve in Fig. 5(b)-α. This is because L_1 places O on the

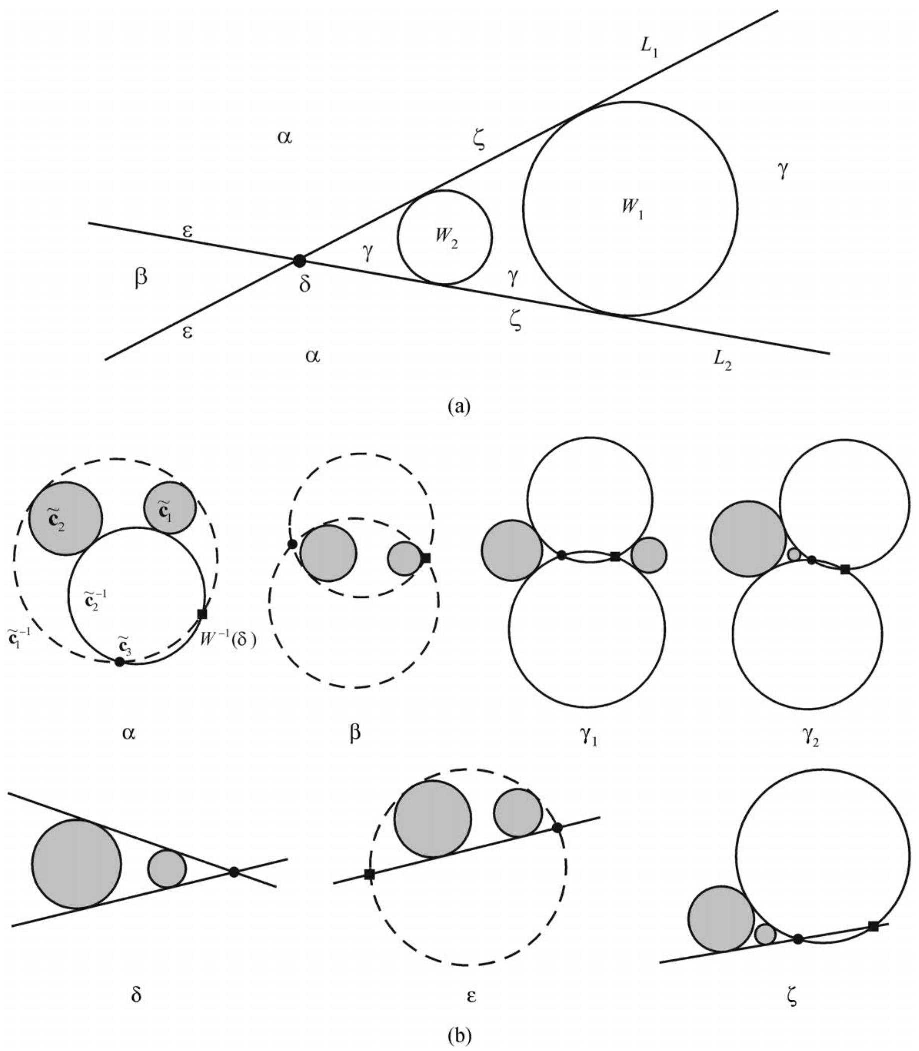


Fig. 5. R_1=R_2>0. (a) the W-plane, (b) the Z-plane.

opposite side of W_1 and W_2. Note that $\tilde{\mathbf{c}}_1$ and $\tilde{\mathbf{c}}_2$ are the inverse-maps of W_1 and W_2. On the other hand, L_2 is inverse-mapped to a circumcircle $\tilde{\mathbf{c}}_2^{-1}$ tangent to $\tilde{\mathbf{c}}_1$ and $\tilde{\mathbf{c}}_2$ in the Z-plane, and is illustrated as a solid curve. This is because L_2 places W_1, W_2 and O on the same side. Since two tangent lines in the W-plane intersect each other at δ, the inverse-mapped circles, regardless whether they are circumcircles or inscribing circles, always intersect each other at $W^{-1}(\delta)$ computed by Eq. (4) shown as a black rectangle in the Z-plane.

