第 三 讲

分离变量法(二)

北京大学物理学院

2007年春

讲授要点

1. 矩形区域内的稳定问题
2. 两端固定弦的受迫振动
   - 方程及边界条件同时齐次化
   - 按相应齐次问题本征函数展开

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References

- 吴崇试，《数学物理方法》，§14.3, 14.5
- 梁昆淼，《数学物理方法》，§8.2
- 胡嗣柱、倪光炯，《数学物理方法》，§10.5

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分离变量法也适用于热传导方程和稳定问题(例如，Laplace方程)的定解问题

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \qquad 0<x<a,\ 0<y<b$$

$$u\Big|_{x=0}=0 \qquad \frac{\partial u}{\partial x}\Big|_{x=a}=0 \qquad 0\le y\le b$$

$$u\Big|_{y=0}=f(x) \qquad \frac{\partial u}{\partial y}\Big|_{y=b}=0 \qquad 0\le x\le a$$

- 仍可用分离变量法求解
- 仍然按照上面总结的四个标准步骤求解

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分离变量

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仍用分离变量法求解．令

$$u(x,y)=X(x)Y(y)$$

分离变量

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☞ 代入方程，分离变量，即得

分离变量

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$$u\big|_{y=0}=f(x) \qquad \frac{\partial u}{\partial y}\bigg|_{y=b}=0 \qquad 0\le x\le a$$

$$\boxed{\frac{X''(x)}{X(x)}=-\frac{Y''(y)}{Y(y)}=-\lambda} \Longrightarrow \boxed{\begin{array}{l} X''(x)+\lambda X(x)=0 \\ Y''(y)-\lambda Y(y)=0 \end{array}}$$

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☞ 代入关于x的一对齐次边界条件

分离变量

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$$\boxed{\begin{aligned}&X(0)Y(y)=0\\&X'(a)Y(y)=0\end{aligned}} \Longrightarrow \boxed{\begin{aligned}&X(0)=0\\&X'(a)=0\end{aligned}}$$

求解本征值问题

$$X''(x) + \lambda X(x) = 0$$
$$X(0) = 0 \qquad X'(a) = 0$$

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☞ 若$\lambda = 0$

微分方程的通解 $X(x) = A_0 x + B_0$

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微分方程的通解 $X(x) = A_0 x + B_0$

边界条件 $\Longrightarrow$ $A_0 = 0,\ B_0 = 0$

说明$\lambda = 0$时只有零解．即$\lambda = 0$不是本征值

求解本征值问题

$$X''(x) + \lambda X(x) = 0$$
$$X(0) = 0 \qquad X'(a) = 0$$

☞ 当$\lambda \neq 0$时

微分方程通解 $X(x) = A\sin\sqrt{\lambda}x + B\cos\sqrt{\lambda}x$

求解本征值问题

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☞ 当$\lambda \neq 0$时

微分方程通解 $X(x) = A \sin \sqrt{\lambda} x + B \cos \sqrt{\lambda} x$

边界条件 $\Longrightarrow$ $\boxed{B = 0 \qquad A \neq 0 \qquad \cos \sqrt{\lambda} a = 0}$

求解本征值问题

$$X''(x) + \lambda X(x) = 0$$
$$X(0) = 0 \qquad X'(a) = 0$$

☞ 当$\lambda \neq 0$时，就求得

$$本\ 征\ 值\quad \lambda_n = \left(\frac{2n+1}{2a}\pi\right)^2, \qquad n = 0, 1, 2, 3, \cdots$$

$$本征函数\ X_n(x) = \sin\frac{2n+1}{2a}\pi x.$$

特解及一般解

方程

$$Y_n''(y) - \lambda_n Y_n(y) = 0$$

$$\lambda_n = \left(\frac{2n+1}{2a}\pi\right)^2, \qquad n = 0, 1, 2, 3, \cdots$$

的解为

$$Y_n(y) = C_n \sinh \frac{2n+1}{2a}\pi y + D_n \cosh \frac{2n+1}{2a}\pi y$$

特解及一般解

因此，既满足Laplace方程、又满足齐次边界条件的特解为

$$u_n(x,y) = \left(C_n \sinh \frac{2n+1}{2a}\pi y + D_n \cosh \frac{2n+1}{2a}\pi y\right) \times \sin \frac{2n+1}{2a}\pi x$$

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将这无穷多个特解叠加起来，就得到一般解

特解及一般解

$$u(x,y)=\sum_{n=0}^{\infty}\left[\left(C_n\sinh\frac{2n+1}{2a}\pi y+D_n\cosh\frac{2n+1}{2a}\pi y\right)\right.$$
$$\left.\times\sin\frac{2n+1}{2a}\pi x\right]$$

利用本征函数的正交性定叠加系数

代入关于y的一对(非齐次)边界条件

$$u\Big|_{y=0} = \sum_{n=0}^{\infty} D_n \sin\frac{2n+1}{2a}\pi x = f(x)$$

$$\frac{\partial u}{\partial y}\Big|_{y=b} = \sum_{n=0}^{\infty} \frac{2n+1}{2a}\pi\Big(C_n \cosh\frac{2n+1}{2a}\pi b + D_n \sinh\frac{2n+1}{2a}\pi b\Big)\sin\frac{2n+1}{2a}\pi x = 0$$

根据本征函数的正交归一性

$$\int_0^a \sin\frac{2n+1}{2a}\pi x \sin\frac{2m+1}{2a}\pi x \mathrm{d}x = \frac{a}{2}\delta_{nm}$$

