第 四 讲

分离变量法(三)

北京大学物理学院

2007年春

讲授要点

1. 非齐次稳定问题
   - 示例
   - 方法的进一步发展
2. 非齐次边界条件的齐次化
   - 基本思路
   - 特殊技巧：方程及边界条件同时齐次化
3. 正交曲面坐标系下的Laplace算符
   - 柱坐标系下的Laplace算符
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- 吴崇试，《数学物理方法》，§14.6, 15.1, 15.2
- 梁昆淼，《数学物理方法》，§8.3
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矩形区域内的稳定问题

设有定解问题

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f(x,y) \qquad 0<x<a,\ 0<y<b$$

$$u\Big|_{x=0}=0 \qquad u\Big|_{x=a}=0 \qquad 0\le y\le b$$

$$u\Big|_{y=0}=0 \qquad u\Big|_{y=b}=0 \qquad 0\le x\le a$$

用按相应齐次问题本征函数展开的办法求解

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可设

$$u(x,y) = \sum_{n=1}^{\infty} Y_n(y)\sin\frac{n\pi}{a}x$$

$$f(x,y) = \sum_{n=1}^{\infty} g_n(y)\sin\frac{n\pi}{a}x$$

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代入方程和边界条件，可得

$$Y_n''(y)-\left(\frac{n\pi}{a}\right)^2 Y_n(y)=g_n(y)$$

$$Y_n(0)=0 \qquad Y_n(b)=0$$

由此即可求出$Y_n(y)$

矩形区域内的稳定问题

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$$u\Big|_{y=0}=0 \qquad u\Big|_{y=b}=0 \qquad 0\le x\le a$$

也可设

$$u(x,y) = \sum_{m=1}^{\infty} X_m(x)\sin\frac{m\pi}{b}y$$

$$f(x,y) = \sum_{m=1}^{\infty} h_m(x)\sin\frac{m\pi}{b}y$$

矩形区域内的稳定问题

设有定解问题

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$$u\Big|_{x=0}=0 \qquad u\Big|_{x=a}=0 \qquad 0\le y\le b$$
$$u\Big|_{y=0}=0 \qquad u\Big|_{y=b}=0 \qquad 0\le x\le a$$

代入方程和边界条件，可得

$$X_m''(x)-\left(\frac{m\pi}{b}\right)^2 X_m(x)=h_m(x)$$
$$X_m(0)=0 \qquad X_m(a)=0$$

由此亦可求出$X_m(y)$

评述

这两种做法没有原则差别．主要的不同是非齐次项$g_n(y)$和$h_m(x)$的函数形式可能不同， 因而在关于$Y_n(y)$和$X_m(x)$的非齐次两个常微分方程

$$Y_n''(y) - \left(\frac{n\pi}{a}\right)^2 Y_n(y) = g_n(y)$$
$$X_m''(x) - \left(\frac{m\pi}{b}\right)^2 X_m(x) = h_m(x)$$

中可能有一个更易于求解

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二重级数展开

设有定解问题

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f(x,y) \qquad 0<x<a,\ 0<y<b$$

$$u\Big|_{x=0}=0 \qquad u\Big|_{x=a}=0 \qquad 0\le y\le b$$

$$u\Big|_{y=0}=0 \qquad u\Big|_{y=b}=0 \qquad 0\le x\le a$$

还可以考虑更进一步的做法，即将$u(x,y)$和$f(x,y)$既按本征函数$\{X_n(x)\}$、又按本征函数$\{Y_m(y)\}$展开(为二重级数)

$$u(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_{nm}\sin\frac{n\pi}{a}x\sin\frac{m\pi}{b}y$$

$$f(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} d_{nm}\sin\frac{n\pi}{a}x\sin\frac{m\pi}{b}y$$

展开系数c_{nm}待求

二重级数展开

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)$$

$$u(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_{nm} \sin\frac{n\pi}{a}x \sin\frac{m\pi}{b}y$$

$$f(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} d_{nm} \sin\frac{n\pi}{a}x \sin\frac{m\pi}{b}y$$