- Case β: When $O \in \beta$, both W_1 and W_2 are on the opposite side of O with respect to both tangent lines L_1 and L_2. Therefore, both L_1 and L_2 should be mapped to inscribing circles, and hence, no circumcircle will result as shown in Fig. 5(b)-β.
- Case γ: When $O \in \gamma$, both W_1 and W_2 are on the same side of O with respect to both tangent lines L_1 and L_2. Hence, both L_1 and L_2 should be mapped to circumcircles only. In this case, two different situations may occur. Note that the region γ consists of three subreigons. The case in Fig. 5(b)-γ_1 occurs when O lies in-between two circles W_1 and W_2, and the case of γ_2 in Fig. 5(b)-γ_2 occurs when O lies in the other subregions of γ.
- Case δ: When $O \equiv \delta$, the inverse-mapping to the Z-plane yields results similar to what is shown in the W-plane. Since the tangent lines in the W-plane pass through the origin O, the (supposedly) inverse-mapped circles should pass through infinity. This means that the radii of the inverse-mapped circles are infinite. Therefore, the mapping results in lines in the Z-plane as shown in Fig. 5(b)-δ. Note that they only intersect at $\tilde{\mathbf{c}}_3$.
- Case ε: When $O \in \varepsilon$, O lies precisely on a ray ε starting from δ. In this case, the corresponding tangent line on which O lies is inverse-mapped to a line in the Z-plane, as was explained above. Then, O should be located on the opposite side of the other tangent line with respect to W_1 and W_2, meaning that there is an inscribing circle as shown in Fig. 5(b)-ε.

- Case ζ: When $O \in \zeta$, O lies precisely on a ray ζ, which is also a ray starting from δ. In this case, the corresponding tangent line inverse-maps to another line in the Z-plane similarly to the above cases. In this case, however, O as well as W_1 and W_2 should be located on the same side of the other tangent line. This means that the tangent line inverse-maps to a circumcircle in the Z-plane as shown in Fig. 5(b)-ζ.

Corollary 3.1. In all cases except Case δ, two circles tangent to two shrunken circles $\tilde{\mathbf{c}}_1$ and $\tilde{\mathbf{c}}_2$ intersect each other at two points $W^{-1}(\delta)$ as well as $\tilde{\mathbf{c}}_3$. In Case δ, the intersection occurs only at $\tilde{\mathbf{c}}_3$.

Proof. Proved in Theorem 3.

Note that some tangent circles to the shrunken circles degenerate into lines in cases δ, ε and ζ. In this case, the desired tangent circles to the generators can be obtained by translating the degenerate lines to the opposite direction of the shrunken circles.

Slightly changing the configuration of generator circles, various types of degeneracy may occur. It turns out that the degeneracy is mainly due to the radii of W_1 and W_2.

4.2. $R_1 > R_2 = 0$

$R_2=0$ means that W_2 degenerates to a point. This case occurs when two smaller generator circles $\mathbf{c}_2$ and $\mathbf{c}_3$ in the Z-plane have identical radii. Some observations follow.

Lemma 4. If two smaller generators in the Z-plane have identical radii (i.e., $r_2=r_3$), $R_2=0$.

Proof. If $r_2=r_3$, $\tilde{\mathbf{c}}_2$ and $\tilde{\mathbf{c}}_3$ shrink to points in the Z-