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$$D_n = \frac{2}{a}\int_0^a f(x)\sin\frac{2n+1}{2a}\pi x\mathrm{d}x$$

$$C_n\cosh\frac{2n+1}{2a}\pi b + D_n\sinh\frac{2n+1}{2a}\pi b = 0$$

$$C_n = -D_n\tanh\frac{2n+1}{2a}\pi b$$

如果知道了$f(x)$的具体形式，还应当进一步算出叠加系数C_n和D_n

☞ 此问题(稳定问题)与时间t无关，因此不出现初始条件

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引言

- 齐次偏微分方程和齐次边界条件在分离变量法中起着关键作用：因为方程和边界条件是齐次的，分离变量才得以实现
- 如果定解问题中的方程和边界条件不是齐次的，还有没有可能应用分离变量法?
- 先讨论方程为非齐次的情形

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定解问题

为了突出对于方程非齐次项的处理，这里研究纯粹由外力引起的两端固定弦的强迫振动，弦的初位移和初速度均为0

定解问题

$$\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = f(x,t) \qquad 0 < x < l,\ t > 0$$

$$u\Big|_{x=0} = 0 \quad u\Big|_{x=l} = 0 \qquad t \geq 0$$

$$u\Big|_{t=0} = 0 \quad \frac{\partial u}{\partial t}\Big|_{t=0} = 0 \qquad 0 \leq x \leq l$$

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基本思想

令$u(x,t)=v(x,t)+w(x,t)$，使得

$$\boxed{\frac{\partial^2 u}{\partial t^2}-a^2\frac{\partial^2 u}{\partial x^2}=f(x,t)} \Longrightarrow \boxed{\begin{array}{l}\dfrac{\partial^2 v}{\partial t^2}-a^2\dfrac{\partial^2 v}{\partial x^2}=f(x,t)\\ \dfrac{\partial^2 w}{\partial t^2}-a^2\dfrac{\partial^2 w}{\partial x^2}=0\end{array}}$$

- 在将非齐次方程齐次化的同时，必须保持原有的齐次边界条件不变
- 解法的关键在于求得特解$v(x,t)$．这适用于$f(x,t)$形式比较简单的情形

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方程及边界条件同时齐次化

作变换$u(x,t)=v(x,t)+w(x,t)$，希望$w(x,t)$满足齐次方程和齐次边界条件

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$$u(x,t)=v(x,t)+w(x,t)\,\|$$

$$\frac{\partial^2 v}{\partial t^2} - a^2\frac{\partial^2 v}{\partial x^2} = f(x,t)$$
$$v\Big|_{x=0} = 0 \qquad v\Big|_{x=l} = 0$$
初始条件(不作要求)

$+$

$$\frac{\partial^2 w}{\partial t^2} - a^2\frac{\partial^2 w}{\partial x^2} = 0$$
$$w\Big|_{x=0} = 0 \qquad w\Big|_{x=l} = 0$$
初始条件

方程及边界条件同时齐次化

由于只要求$v(x,t)$满足原定解问题中的方程及边界条件

$$\frac{\partial^2 v}{\partial t^2} - a^2\frac{\partial^2 v}{\partial x^2} = f(x,t)$$

$$v\Big|_{x=0} = 0 \quad v\Big|_{x=l} = 0$$

而对初始条件无要求，故解$v(x,t)$存在而不唯一．我们只需在可能的条件下选择一个容易求得的$v(x,t)$

故称此法为方程及边界条件同时齐次化

方程及边界条件同时齐次化

而$w(x,t)$满足齐次方程和齐次边界条件

$$\frac{\partial^2 w}{\partial t^2} - a^2\frac{\partial^2 w}{\partial x^2} = 0$$

$$w\Big|_{x=0} = 0 \quad w\Big|_{x=l} = 0$$

且满足边界条件

$$w\Big|_{t=0} = -v\Big|_{t=0} \qquad \frac{\partial w}{\partial t}\Big|_{t=0} = -\frac{\partial v}{\partial t}\Big|_{t=0}$$

故称此法为方程及边界条件同时齐次化

方程及边界条件同时齐次化

一旦求得了$v(x,t)$，就可以求出$w(x,t)$的一般解

$$w(x,t)=\sum_{n=1}^{\infty}\left(C_n\sin\frac{n\pi}{l}at+D_n\cos\frac{n\pi}{l}at\right)\sin\frac{n\pi}{l}x$$

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而后根据边界条件定出叠加系数C_n和D_n

$$\boxed{\begin{aligned} w\Big|_{t=0} &= -v\Big|_{t=0} \\ \frac{\partial w}{\partial t}\Big|_{t=0} &= -\frac{\partial v}{\partial t} \end{aligned}} \quad\Longrightarrow\quad \boxed{C_n,\ D_n}$$

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一旦求得了$v(x,t)$，就可以求出$w(x,t)$的一般解

$$w(x,t)=\sum_{n=1}^{\infty}\left(C_n\sin\frac{n\pi}{l}at+D_n\cos\frac{n\pi}{l}at\right)\sin\frac{n\pi}{l}x$$

$$C_n=-\frac{2}{n\pi a}\int_0^l\frac{\partial v(x,t)}{\partial t}\Big|_{t=0}\sin\frac{n\pi}{l}x\mathrm{d}x$$

$$D_n=-\frac{2}{l}\int_0^l v(x,0)\sin\frac{n\pi}{l}x\mathrm{d}x$$

方程及边界条件同时齐次化

一旦求得了$v(x,t)$，就可以求出$w(x,t)$的一般解

$$w(x,t)=\sum_{n=1}^{\infty}\left(C_n\sin\frac{n\pi}{l}at+D_n\cos\frac{n\pi}{l}at\right)\sin\frac{n\pi}{l}x$$

$$C_n=-\frac{2}{n\pi a}\int_0^l\frac{\partial v(x,t)}{\partial t}\Big|_{t=0}\sin\frac{n\pi}{l}x\mathrm{d}x$$

$$D_n=-\frac{2}{l}\int_0^l v(x,0)\sin\frac{n\pi}{l}x\mathrm{d}x$$

再将$v(x,t)$和$w(x,t)$相加，就求得了$u(x,t)$

讨论

- 这种解法称为方程和边界条件的同时齐次化
- 在将非齐次方程齐次化的同时，必须保持原有的齐次边界条件不变
- 解法的关键在于求得特解$v(x,t)$．因此此法只适用于$f(x,t)$形式比较简单的情形
- 齐次初始条件的限制可以取消
- 齐次边界条件的限制是否也可以取消？

讨论

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讨论

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- 齐次边界条件的限制是否也可以取消?