★ 因为$f(x,y)$已知，故c_{nm}已知

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★ 因为$f(x,y)$已知，故c_{nm}已知

★ 在作二重级数展开时，已经考虑了边界条件

二重级数展开

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因此只需将上面的展开式代入方程

$$-\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_{nm} \left[\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2\right] \sin\frac{n\pi}{a}x \sin\frac{m\pi}{b}y$$

$$= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} d_{nm} \sin\frac{n\pi}{a}x \sin\frac{m\pi}{b}y$$

二重级数展开

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$$f(x,y)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}d_{nm}\sin\frac{n\pi}{a}x\sin\frac{m\pi}{b}y$$

根据本征函数的正交性，比较系数，即得

$$-c_{nm}\left[\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2\right]=d_{nm}$$

二重级数展开

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★ 优点：无需解非齐次常微分方程

二重级数展开

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因此

$$c_{nm}=-\frac{d_{nm}}{\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2}$$

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$$u(x,y)=-\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{d_{nm}}{\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2}\sin\frac{n\pi}{a}x\sin\frac{m\pi}{b}y$$

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★ 这种方法，实际上扩充了“相应齐次问题本征函数”的概念

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★ 缺点：结果是二重级数．事实上，尚可求和

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引言

到目前为止，除了在稳定问题中需要有一部分边界条件用于定叠加系数、因而允许是非齐次的以外， 在应用分离变量法解偏微分方程定解问题时，我们总是要求边界条件是齐次的

引言

为什么边界条件必须是齐次的?

- 非齐次边界条件不能分离变量
- 只有满足齐次方程和齐次边界条件的特解叠加起来才仍能满足齐次方程和齐次边界条件
- 但最根本的原因涉及到本征函数的完备性

非齐次边界条件如何处理?

仍以波动方程的定解问题为例

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非齐次边界条件如何处理?

仍以波动方程的定解问题为例

定解问题

为了突出非齐次边界条件，假定方程和初始条件均为齐次

定解问题

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = 0 \qquad 0 < x < l,\ t > 0$$

$$u\Big|_{x=0} = \mu(t) \quad u\Big|_{x=l} = \nu(t) \qquad t \geq 0$$

$$u\Big|_{t=0} = 0 \qquad \frac{\partial u}{\partial t}\Big|_{t=0} = 0 \qquad 0 \leq x \leq l$$

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$$u\Big|_{t=0} = 0 \qquad \frac{\partial u}{\partial t}\Big|_{t=0} = 0 \qquad 0 \leq x \leq l$$

基本思想

令$u(x,t) = v(x,t) + w(x,t)$，使得

$$\boxed{\begin{aligned} u\Big|_{x=0} &= \mu(t) \\ u\Big|_{x=l} &= \nu(t) \end{aligned}} \Longrightarrow \begin{array}{c} \boxed{\begin{aligned} v\Big|_{x=0} &= \mu(t) \\ v\Big|_{x=l} &= \nu(t) \end{aligned}} \\ + \\ \boxed{\begin{aligned} w\Big|_{x=0} &= 0 \\ w\Big|_{x=l} &= 0 \end{aligned}} \end{array}$$

唯一的要求就是使得$w(x,t)$满足齐次边界条件

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唯一的要求就是使得$w(x,t)$满足齐次边界条件

标准步骤

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$$u\Big|_{x=0} = \mu(t) \qquad u\Big|_{x=l} = \nu(t) \qquad t \geq 0$$

$$u\Big|_{t=0} = 0 \qquad \frac{\partial u}{\partial t}\Big|_{t=0} = 0 \qquad 0 \leq x \leq l$$