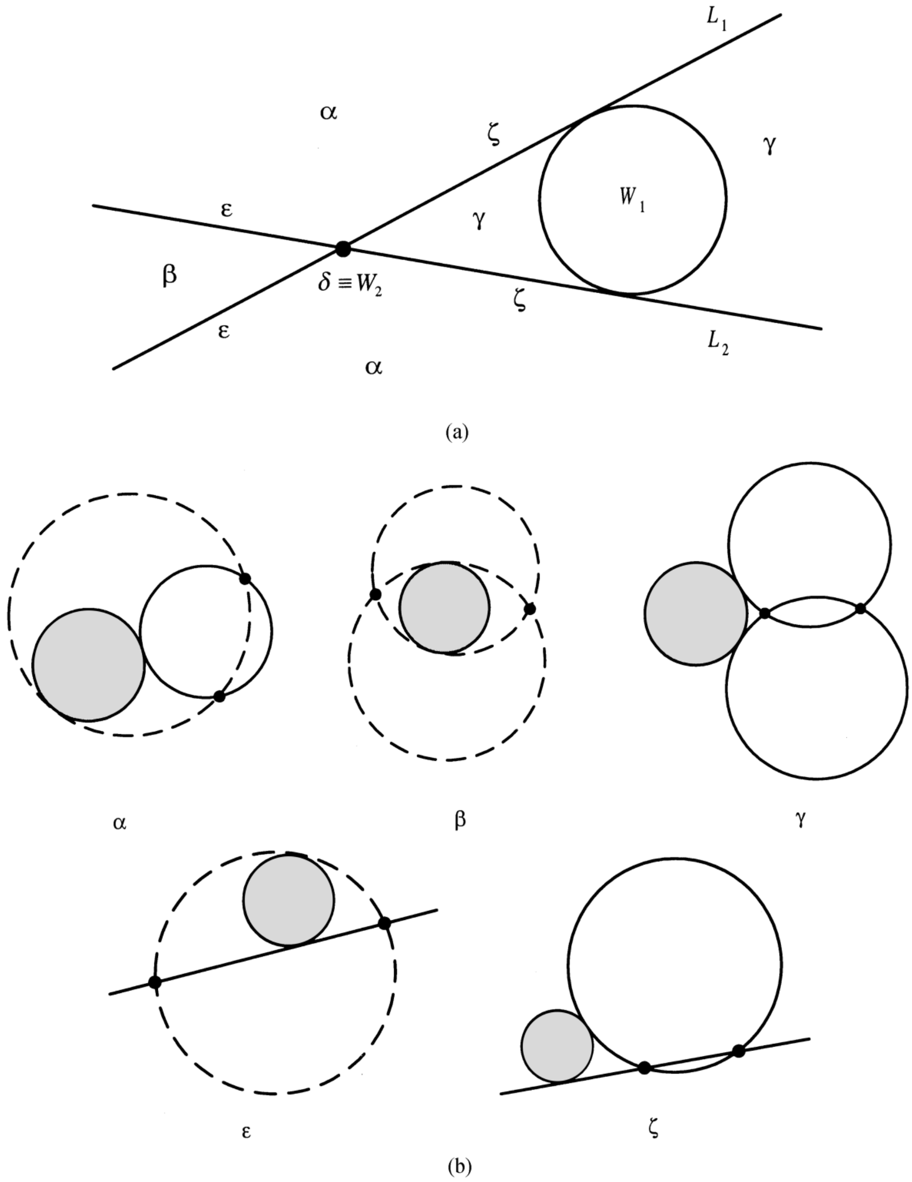


Fig. 6. $R_1 > R_2 = 0$ (a) the W-plane, (b) the Z-plane.

plane simultaneously. Applying the mapping $W(z)$ to $\tilde{\mathbf{c}}_2$ now yields also a point W_2 in the *W*-plane.

Lemma 5. If $r_2=r_3$, $W_2 \equiv \delta$.

Proof. Since the lines tangent to W_1 should be also tangent to W_2, both tangent lines should pass through the point simultaneously. Therefore, W_2 should be the intersection point δ between two tangent lines as shown in Fig. 6(a).

Corollary 5.1. If $r_2=r_3$, the region γ consists of two subregions.

Proof. Obvious.

Corollary 5.2. Case δ does not occur.

Proof. $O \notin (L_1 \cup L_2)$ by Lemma 1. Therefore, O cannot lie on $W_2 \equiv \delta$ which is the intersection between L_1 and L_2.

Theorem 6. If $r_2=r_3$, then there are five cases as follows.