讨论

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- 解法的关键在于求得特解$v(x,t)$．因此此法只适用于$f(x,t)$形式比较简单的情形
- 齐次初始条件的限制可以取消
- 齐次边界条件的限制是否也可以取消?

举例(一)

例3.1　求解定解问题(其中$f(x)$为已知函数)

$$\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = f(x) \qquad 0 < x < l,\ t > 0$$

$$u\Big|_{x=0} = 0 \qquad u\Big|_{x=l} = 0 \qquad t \geq 0$$

$$u\Big|_{t=0} = 0 \qquad \frac{\partial u}{\partial t}\Big|_{t=0} = 0 \qquad 0 \leq x \leq l$$

Answer

举例(一)

例3.1　求解定解问题(其中$f(x)$为已知函数)

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = f(x) && 0 < x < l,\ t > 0 \\
&u\Big|_{x=0} = 0 \qquad u\Big|_{x=l} = 0 && t \geq 0 \\
&u\Big|_{t=0} = 0 \qquad \frac{\partial u}{\partial t}\Big|_{t=0} = 0 && 0 \leq x \leq l
\end{aligned}
$$

Answer

只给出解题的主要思路

举例(一)

例3.1 求解定解问题(其中$f(x)$为已知函数)

$$\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = f(x) \qquad 0 < x < l,\ t > 0$$

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$$u\Big|_{t=0} = 0 \qquad \frac{\partial u}{\partial t}\Big|_{t=0} = 0 \qquad 0 \leq x \leq l$$

Answer

由于方程的非齐次项只是x的函数，就可以把齐次化函数v也取为只是x的函数

$$u(x,t) = v(x) + w(x,t)$$

举例(一)

例3.1　求解定解问题(其中$f(x)$为已知函数)

$$\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = f(x) \qquad 0 < x < l,\ t > 0$$

$$u\Big|_{x=0} = 0 \qquad u\Big|_{x=l} = 0 \qquad t \geq 0$$

$$u\Big|_{t=0} = 0 \qquad \frac{\partial u}{\partial t}\Big|_{t=0} = 0 \qquad 0 \leq x \leq l$$

Answer

$$v''(x) = -\frac{1}{a^2}f(x)$$

$$v(0) = 0$$

$$v(l) = 0$$

举例(一)

例3.1　求解定解问题(其中$f(x)$为已知函数)

$$\frac{\partial^2 u}{\partial t^2}-a^2\frac{\partial^2 u}{\partial x^2}=f(x) \qquad 0<x<l,\ t>0$$

$$u\big|_{x=0}=0 \qquad u\big|_{x=l}=0 \qquad t\geq 0$$

$$u\big|_{t=0}=0 \qquad \frac{\partial u}{\partial t}\bigg|_{t=0}=0 \qquad 0\leq x\leq l$$

Answer

$$v''(x)=-\frac{1}{a^2}f(x)$$
$$v(0)=0$$
$$v(l)=0$$

$$\frac{\partial^2 w}{\partial t^2}-a^2\frac{\partial^2 w}{\partial x^2}=0$$

$$w\big|_{x=0}=0 \qquad w\big|_{x=l}=0$$

$$w\big|_{t=0}=-v(x) \qquad \frac{\partial w}{\partial t}\bigg|_{t=0}=0$$

举例(二)

例3.2 求解定解问题

$$\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = A_0 \sin\omega t \qquad 0 < x < l,\ t > 0$$

$$u\Big|_{x=0} = 0 \qquad u\Big|_{x=l} = 0 \qquad t \geq 0$$

$$u\Big|_{t=0} = 0 \qquad \frac{\partial u}{\partial t}\Big|_{t=0} = 0 \qquad 0 \leq x \leq l$$

Discussion

举例(二)

例3.2　求解定解问题

$$\frac{\partial^2 u}{\partial t^2}-a^2\frac{\partial^2 u}{\partial x^2}=A_0\sin\omega t \qquad 0<x<l,\ t>0$$

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$$u\Big|_{t=0}=0 \qquad \frac{\partial u}{\partial t}\Big|_{t=0}=0 \qquad 0\leq x\leq l$$

Discussion

讨论：能否设$u(x,t)=v(t)+w(x,t)$?

举例(二)

例3.2　求解定解问题

$$\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = A_0 \sin \omega t \qquad 0 < x < l,\ t > 0$$

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Discussion

$$\boxed{u(x,t) = v(x,t) + w(x,t)} \Downarrow$$

举例(二)

例3.2　求解定解问题

$$\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = A_0 \sin \omega t \qquad 0 < x < l,\ t > 0$$

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Discussion

$$\boxed{u(x,t) = v(x,t) + w(x,t)} \Downarrow$$

$$\frac{\partial^2 v}{\partial t^2} - a^2\frac{\partial^2 v}{\partial x^2} = A_0 \sin \omega t$$

$$v\Big|_{x=0} = 0 \qquad v\Big|_{x=l} = 0$$

初始条件(不作要求)

$+$

$$\frac{\partial^2 w}{\partial t^2} - a^2\frac{\partial^2 w}{\partial x^2} = 0$$

$$w\Big|_{x=0} = 0 \qquad w\Big|_{x=l} = 0$$

初始条件

举例(二)

例3.2　求解定解问题

$$\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = A_0 \sin\omega t \qquad 0 < x < l,\ t > 0$$

$$u\big|_{x=0} = 0 \qquad u\big|_{x=l} = 0 \qquad t \geq 0$$

$$u\big|_{t=0} = 0 \qquad \frac{\partial u}{\partial t}\Big|_{t=0} = 0 \qquad 0 \leq x \leq l$$