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- 边界条件齐次化，即寻找$v(x,t)$使满足非齐次边界条件

$$v\Big|_{x=0} = \mu(t), \qquad v\Big|_{x=l} = \nu(t)$$

标准步骤

$$\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = 0 \qquad 0 < x < l,\ t > 0$$

$$u\Big|_{x=0} = \mu(t) \qquad u\Big|_{x=l} = \nu(t) \qquad t \geq 0$$

$$u\Big|_{t=0} = 0 \qquad \frac{\partial u}{\partial t}\Big|_{t=0} = 0 \qquad 0 \leq x \leq l$$

- 令$u(x,t)=v(x,t)+w(x,t)$，列出$w(x,t)$满足的定解问题

$$\frac{\partial^2 w}{\partial t^2} - a^2\frac{\partial^2 w}{\partial x^2} = -\left(\frac{\partial^2 v}{\partial t^2} - a^2\frac{\partial^2 v}{\partial x^2}\right) \qquad 0 < x < l,\ t > 0$$

$$w(x,t)\Big|_{x=0} = 0 \quad w(x,t)\Big|_{x=l} = 0 \qquad t > 0$$

$$w\Big|_{t=0} = -v\Big|_{t=0} \quad \frac{\partial w}{\partial t}\Big|_{t=0} = -\frac{\partial v}{\partial t}\Big|_{t=0} \qquad 0 < x < l$$

标准步骤

$$\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = 0 \qquad 0 < x < l,\ t > 0$$

$$u\Big|_{x=0} = \mu(t) \qquad u\Big|_{x=l} = \nu(t) \qquad t \geq 0$$

$$u\Big|_{t=0} = 0 \qquad \frac{\partial u}{\partial t}\Big|_{t=0} = 0 \qquad 0 \leq x \leq l$$

- 选择正确方法，解出$w(x,t)$

如何选取齐次化函数$v(x,t)$?

- 因为仅要求$v(x,t)$满足边界条件

$$v(x,t)\big|_{x=0}=\mu(t)\quad v(x,t)\big|_{x=l}=\nu(t)$$

所以有相当大的选择余地

- 不妨把t看成是参数，这就只要求在(x,y)平面上的曲线$y=v(x,t)$通过给定的两点$(0,\mu(t))$和$(l,\nu(t))$即可
- 例如，可取直线$v(x,t)=A(t)x+B(t)$
- 也可取抛物线$v(x,t)=A(t)x^2+B(t)$或$v(x,t)=A(t)(l-x)^2+B(t)x^2$

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举例

例4.1　求解定解问题

$$\frac{\partial u}{\partial t} - \kappa\frac{\partial^2 u}{\partial x^2} = 0 \qquad 0 < x < l,\ t > 0$$

$$u\Big|_{x=0} = A\sin\omega t \qquad u\Big|_{x=l} = 0 \qquad t \geq 0$$

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Answer

举例

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Answer

依上法，取齐次化函数为x的线性函数，则

$$v(x,t) = A\left(1 - \frac{x}{l}\right)\sin \omega t$$

举例

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Answer

令$u(x,t)=v(x,t)+w(x,t)$，列出$w(x,t)$满足的定解问题

$$\frac{\partial w}{\partial t}-\kappa\frac{\partial^2 w}{\partial x^2}=-A\omega\left(1-\frac{x}{l}\right)\cos\omega t \qquad 0<x<l,\ t>0$$
$$w\big|_{x=0}=0 \qquad w\big|_{x=l}=0 \qquad t\geq 0$$
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Answer

求解$w(x,t)$ （以下从略）

讨论(一)

- 选择不同的齐次化函数$v(x,t)$，导出的$w(x,t)$的定解问题当然也就不同，求出的$w(x,t)$也就不同
- 定解问题的解的存在唯一性，保证了最后给出的$u(x,t)$一定是相同的，尽管表达式的形式可能有所不同
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- 最理想的情况是：不论原来$u(x,t)$的方程是不是齐次的，最终$w(x,t)$的方程是齐次的
- 就上面的定解问题而言，这意味着要求齐次化函数$v(x,t)$也是方程的解

$$\frac{\partial^2 v}{\partial t^2} - a^2 \frac{\partial^2 v}{\partial x^2} = 0$$