- Case α: If $O \in \alpha$, one tangent line maps to a circumcircle while the other tangent line maps to an inscribing circle. (Fig. 6(b)-α)
- Case β: If $O \in \beta$, both tangent lines map to inscribing circles. (Fig. 6(b)-β)
- Case γ: If $O \in \gamma$, both tangent lines map to circumcircles. (Fig. 6(b)-γ)
- Case ε: If $O \in \varepsilon$, a tangent line on which O lies maps to a line, while the other tangent line maps to an inscribing circle. (Fig. 6(b)-ε)
- Case ζ: If $O \in \zeta$, the tangent line on which O lies maps to a line while the other tangent line maps to a circumcircle. (Fig. 6(b)-ζ)

Proof. (identical to the proof of Theorem 1.)

Corollary 6.1. The inscribing circle and the circumcircle of the shrunken generators in the Z-plane intersect at both $\tilde{\mathbf{c}}_2$ as well as $\tilde{\mathbf{c}}_3$

Proof. Obvious.

Note that the line segment connecting two dots intersect or are tangent to $\tilde{\mathbf{c}}_1$ for Cases β and ε, while the line segment connecting two dots does not intersect $\tilde{\mathbf{c}}_1$ for Cases γ and ζ.

4.3. $R_1=R_2>0$

This is the case in which W_1 and W_2 have identical

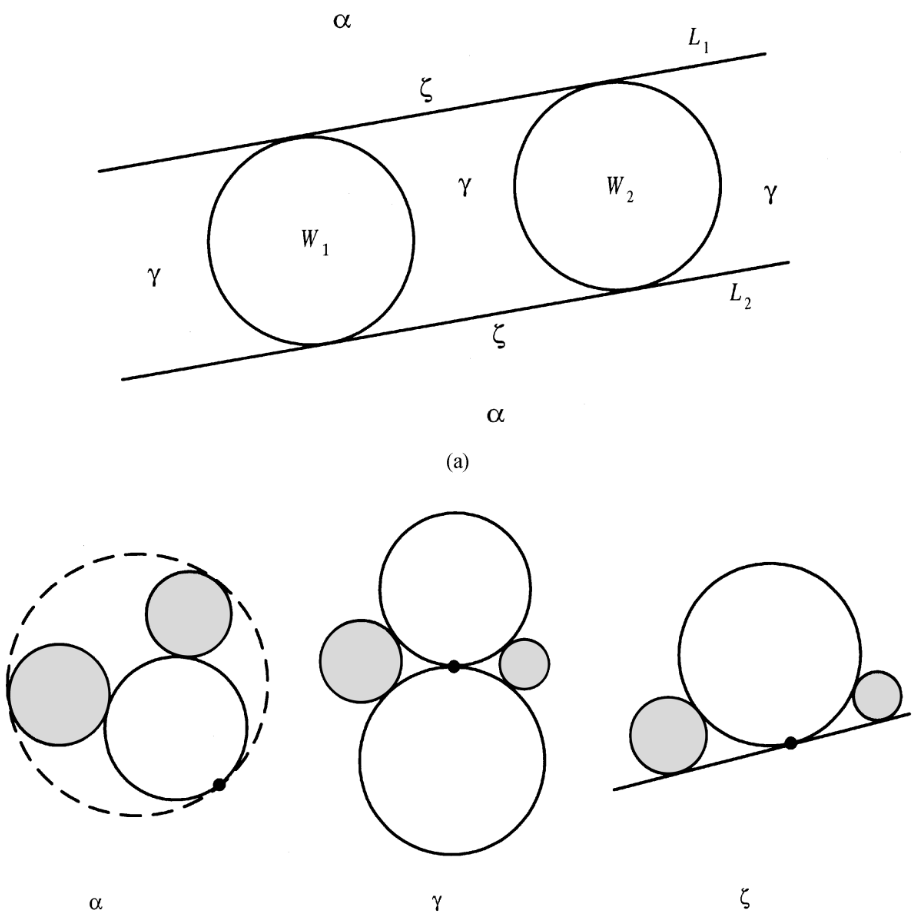


Fig. 7. $R_1=R_2>0$ (a) the *W*-plane, (b) the *Z*-plane.

non-zero radii, as shown in Fig. 7(a). Note that $R_1=R_2$, in general, does not guarantee $r_1=r_2$. In other words, even though two generator circles in the Z-plane have identical radii, the radii of mapped circles in the W-plane are not necessarily identical, and vice versa. Note that two exterior tangent lines in the W-plane are parallel in this case. The following observations hold.