Answer

可将齐次化函数$v(x,t)$取为$v(x,t) = f(x)\sin\omega t$

$$\frac{\partial^2 v}{\partial t^2} - a^2\frac{\partial^2 v}{\partial x^2} = A_0 \sin\omega t \implies -\omega^2 f(x) - a^2 f''(x) = A_0$$

$$v\big|_{x=0} = 0 \implies f(0) = 0$$

$$v\big|_{x=l} = 0 \implies f(l) = 0$$

举例(二)

例3.2 求解定解问题

$$\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = A_0 \sin\omega t \qquad 0<x<l,\ t>0$$

$$u\big|_{x=0} = 0 \qquad u\big|_{x=l} = 0 \qquad t \geq 0$$

$$u\big|_{t=0} = 0 \qquad \frac{\partial u}{\partial t}\bigg|_{t=0} = 0 \qquad 0 \leq x \leq l$$

Answer

可将齐次化函数$v(x,t)$取为$v(x,t)=f(x)\sin\omega t$

$$\boxed{\frac{\partial^2 v}{\partial t^2} - a^2\frac{\partial^2 v}{\partial x^2} = A_0 \sin\omega t} \implies \boxed{-\omega^2 f(x) - a^2 f''(x) = A_0}$$

$$\boxed{v\big|_{x=0}=0} \implies \boxed{f(0)=0}$$

$$\boxed{v\big|_{x=l}=0} \implies \boxed{f(l)=0}$$

举例(二)

例3.2　求解定解问题

$$\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = A_0 \sin\omega t \qquad 0 < x < l,\ t > 0$$

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Answer

可将齐次化函数$v(x,t)$取为$v(x,t) = f(x)\sin\omega t$

$$\boxed{\frac{\partial^2 v}{\partial t^2} - a^2\frac{\partial^2 v}{\partial x^2} = A_0 \sin\omega t} \Longrightarrow \boxed{-\omega^2 f(x) - a^2 f''(x) = A_0}$$

$$\boxed{v\big|_{x=0} = 0} \Longrightarrow \boxed{f(0) = 0}$$

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举例(二)

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$$\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = A_0 \sin\omega t \qquad 0 < x < l,\ t > 0$$

$$u\Big|_{x=0} = 0 \qquad u\Big|_{x=l} = 0 \qquad t \geq 0$$

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Answer

可将齐次化函数$v(x,t)$取为$v(x,t) = f(x)\sin\omega t$

$$\boxed{\frac{\partial^2 v}{\partial t^2} - a^2\frac{\partial^2 v}{\partial x^2} = A_0 \sin\omega t} \Longrightarrow \boxed{-\omega^2 f(x) - a^2 f''(x) = A_0}$$

$$\boxed{v\Big|_{x=0} = 0} \Longrightarrow \boxed{f(0) = 0}$$

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举例(二)

例3.2　求解定解问题

$$\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = A_0 \sin\omega t \qquad 0<x<l,\ t>0$$

$$u\Big|_{x=0} = 0 \qquad u\Big|_{x=l} = 0 \qquad t \geq 0$$

$$u\Big|_{t=0} = 0 \qquad \frac{\partial u}{\partial t}\Big|_{t=0} = 0 \qquad 0 \leq x \leq l$$

Answer

可将齐次化函数$v(x,t)$取为$v(x,t) = f(x)\sin\omega t$
非齐次常微分方程的通解为

$$f(x) = -\frac{A_0}{\omega^2} + A\sin\frac{\omega}{a}x + B\cos\frac{\omega}{a}x$$

举例(二)

例3.2 求解定解问题

$$\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = A_0 \sin\omega t \qquad 0<x<l,\ t>0$$

$$u\Big|_{x=0}=0 \qquad u\Big|_{x=l}=0 \qquad t\geq 0$$

$$u\Big|_{t=0}=0 \qquad \frac{\partial u}{\partial t}\Big|_{t=0}=0 \qquad 0\leq x\leq l$$

Answer

可将齐次化函数$v(x,t)$取为$v(x,t)=f(x)\sin\omega t$

非齐次常微分方程的通解为

$$f(x) = -\frac{A_0}{\omega^2} + A\sin\frac{\omega}{a}x + B\cos\frac{\omega}{a}x$$

边界条件 $\Longrightarrow \quad B=\frac{A_0}{\omega^2} \quad A=\frac{A_0}{\omega^2}\tan\frac{\omega l}{2a}$

举例(二)

例3.2 求解定解问题

$$\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = A_0 \sin\omega t \qquad 0 < x < l,\ t > 0$$

$$u\Big|_{x=0} = 0 \qquad u\Big|_{x=l} = 0 \qquad t \geq 0$$

$$u\Big|_{t=0} = 0 \qquad \frac{\partial u}{\partial t}\Big|_{t=0} = 0 \qquad 0 \leq x \leq l$$

Answer

可将齐次化函数$v(x,t)$取为$v(x,t) = f(x)\sin\omega t$

$$\begin{aligned} f(x) &= -\frac{A_0}{\omega^2}\left[\left(1-\cos\frac{\omega}{a}x\right) - \tan\frac{\omega l}{2a}\sin\frac{\omega}{a}x\right] \\ &= -\frac{A_0}{\omega^2}\left[1 - \frac{\cos(\omega(x-l/2)/a)}{\cos(\omega l/2a)}\right] \end{aligned}$$

举例(二)

例3.2 求解定解问题

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = A_0\sin\omega t && && 0<x<l,\ t>0\\
&u\Big|_{x=0} = 0 && u\Big|_{x=l} = 0 && t\geq 0\\
&u\Big|_{t=0} = 0 && \frac{\partial u}{\partial t}\Big|_{t=0} = 0 && 0\leq x\leq l
\end{aligned}
$$

Answer

再求$w(x,t)$

举例(二)