- 对于某些特殊的$\mu(t)$和$\nu(t)$，可以做到这一点
- 这种方法称为

将方程和边界条件同时齐次化

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讲授要点

举例

例4.2 求解定解问题

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Answer

举例

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Answer

找齐次化函数$v(x,t)$，将方程和边界条件同时齐次化

举例

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Answer

考虑到非齐次边界条件的具体形式，取齐次化函数为

$$v(x,t) = f(x) \sin \omega t$$

举例

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Answer

$$\begin{array}{l} \dfrac{\partial^2 v}{\partial t^2} - a^2\dfrac{\partial^2 v}{\partial x^2} = 0 \\ v\Big|_{x=0} = 0 \quad \dfrac{\partial v}{\partial x}\Big|_{x=l} = A\sin\omega t \end{array} \Rightarrow \begin{array}{l} f''(x)+\left(\dfrac{\omega}{a}\right)^2 f(x)=0 \\ f(0) = 0 \quad f'(l) = A \end{array}$$

举例

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Answer

由此即可求出$f(x)$

$$f(x) = \frac{Aa}{\omega}\frac{1}{\cos(\omega l/a)}\sin\frac{\omega}{a}x$$

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Answer

由此即可求出$f(x)$及$v(x,t)$

$$f(x) = \frac{Aa}{\omega}\frac{1}{\cos(\omega l/a)}\sin\frac{\omega}{a}x$$

$$v(x,t) = \frac{Aa}{\omega}\frac{1}{\cos(\omega l/a)}\sin\frac{\omega}{a}x\ \sin\omega t$$

举例

求$w(x,t)$

令$u(x,t)=v(x,t)+w(x,t)$，则$w(x,t)$满足的定解问题为

$$\frac{\partial^2 u}{\partial t^2}-a^2\frac{\partial^2 u}{\partial x^2}=0\Longrightarrow\frac{\partial^2 w}{\partial t^2}-a^2\frac{\partial^2 w}{\partial x^2}=0$$

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举例

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因为$w(x,t)$满足齐次方程和齐次边界条件

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因为$w(x,t)$满足齐次方程和齐次边界条件，所以一般解为

$$w(x,t)=\sum_{n=0}^{\infty}\left(C_n\sin\frac{2n+1}{2l}\pi at+D_n\cos\frac{2n+1}{2l}\pi at\right)\sin\frac{2n+1}{2l}\pi x$$

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根据初始条件，可以定出

$$C_n = -\frac{4A}{\pi\cos\frac{\omega l}{a}}\frac{1}{2n+1}\int_0^l \sin\frac{\omega}{a}x\sin\frac{2n+1}{2l}\pi x\mathrm{d}x$$

$$= (-)^n\frac{4A\omega a}{(2n+1)\pi a}\frac{1}{\omega^2-[(2n+1)\pi a/2l]^2}$$

$$D_n = 0$$

正交曲面坐标系下的Laplace算符

讲授要点

1. 非齐次稳定问题
   - 示例
   - 方法的进一步发展
2. 非齐次边界条件的齐次化
   - 基本思路
   - 特殊技巧：方程及边界条件同时齐次化
3. 正交曲面坐标系下的Laplace算符
   - 柱坐标系下的Laplace算符
   - 球坐标系下的Laplace算符

平面极坐标系下的二维Laplace算符

- 在二维直角坐标系下，Laplace算符为

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

- 平面极坐标(r, ϕ)和直角坐标(x, y)的关系是

$$x = r\cos\phi, \qquad y = r\sin\phi$$

- 因此

$$\mathrm{d}r = \cos\phi\,\mathrm{d}x + \sin\phi\,\mathrm{d}y \qquad \mathrm{d}\phi = -\frac{\sin\phi}{r}\mathrm{d}x + \frac{\cos\phi}{r}\mathrm{d}y$$