Lemma 7. If $R_1=R_2>0$, the regions β, δ, and ε disappear.
Proof. Obvious.

Therefore, the cases left are Cases α, γ, and ζ and are illustrated in Fig. 7(b)-α, γ, and ζ. Since detailed descriptions for the cases are similar to the above-mentioned cases, they are not further elaborated here, but stated as a theorem without a proof.

Theorem 8. If $R_1=R_2>0$, there are three cases as follows.

- Case α: If $O\in\alpha$, one tangent line maps to a circumcircle while the other tangent line maps to an inscribing circle. (Fig. 7(b)-α)
- Case γ: If $O\in\gamma$, both tangent lines map to circumcircles. (Fig. 7(b)-γ)
- Case ζ: If $O\in\zeta$, the tangent line on which O lies maps to a line while the other tangent line maps to a circumcircle. (Fig. 7(b)-ζ)

Corollary 8.1. The inverse-mapped tangent circles in the Z-plane intersect only once at z_3.
Proof. Since δ vanishes in this configuration, there is only one intersection between the inverse-mapped circles, which is $\tilde{\mathbf{c}}_3$.

In addition, both tangent circles are always tangent to each other at z_3.

4.4. $R_1=R_2=0$

This is the case in which both W_1 and W_2 have zero radii, and therefore $L_1\equiv L_2$. This case occurs only when all generator circles in the Z-plane have identical radii.