例3.2 求解定解问题

$$\begin{array}{lll} \dfrac{\partial^2 u}{\partial t^2} - a^2\dfrac{\partial^2 u}{\partial x^2} = A_0\sin\omega t & & 0<x<l,\ t>0 \\ u\Big|_{x=0} = 0 & u\Big|_{x=l} = 0 & t\ge 0 \\ u\Big|_{t=0} = 0 & \dfrac{\partial u}{\partial t}\Big|_{t=0} = 0 & 0\le x\le l \end{array}$$

Answer

定解问题为

$$\begin{array}{lll} \dfrac{\partial^2 w}{\partial t^2} - a^2\dfrac{\partial^2 w}{\partial x^2} = 0 & & 0<x<l,\ t>0 \\ w\Big|_{x=0} = 0 & w\Big|_{x=l} = 0 & t\ge 0 \\ w\Big|_{t=0} = 0 & \dfrac{\partial w}{\partial t}\Big|_{t=0} = -\omega f(x) & 0\le x\le l \end{array}$$

举例(二)

例3.2 求解定解问题

$$\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = A_0 \sin\omega t \qquad 0 < x < l,\ t > 0$$

$$u\big|_{x=0} = 0 \qquad u\big|_{x=l} = 0 \qquad t \geq 0$$

$$u\big|_{t=0} = 0 \qquad \frac{\partial u}{\partial t}\bigg|_{t=0} = 0 \qquad 0 \leq x \leq l$$

Answer

一般解为

$$w(x,t) = \sum_{n=1}^{\infty}\left[C_n \sin\frac{n\pi}{l}at + D_n \cos\frac{n\pi}{l}at\right]\sin\frac{n\pi}{l}x$$

举例(二)

例3.2　求解定解问题

$$\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = A_0 \sin\omega t \qquad 0 < x < l,\ t > 0$$

$$u\Big|_{x=0} = 0 \qquad u\Big|_{x=l} = 0 \qquad t \geq 0$$

$$u\Big|_{t=0} = 0 \qquad \frac{\partial u}{\partial t}\Big|_{t=0} = 0 \qquad 0 \leq x \leq l$$

Answer

利用初始条件可以定出

$$\boxed{C_n = -\frac{2A_0\omega l^3}{\pi^2 a}\frac{1-(-)^n}{n^2}\frac{1}{(n\pi a)^2-(\omega l)^2}} \qquad \boxed{D_n = 0}$$

举例(二)

例3.2　求解定解问题

$$\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = A_0 \sin\omega t \qquad 0 < x < l,\ t > 0$$

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Answer

利用初始条件可以定出

$$\boxed{C_n = -\frac{2A_0\omega l^3}{\pi^2 a}\frac{1-(-)^n}{n^2}\frac{1}{(n\pi a)^2-(\omega l)^2}} \qquad \boxed{D_n = 0}$$

只有$n =$ 奇数时，C_n才不为0

举例(二)

例3.2　求解定解问题

$$\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = A_0 \sin\omega t \qquad 0<x<l,\ t>0$$

$$u\big|_{x=0} = 0 \qquad u\big|_{x=l} = 0 \qquad t\geq 0$$

$$u\big|_{t=0} = 0 \qquad \frac{\partial u}{\partial t}\bigg|_{t=0} = 0 \qquad 0\leq x\leq l$$

Answer

$$w(x,t) = -\frac{4A_0\omega l^3}{\pi^2 a}\sum_{n=0}^{\infty}\left[\frac{1}{(2n+1)^2}\frac{1}{[(2n+1)\pi a]^2-(\omega l)^2} \times \sin\frac{2n+1}{l}\pi x \sin\frac{2n+1}{l}\pi at\right]$$

举例(二)

例3.2 求解定解问题

$$\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = A_0 \sin \omega t \qquad 0 < x < l,\ t > 0$$
$$u\Big|_{x=0} = 0 \qquad u\Big|_{x=l} = 0 \qquad t \geq 0$$
$$u\Big|_{t=0} = 0 \qquad \frac{\partial u}{\partial t}\Big|_{t=0} = 0 \qquad 0 \leq x \leq l$$

Answer

$$u(x,t) = -\frac{A_0}{\omega^2}\left[1 - \frac{\cos \omega(x-l/2)/a}{\cos(\omega l/2a)}\right]\sin \omega t - \frac{4A_0\omega l^3}{\pi^2 a}\sum_{n=0}^{\infty}\left[\frac{1}{(2n+1)^2}\frac{1}{[(2n+1)\pi a]^2-(\omega l)^2} \times \sin\frac{2n+1}{l}\pi x \sin\frac{2n+1}{l}\pi a t\right]$$

举例(三)

例3.3　求解定解问题

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=xy \qquad 0<x<a,\ 0<y<b$$

$$u\big|_{x=0}=0 \qquad u\big|_{x=a}=0 \qquad 0\le y\le b$$

$$u\big|_{y=0}=\phi(x) \qquad u\big|_{y=b}=\psi(x) \qquad 0\le x\le a$$

Answer

举例(三)

例3.3　求解定解问题

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=xy \qquad 0<x<a,\ 0<y<b$$

$$u\Big|_{x=0}=0 \qquad u\Big|_{x=a}=0 \qquad 0\le y\le b$$

$$u\Big|_{y=0}=\phi(x) \qquad u\Big|_{y=b}=\psi(x) \qquad 0\le x\le a$$

Answer

容易求出方程的特解

$$v(x,y)=\frac{1}{6}x^3y+Axy$$

举例(三)

例3.3　求解定解问题

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=xy \qquad 0<x<a,\ 0<y<b$$

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$$u\Big|_{y=0}=\phi(x) \qquad u\Big|_{y=b}=\psi(x) \qquad 0\le x\le a$$

Answer

容易求出方程的特解

$$v(x,y)=\frac{1}{6}x^3y+Axy$$

此特解已满足边界条件$v(x,y)\Big|_{x=0}=0$

举例(三)