即

$$\frac{\partial r}{\partial x} = \cos\phi, \qquad \frac{\partial \phi}{\partial x} = -\frac{\sin\phi}{r}$$

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平面极坐标系下的二维Laplace算符

- 按照复合函数的求导法则

$$\frac{\partial}{\partial x}=\frac{\partial r}{\partial x}\frac{\partial}{\partial r}+\frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi}=\cos\phi\frac{\partial}{\partial r}-\frac{\sin\phi}{r}\frac{\partial}{\partial \phi}$$

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平面极坐标系下的二维Laplace算符

进一步就能得到

$$\frac{\partial^2}{\partial x^2} = \left(\cos\phi\frac{\partial}{\partial r} - \frac{\sin\phi}{r}\frac{\partial}{\partial\phi}\right)\left(\cos\phi\frac{\partial}{\partial r} - \frac{\sin\phi}{r}\frac{\partial}{\partial\phi}\right)$$

平面极坐标系下的二维Laplace算符

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$$\begin{aligned}\frac{\partial^2}{\partial x^2} &= \left(\cos\phi\frac{\partial}{\partial r} - \frac{\sin\phi}{r}\frac{\partial}{\partial\phi}\right)\left(\cos\phi\frac{\partial}{\partial r} - \frac{\sin\phi}{r}\frac{\partial}{\partial\phi}\right) \\ &= \cos^2\phi\frac{\partial^2}{\partial r^2} - \frac{2\sin\phi\cos\phi}{r}\frac{\partial^2}{\partial r\partial\phi} + \frac{\sin^2\phi}{r^2}\frac{\partial^2}{\partial\phi^2} \\ &\quad + \frac{\sin^2\phi}{r}\frac{\partial}{\partial r} + \frac{2\sin\phi\cos\phi}{r^2}\frac{\partial}{\partial\phi}\end{aligned}$$

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&= \cos^2\phi\frac{\partial^2}{\partial r^2} - \frac{2\sin\phi\cos\phi}{r}\frac{\partial^2}{\partial r\partial\phi} + \frac{\sin^2\phi}{r^2}\frac{\partial^2}{\partial\phi^2} \\
&\quad + \frac{\sin^2\phi}{r}\frac{\partial}{\partial r} + \frac{2\sin\phi\cos\phi}{r^2}\frac{\partial}{\partial\phi} \\
\frac{\partial^2}{\partial y^2} &= \left(\sin\phi\frac{\partial}{\partial r} + \frac{\cos\phi}{r}\frac{\partial}{\partial\phi}\right)\left(\sin\phi\frac{\partial}{\partial r} + \frac{\cos\phi}{r}\frac{\partial}{\partial\phi}\right)
\end{aligned}$$

平面极坐标系下的二维Laplace算符

进一步就能得到

$$\begin{aligned}\frac{\partial^2}{\partial x^2} &= \left(\cos\phi\frac{\partial}{\partial r} - \frac{\sin\phi}{r}\frac{\partial}{\partial\phi}\right)\left(\cos\phi\frac{\partial}{\partial r} - \frac{\sin\phi}{r}\frac{\partial}{\partial\phi}\right)\\ &= \cos^2\phi\frac{\partial^2}{\partial r^2} - \frac{2\sin\phi\cos\phi}{r}\frac{\partial^2}{\partial r\partial\phi} + \frac{\sin^2\phi}{r^2}\frac{\partial^2}{\partial\phi^2}\\ &\quad + \frac{\sin^2\phi}{r}\frac{\partial}{\partial r} + \frac{2\sin\phi\cos\phi}{r^2}\frac{\partial}{\partial\phi}\\ \frac{\partial^2}{\partial y^2} &= \left(\sin\phi\frac{\partial}{\partial r} + \frac{\cos\phi}{r}\frac{\partial}{\partial\phi}\right)\left(\sin\phi\frac{\partial}{\partial r} + \frac{\cos\phi}{r}\frac{\partial}{\partial\phi}\right)\\ &= \sin^2\phi\frac{\partial^2}{\partial r^2} + \frac{2\sin\phi\cos\phi}{r}\frac{\partial^2}{\partial r\partial\phi} + \frac{\cos^2\phi}{r^2}\frac{\partial^2}{\partial\phi^2}\\ &\quad + \frac{\cos^2\phi}{r}\frac{\partial}{\partial r} - \frac{2\sin\phi\cos\phi}{r^2}\frac{\partial}{\partial\phi}\end{aligned}$$