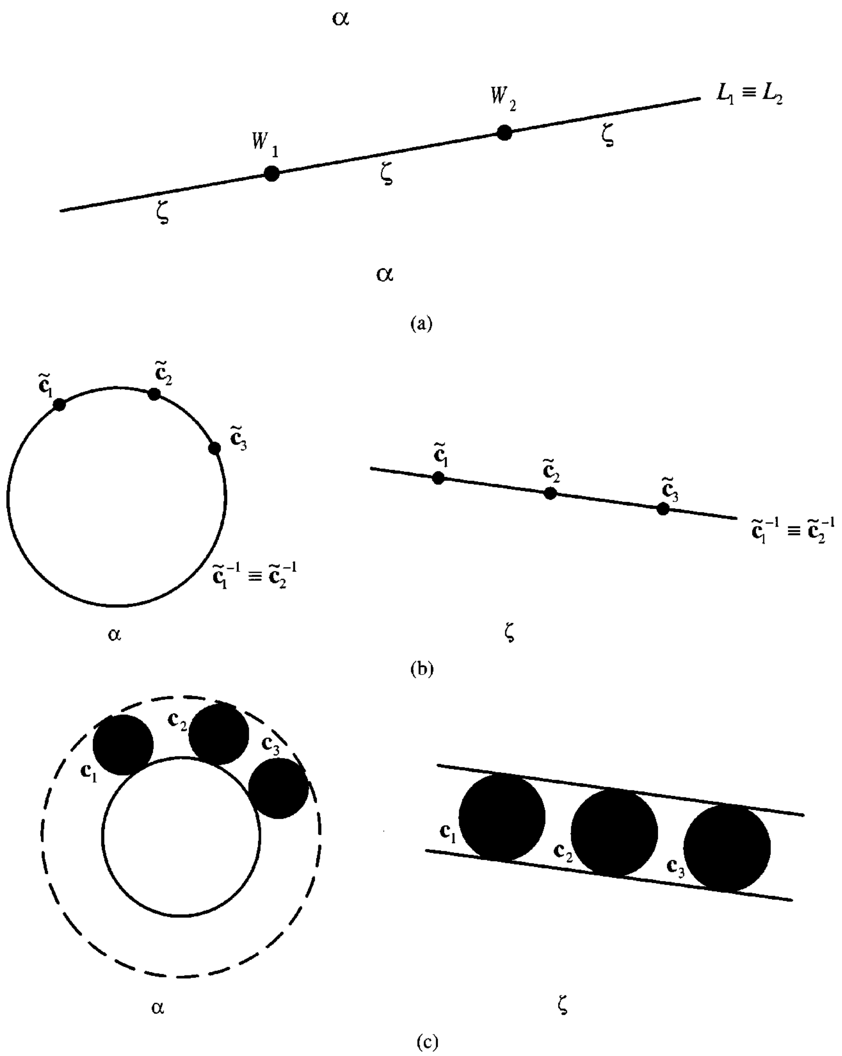


Fig. 8. $R_1=R_2=0$, (a) the W-plane, (b) the Z-plane: shrunken circles, (c) the Z-plane: generator circles.

Lemma 9. If $r_1=r_2=r_3$, $R_1=R_2=0$.

Proof. Since the radii of all generators are identical, all shrunken circles $\tilde{\mathbf{c}}_1$, $\tilde{\mathbf{c}}_2$ and $\tilde{\mathbf{c}}_3$ simultaneously degenerate to points z_1, z_2 and z_3, respectively, in the Z-plane. Since $W(z)$ is conformal, $\tilde{\mathbf{c}}_1$ and $\tilde{\mathbf{c}}_2$ as well as $\tilde{\mathbf{c}}_3$ are all mapped to points in the W-plane.

Corollary 9.1. If $r_1=r_2=r_3$, $L_1 \equiv L_2$.

Proof. Since both W_1 and W_2 degenerate to points, both L_1 and L_2 should reduce to an identical line.

Lemma 10. If $r_1=r_2=r_3$, the regions α and ζ only remain.

Proof. Since $L_1 \equiv L_2$, the regions β, γ, δ and ε become simultaneously null.

The interpretations of the remaining regions stay the same as before, and therefore the following theorem holds. Note that the black circles are generators, not shrunken circles, in Fig. 8.

Theorem 11. If $r_1=r_2=r_3$, there are two cases as follows.

- Case α: If $O \in \alpha$, both tangent lines map to a circle passing through three shrunken circles degenerated to points. (Fig. 8(b)-α)
- Case ζ: If $O \in \zeta$, both tangent lines map to a line passing through three shrunken circles degenerated to points. (Fig. 8(b)-ζ)

Proof. Obvious

Corollary 11.1. If $r_1=r_2=r_3$, the circumcircle and inscribing circle do not intersect.

Proof. Obvious.

In Case α, there are one inscribing circle and one circumcircle of the generator circles. The inscribing circle can be obtained by increasing the radius of the circle $\tilde{\mathbf{c}}_1^{-1}$ by the radius r_3 of generators. Similarly, the circumcircle can be obtained by decreasing the radius of the circle $\tilde{\mathbf{c}}_2^{-1}$ by the same radius, as shown in Fig. 8(c)-α. Note that these two circles are concentric. In Case ζ, the centers of three generator circles, whose radii are identical, are collinear. Hence, there are two lines, not circles, tangent to the generators, and these tangent lines can be obtained by translating the line $\tilde{\mathbf{c}}_1^{-1}$ to both directions by the radius r_3 of generators. Hence, the final tangent lines are parallel, as shown in Fig. 8(c)-ζ.

5. Conclusions

Presented in this paper is an algorithm to compute the circumcircle(s) of a set of three generator circles in a plane. This problem is a part of the well-known Apollonius' 10th Problem and is frequently encountered in various geometric computations such as the Voronoi diagram for circles. It turns out that this seemingly trivial problem is not at all easy to solve in a general setting. In addition, there can be several degenerate configurations of the generators.

Even though the problem is quite complicated in Euclidean space, it turns out that it can be rather easily solved by employing a complex system. Following Rokne's approach, we have adopted Möbius transformation to reduce the given problem into the problem of finding tangent lines of two circles in a mapped space. Then, we formulate a point location problem so that all of the degenerate configurations of generators can be handled in a unified way.

It turns out that the proposed approach incorporates all variations of degeneracies in a single framework, is easy to program, numerically robust, and computationally very efficient.

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