例3.3　求解定解问题

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Answer

容易求出方程的特解

$$v(x,y)=\frac{1}{6}x^3y+Axy$$

此特解已满足边界条件$v(x,y)\Big|_{x=0}=0$

$$\boxed{v(x,y)\Big|_{x=a}=0} \implies \boxed{A=-\frac{1}{6}a^2}$$

举例(三)

例3.3　求解定解问题

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=xy \qquad 0<x<a,\ 0<y<b$$

$$u\Big|_{x=0}=0 \qquad u\Big|_{x=a}=0 \qquad 0\le y\le b$$

$$u\Big|_{y=0}=\phi(x) \qquad u\Big|_{y=b}=\psi(x) \qquad 0\le x\le a$$

Answer

容易求出方程的特解

$$v(x,y)=\frac{1}{6}x^3y+Axy$$

此特解已满足边界条件$v(x,y)\Big|_{x=0}=0$

$$v(x,y)=\frac{1}{6}\left(x^2-a^2\right)xy$$

举例(三)

例3.3 求解定解问题

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=xy \qquad 0<x<a,\ 0<y<b$$

$$u\Big|_{x=0}=0 \qquad u\Big|_{x=a}=0 \qquad 0\le y\le b$$

$$u\Big|_{y=0}=\phi(x) \qquad u\Big|_{y=b}=\psi(x) \qquad 0\le x\le a$$

Answer

令$u(x,y)=\frac{1}{6}\left(x^2-a^2\right)xy+w(x,y)$

$$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=0$$

$$w\Big|_{x=0}=0 \qquad w\Big|_{x=a}=0$$

$$w\Big|_{y=0}=\phi(x) \qquad w\Big|_{y=b}=\psi(x)-\frac{b}{6}\left(x^2-a^2\right)x$$

举例(三)

例3.3 求解定解问题

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=xy \qquad 0<x<a,\ 0<y<b$$

$$u\Big|_{x=0}=0 \qquad u\Big|_{x=a}=0 \qquad 0\le y\le b$$

$$u\Big|_{y=0}=\phi(x) \qquad u\Big|_{y=b}=\psi(x) \qquad 0\le x\le a$$

Answer

令$u(x,y)=\frac{1}{6}\left(x^2-a^2\right)xy+w(x,y)$

$$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=0$$

$$w\Big|_{x=0}=0 \qquad w\Big|_{x=a}=0$$

$$w\Big|_{y=0}=\phi(x) \qquad w\Big|_{y=b}=\psi(x)-\frac{b}{6}\left(x^2-a^2\right)x$$

以下从略

重新审视分离变量法

- 所谓分离变量法，只是提供了一种求特解的方法：在求解过程中得到的特解是分离变量形式的
- 一旦叠加后，得到的一般解就不再是分离变量的
- 可以从另一个角度审视分离变量法：间接说明了本征函数组的完备性
- 分离变量法提供了一种求完备函数组的方法

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讲授要点

1. 矩形区域内的稳定问题
2. 两端固定弦的受迫振动
   - 方程及边界条件同时齐次化
   - 按相应齐次问题本征函数展开

中心思想

仍以两端固定弦的受迫振动为例

$$\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = f(x,t) \qquad 0 < x < l,\ t > 0$$

$$u\Big|_{x=0} = 0 \quad u\Big|_{x=l} = 0 \qquad t \geq 0$$

$$u\Big|_{t=0} = 0 \quad \frac{\partial u}{\partial t}\Big|_{t=0} = 0 \qquad 0 \leq x \leq l$$

中心思想

仍以两端固定弦的受迫振动为例

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如果方程非齐次项$f(x,t)$的形式比较复杂，难以求得非齐次方程的特解，就可以采用下面的解法

中心思想

中心思想是设法找到一组本征函数$\{X_n(x), n = 1, 2, \cdots\}$，只要这组本征函数是完备的， 就可以将解$u(x, t)$及非齐次方程的非齐次项$f(x, t)$均按本征函数展开

$$u(x, t) = \sum_{n=1}^{\infty} T_n(t) X_n(x)$$

$$f(x, t) = \sum_{n=1}^{\infty} g_n(t) X_n(x)$$

然后再设法求出$T_n(t)$即可

中心思想

由于$T_n(t)$是一元函数，它满足的是常微分方程(组)，有可能比求解偏微分方程来得简单

如何选取本征函数

最简单的做法是选择$\{X_n(x)\}$为相应齐次定解问题的本征函数，即满足由相应的齐次偏微分方程和齐次边界条件

$$\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = 0 \qquad 0 < x < l,\ t > 0$$

$$u\Big|_{x=0} = 0 \quad u\Big|_{x=l} = 0 \qquad t \geq 0$$

分离变量而得到的本征值问题

$$X_n''(x) + \lambda_n X_n(x) = 0$$

$$X_n(0) = 0 \qquad X_n(l) = 0$$

如何选取本征函数

最简单的做法是选择$\{X_n(x)\}$为相应齐次定解问题的本征函数，即

$$\text{本征值} \qquad \lambda_n = \left(\frac{n\pi}{l}\right)^2 \qquad n = 1, 2, 3, \cdots$$

$$\text{本征函数} \qquad X_n(x) = \sin\frac{n\pi}{l}x$$

如何选取本征函数

最简单的做法是选择$\{X_n(x)\}$为相应齐次定解问题的本征函数，即

$$\text{本征值} \qquad \lambda_n = \left(\frac{n\pi}{l}\right)^2 \qquad n = 1, 2, 3, \cdots$$

$$\text{本征函数} \qquad X_n(x) = \sin\frac{n\pi}{l}x$$

所以这种解法称为

按相应齐次问题的本征函数展开法

解题梗概

两端固定弦的受迫振动

$$\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = f(x,t) \qquad 0 < x < l,\ t > 0$$

$$u\Big|_{x=0} = 0 \qquad u\Big|_{x=l} = 0 \qquad t \geq 0$$

$$u\Big|_{t=0} = 0 \qquad \frac{\partial u}{\partial t}\Big|_{t=0} = 0 \qquad 0 \leq x \leq l$$