平面极坐标系下的二维Laplace算符

$$\frac{\partial^2}{\partial x^2} = \cos^2\phi \frac{\partial^2}{\partial r^2} - \frac{2\sin\phi\cos\phi}{r}\frac{\partial^2}{\partial r\partial\phi} + \frac{\sin^2\phi}{r^2}\frac{\partial^2}{\partial\phi^2} + \frac{\sin^2\phi}{r}\frac{\partial}{\partial r} + \frac{2\sin\phi\cos\phi}{r^2}\frac{\partial}{\partial\phi}$$

$$\frac{\partial^2}{\partial y^2} = \sin^2\phi \frac{\partial^2}{\partial r^2} + \frac{2\sin\phi\cos\phi}{r}\frac{\partial^2}{\partial r\partial\phi} + \frac{\cos^2\phi}{r^2}\frac{\partial^2}{\partial\phi^2} + \frac{\cos^2\phi}{r}\frac{\partial}{\partial r} - \frac{2\sin\phi\cos\phi}{r^2}\frac{\partial}{\partial\phi}$$

平面极坐标系下的二维Laplace算符

$$\frac{\partial^2}{\partial x^2} = \cos^2\phi\frac{\partial^2}{\partial r^2} - \frac{2\sin\phi\cos\phi}{r}\frac{\partial^2}{\partial r\partial\phi} + \frac{\sin^2\phi}{r^2}\frac{\partial^2}{\partial\phi^2} + \frac{\sin^2\phi}{r}\frac{\partial}{\partial r} + \frac{2\sin\phi\cos\phi}{r^2}\frac{\partial}{\partial\phi}$$

$$\frac{\partial^2}{\partial y^2} = \sin^2\phi\frac{\partial^2}{\partial r^2} + \frac{2\sin\phi\cos\phi}{r}\frac{\partial^2}{\partial r\partial\phi} + \frac{\cos^2\phi}{r^2}\frac{\partial^2}{\partial\phi^2} + \frac{\cos^2\phi}{r}\frac{\partial}{\partial r} - \frac{2\sin\phi\cos\phi}{r^2}\frac{\partial}{\partial\phi}$$

- 于是就得到平面极坐标系下的Laplace算符

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}$$

平面极坐标系下的二维Laplace算符

$$\begin{aligned}\frac{\partial^2}{\partial x^2} &= \cos^2\phi\frac{\partial^2}{\partial r^2} - \frac{2\sin\phi\cos\phi}{r}\frac{\partial^2}{\partial r\partial\phi} + \frac{\sin^2\phi}{r^2}\frac{\partial^2}{\partial\phi^2} \\ &\quad + \frac{\sin^2\phi}{r}\frac{\partial}{\partial r} + \frac{2\sin\phi\cos\phi}{r^2}\frac{\partial}{\partial\phi} \\ \frac{\partial^2}{\partial y^2} &= \sin^2\phi\frac{\partial^2}{\partial r^2} + \frac{2\sin\phi\cos\phi}{r}\frac{\partial^2}{\partial r\partial\phi} + \frac{\cos^2\phi}{r^2}\frac{\partial^2}{\partial\phi^2} \\ &\quad + \frac{\cos^2\phi}{r}\frac{\partial}{\partial r} - \frac{2\sin\phi\cos\phi}{r^2}\frac{\partial}{\partial\phi}\end{aligned}$$

- 于是就得到平面极坐标系下的Laplace算符

$$\nabla^2 \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}$$

平面极坐标系下的二维Laplace算符

平面极坐标系下的Laplace算符

$$\nabla^2 \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}$$

☞ 思考题　此结果有何限制条件?

平面极坐标系下的二维Laplace算符

平面极坐标系下的Laplace算符

$$\nabla^2 \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}$$

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柱坐标系下的Laplace算符

平面极坐标系下的Laplace算符

$$\nabla^2 \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}$$

- 过渡到三维情形，增加一项$\dfrac{\partial^2}{\partial z^2}$
- 所以柱坐标系下的Laplace算符是

$$\nabla^2 \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}+\frac{\partial^2}{\partial z^2}$$

柱坐标系下的Laplace算符

平面极坐标系下的Laplace算符

$$\nabla^2 \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}$$

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柱坐标系下的Laplace算符

平面极坐标系下的Laplace算符

$$\nabla^2 \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}$$

- 过渡到三维情形，增加一项 $\dfrac{\partial^2}{\partial z^2}$
- 所以柱坐标系下的Laplace算符是

$$\nabla^2 \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}+\frac{\partial^2}{\partial z^2}$$

讲授要点

1. 非齐次稳定问题
   - 示例
   - 方法的进一步发展
2. 非齐次边界条件的齐次化
   - 基本思路
   - 特殊技巧：方程及边界条件同时齐次化
3. 正交曲面坐标系下的Laplace算符
   - 柱坐标系下的Laplace算符
   - 球坐标系下的Laplace算符

球坐标系下的Laplace算符

- 球坐标(r,θ,ϕ)和直角坐标(x,y,z)的关系是

$$x=r\sin\theta\cos\phi,\quad y=r\sin\theta\sin\phi,\quad z=r\cos\theta$$

- 由此可以解出

$$\begin{aligned}
\mathrm{d}r &= \sin\theta\cos\phi\mathrm{d}x+\sin\theta\sin\phi\mathrm{d}y+\cos\theta\mathrm{d}z\\
\mathrm{d}\theta &= \frac{\cos\theta\cos\phi}{r}\mathrm{d}x+\frac{\cos\theta\sin\phi}{r}\mathrm{d}y-\frac{\sin\theta}{r}\mathrm{d}z\\
\mathrm{d}\phi &= -\frac{\sin\phi}{r\sin\theta}\mathrm{d}x+\frac{\cos\phi}{r\sin\theta}\mathrm{d}y
\end{aligned}$$

球坐标系下的Laplace算符

- 球坐标(r,θ,ϕ)和直角坐标(x,y,z)的关系是

$$x=r\sin\theta\cos\phi,\quad y=r\sin\theta\sin\phi,\quad z=r\cos\theta$$

- 由此可以解出

$$\begin{aligned}
\mathrm{d}r &= \sin\theta\cos\phi\mathrm{d}x+\sin\theta\sin\phi\mathrm{d}y+\cos\theta\mathrm{d}z\\
\mathrm{d}\theta &= \frac{\cos\theta\cos\phi}{r}\mathrm{d}x+\frac{\cos\theta\sin\phi}{r}\mathrm{d}y-\frac{\sin\theta}{r}\mathrm{d}z\\
\mathrm{d}\phi &= -\frac{\sin\phi}{r\sin\theta}\mathrm{d}x+\frac{\cos\phi}{r\sin\theta}\mathrm{d}y
\end{aligned}$$

球坐标系下的Laplace算符

因此

$$\begin{aligned}\frac{\partial}{\partial x} &= \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi} \\ &= \sin\theta\ \cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial \theta} - \frac{\sin\phi}{r\ \sin\theta}\frac{\partial}{\partial \phi}\end{aligned}$$