1. 求出相应齐次问题的本征函数$X_n(x)$
2. 将$u(x,t)$及$f(x,t)$按相应齐次问题本征函数$X_n(x)$展开
3. 将$u(x,t)$及$f(x,t)$代入偏微分方程，根据本征函数的正交性，导出$T_n(t)$满足的常微分方程
4. 将$u(x,t)$代入初始条件，根据本征函数的正交性，导出$T_n(t)$满足的初始条件
5. 求出$T_n(t)$

解题梗概

两端固定弦的受迫振动

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5. 求出$T_n(t)$

(一)求出相应齐次问题的本征函数$X_n(x)$

两端固定弦的受迫振动

$$\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = f(x,t) \qquad 0<x<l,\ t>0$$

$$u\Big|_{x=0} = 0 \qquad u\Big|_{x=l} = 0 \qquad t \geq 0$$

$$u\Big|_{t=0} = 0 \qquad \frac{\partial u}{\partial t}\Big|_{t=0} = 0 \qquad 0 \leq x \leq l$$

求出相应齐次问题的本征函数$X_n(x)$

$$X_n = \sin\frac{n\pi}{l}x \quad n = 1,2,3,\cdots$$

(二)将$u(x,t)$及$f(x,t)$按相应齐次问题的本征函数展开

两端固定弦的受迫振动

$$\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = f(x,t) \qquad 0<x<l,\ t>0$$

$$u\big|_{x=0}=0 \qquad u\big|_{x=l}=0 \qquad t\geq 0$$

$$u\big|_{t=0}=0 \qquad \frac{\partial u}{\partial t}\bigg|_{t=0}=0 \qquad 0\leq x\leq l$$

将$u(x,t)$及$f(x,t)$均按本征函数$\{X_n(x)\}$展开

$$u(x,t)=\sum_{n=1}^{\infty}T_n(t)X_n(x) \qquad f(x,t)=\sum_{n=1}^{\infty}g_n(t)X_n(x)$$

(三)$\cdots$ 导出$T_n(t)$满足的常微分方程

两端固定弦的受迫振动

$$\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = f(x,t) \qquad 0<x<l,\ t>0$$

$$u\big|_{x=0} = 0 \qquad u\big|_{x=l} = 0 \qquad t \geq 0$$

$$u\big|_{t=0} = 0 \qquad \frac{\partial u}{\partial t}\bigg|_{t=0} = 0 \qquad 0 \leq x \leq l$$

将$u(x,t)$及$f(x,t)$代入偏微分方程

$$\sum_{n=1}^{\infty} T_n''(t)X_n(x) - a^2\sum_{n=1}^{\infty} T_n(t)X_n''(x) = \sum_{n=1}^{\infty} g_n(t)X_n(x)$$

(三)··· 导出$T_n(t)$满足的常微分方程

两端固定弦的受迫振动

$$\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = f(x,t) \qquad 0<x<l,\ t>0$$

$$u\Big|_{x=0} = 0 \qquad u\Big|_{x=l} = 0 \qquad t\geq 0$$

$$u\Big|_{t=0} = 0 \qquad \frac{\partial u}{\partial t}\Big|_{t=0} = 0 \qquad 0\leq x\leq l$$

利用$X_n(x)$所满足的常微分方程，又化成

$$\sum_{n=1}^{\infty} T_n''(t)X_n(x) + a^2\sum_{n=1}^{\infty}\lambda_n T_n(t)X_n(x) = \sum_{n=1}^{\infty} g_n(t)X_n(x)$$

(三)$\cdots$ 导出$T_n(t)$满足的常微分方程

两端固定弦的受迫振动

$$\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = f(x,t) \qquad 0 < x < l,\ t > 0$$

$$u\Big|_{x=0} = 0 \qquad u\Big|_{x=l} = 0 \qquad t \geq 0$$

$$u\Big|_{t=0} = 0 \qquad \frac{\partial u}{\partial t}\Big|_{t=0} = 0 \qquad 0 \leq x \leq l$$

根据本征函数的正交性比较系数，就得到$T_n(t)$满足的常微分方程

$$\boxed{\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = f(x,t)} \Rightarrow \boxed{T_n''(t) + \lambda_n a^2 T_n(t) = g_n(t)}$$

(四)将$u(x,t)$代入初始条件，导出$T_n(t)$满足的初始条件

两端固定弦的受迫振动

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = f(x,t) \qquad 0 < x < l,\ t > 0$$

$$u\Big|_{x=0} = 0 \qquad u\Big|_{x=l} = 0 \qquad t \geq 0$$

$$u\Big|_{t=0} = 0 \qquad \frac{\partial u}{\partial t}\Big|_{t=0} = 0 \qquad 0 \leq x \leq l$$

将$u(x,t)$的展开式代入初始条件，也可得到

$$\boxed{\sum_{n=1}^{\infty} T_n(0) X_n(x) = 0} \implies \boxed{T_n(0) = 0}$$

$$\boxed{\sum_{n=1}^{\infty} T'_n(0) X_n(0) = 0} \implies \boxed{T'_n(0) = 0}$$

(五)求出$T_n(t)$

两端固定弦的受迫振动

$$\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = f(x,t) \qquad 0<x<l,\ t>0$$

$$u\Big|_{x=0} = 0 \qquad u\Big|_{x=l} = 0 \qquad t\geq 0$$

$$u\Big|_{t=0} = 0 \qquad \frac{\partial u}{\partial t}\Big|_{t=0} = 0 \qquad 0\leq x\leq l$$

$T_n(t)$满足的常微分方程初值问题

$$\boxed{\begin{array}{l} T_n''(t) + \lambda_n a^2 T_n(t) = g_n(t) \\ T_n(0) = 0 \qquad T_n'(0) = 0 \end{array}}$$