球坐标系下的Laplace算符

因此

$$\begin{aligned}\frac{\partial}{\partial x} &= \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi} \\ &= \sin\theta\ \cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial \theta} - \frac{\sin\phi}{r\ \sin\theta}\frac{\partial}{\partial \phi} \\ \frac{\partial}{\partial y} &= \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial \phi} \\ &= \sin\theta\ \sin\phi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial \theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial \phi}\end{aligned}$$

球坐标系下的Laplace算符

因此

$$\begin{aligned}
\frac{\partial}{\partial x} &= \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi} \\
&= \sin\theta\ \cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial \theta} - \frac{\sin\phi}{r\ \sin\theta}\frac{\partial}{\partial \phi} \\
\frac{\partial}{\partial y} &= \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial \phi} \\
&= \sin\theta\ \sin\phi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial \theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial \phi} \\
\frac{\partial}{\partial z} &= \frac{\partial r}{\partial z}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial z}\frac{\partial}{\partial \theta} \\
&= \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}
\end{aligned}$$

球坐标系下的Laplace算符

$$\begin{aligned}\frac{\partial^2}{\partial x^2} &= \left(\sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)^2 \\ &= \sin^2\theta\cos^2\phi\frac{\partial^2}{\partial r^2} + \frac{\cos^2\theta\cos^2\phi}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\sin^2\phi}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \\ &\quad + \frac{2\sin\theta\cos\theta\cos^2\phi}{r}\frac{\partial^2}{\partial r\partial\theta} - \frac{2\sin\phi\cos\phi}{r}\frac{\partial^2}{\partial r\partial\phi} \\ &\quad - \frac{2\cos\theta\sin\phi\cos\phi}{r^2\sin\theta}\frac{\partial^2}{\partial\theta\partial\phi} + \frac{\cos^2\theta\cos^2\phi + \sin^2\phi}{r}\frac{\partial}{\partial r} \\ &\quad + \frac{-2\sin^2\theta\cos\theta\cos^2\phi + \cos\theta\sin^2\phi}{r^2\sin\theta}\frac{\partial}{\partial\theta} \\ &\quad + \frac{2\sin\phi\cos\phi}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}\end{aligned}$$

球坐标系下的Laplace算符

$$
\begin{aligned}
\frac{\partial^2}{\partial y^2} = & \left(\sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)^2 \\
= & \sin^2\theta\sin^2\phi\frac{\partial^2}{\partial r^2} + \frac{\cos^2\theta\sin^2\phi}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\cos^2\phi}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \\
& + \frac{2\sin\theta\cos\theta\sin^2\phi}{r}\frac{\partial^2}{\partial r\partial\theta} + \frac{2\sin\phi\cos\phi}{r}\frac{\partial^2}{\partial r\partial\phi} \\
& + \frac{2\cos\theta\sin\phi\cos\phi}{r^2\sin\theta}\frac{\partial^2}{\partial\theta\partial\phi} + \frac{\cos^2\theta\sin^2\phi + \cos^2\phi}{r}\frac{\partial}{\partial r} \\
& + \frac{-2\sin^2\theta\cos\theta\sin^2\phi + \cos\theta\cos^2\phi}{r^2\sin\theta}\frac{\partial}{\partial\theta} \\
& - \frac{2\sin\phi\cos\phi}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}
\end{aligned}
$$

球坐标系下的Laplace算符

$$\begin{aligned}\frac{\partial^2}{\partial z^2} &= \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)^2 \\ &= \cos^2\theta\frac{\partial^2}{\partial r^2} + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial r\partial\theta} \\ &\quad + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{\sin^2\theta}{r}\frac{\partial}{\partial r}\end{aligned}$$

球坐标系下的Laplace算符

最后就得到球坐标系下的Laplace算符

$$\begin{aligned}\nabla^2 &\equiv \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \\ &\equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\end{aligned}$$

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球坐标系下的Laplace算符

最后就得到球坐标系下的Laplace算符

$$\begin{aligned}\nabla^2 &\equiv \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \\ &\equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\end{aligned}$$

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