(五)求出$T_n(t)$

两端固定弦的受迫振动

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = f(x,t) \qquad 0 < x < l,\ t > 0$$

$$u\Big|_{x=0} = 0 \qquad u\Big|_{x=l} = 0 \qquad t \geq 0$$

$$u\Big|_{t=0} = 0 \qquad \frac{\partial u}{\partial t}\Big|_{t=0} = 0 \qquad 0 \leq x \leq l$$

可用常数变易法求出

$$T_n(t) = \frac{l}{n\pi a}\int_0^t g_n(\tau)\sin\frac{n\pi}{l}a(t-\tau)\,\mathrm{d}\tau$$

举例：重解例3.2

例3.2　求解定解问题

$$\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = A_0 \sin \omega t \qquad 0 < x < l,\ t > 0$$

$$u\Big|_{x=0} = 0 \qquad u\Big|_{x=l} = 0 \qquad t \geq 0$$

$$u\Big|_{t=0} = 0 \qquad \frac{\partial u}{\partial t}\Big|_{t=0} = 0 \qquad 0 \leq x \leq l$$

Answer

举例：重解例3.2

例3.2　求解定解问题

$$\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = A_0 \sin\omega t \qquad 0<x<l,\ t>0$$

$$u\Big|_{x=0} = 0 \qquad u\Big|_{x=l} = 0 \qquad t\geq 0$$

$$u\Big|_{t=0} = 0 \qquad \frac{\partial u}{\partial t}\Big|_{t=0} = 0 \qquad 0\leq x\leq l$$

Answer

将$u(x,t)$及$f(x,t)$按相应齐次问题本征函数$\left\{\sin\frac{n\pi}{l}x\right\}$展开

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t)\sin\frac{n\pi}{l}x$$

$$A_0\sin\omega t = \frac{2A_0}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin\frac{n\pi}{l}x\sin\omega t$$

举例：重解例3.2

例3.2　求解定解问题

$$\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = A_0\sin\omega t \qquad 0<x<l,\ t>0$$

$$u\Big|_{x=0}=0 \qquad u\Big|_{x=l}=0 \qquad t\ge 0$$

$$u\Big|_{t=0}=0 \qquad \frac{\partial u}{\partial t}\Big|_{t=0}=0 \qquad 0\le x\le l$$

Answer

代入方程，导出$T_n(t)$满足的微分方程

$$T''(t)+\left(\frac{n\pi}{l}a\right)^2T_n(t)=\frac{2A_0}{\pi}\frac{1-(-1)^n}{n}\sin\omega t$$

举例：重解例3.2

例3.2　求解定解问题

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = A_0 \sin \omega t \qquad 0 < x < l,\ t > 0$$

$$u\Big|_{x=0} = 0 \qquad u\Big|_{x=l} = 0 \qquad t \geq 0$$

$$u\Big|_{t=0} = 0 \qquad \frac{\partial u}{\partial t}\Big|_{t=0} = 0 \qquad 0 \leq x \leq l$$

Answer

将$u(x,t)$代入初始条件，导出$T_n(t)$满足的初始条件

$$T(0) = 0 \qquad T'(0) = 0$$

举例：重解例3.2

例3.2 求解定解问题

$$\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = A_0 \sin\omega t \qquad 0 < x < l,\ t > 0$$

$$u\Big|_{x=0} = 0 \qquad u\Big|_{x=l} = 0 \qquad t \geq 0$$

$$u\Big|_{t=0} = 0 \qquad \frac{\partial u}{\partial t}\Big|_{t=0} = 0 \qquad 0 \leq x \leq l$$

Answer

求解$T_n(t)$满足的常微分方程初值问题

$$T''(t) + \left(\frac{n\pi}{l}a\right)^2 T_n(t) = \frac{2A_0}{\pi}\frac{1-(-1)^n}{n}\sin\omega t$$

$$T(0) = 0 \qquad T'(0) = 0$$

举例：重解例3.2

例3.2 求解定解问题

$$\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = A_0 \sin\omega t \qquad 0 < x < l,\ t > 0$$

$$u\Big|_{x=0} = 0 \qquad u\Big|_{x=l} = 0 \qquad t \geq 0$$

$$u\Big|_{t=0} = 0 \qquad \frac{\partial u}{\partial t}\Big|_{t=0} = 0 \qquad 0 \leq x \leq l$$

Answer

因此

$$\begin{aligned} T_n(t) &= \frac{2A_0 l^2}{\pi}\frac{1-(-1)^n}{n}\frac{1}{(n\pi a)^2-(\omega l)^2}\sin\omega t \\ &\quad -\frac{2A_0\omega l^3}{\pi^2 a}\frac{1-(-1)^n}{n^2}\frac{1}{(n\pi a)^2-(\omega l)^2}\sin\frac{n\pi}{l}at \end{aligned}$$

举例：重解例3.2

例3.2　求解定解问题

$$\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = A_0 \sin\omega t \qquad 0 < x < l,\ t > 0$$

$$u\Big|_{x=0} = 0 \qquad u\Big|_{x=l} = 0 \qquad t \geq 0$$

$$u\Big|_{t=0} = 0 \qquad \frac{\partial u}{\partial t}\Big|_{t=0} = 0 \qquad 0 \leq x \leq l$$

Answer

$$\begin{aligned} u(x,t) =& \frac{4A_0 l^2}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\frac{\sin\omega t}{[(2n+1)\pi a]^2-(\omega l)^2}\sin\frac{2n+1}{l}\pi x \\ &- \frac{4A_0\omega l^3}{\pi^2 a}\sum_{n=0}^{\infty}\left[\frac{1}{(2n+1)^2}\frac{1}{[(2n+1)\pi a]^2-(\omega l)^2}\right. \\ &\left.\times \sin\frac{2n+1}{l}\pi x\ \sin\frac{2n+1}{l}\pi a t\right] \end{aligned}$$