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Some unbounded functions of intermittent maps for which the central limit theorem holds

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Abstract. We compute some dependence coefficients for the stationary Markov chain whose transition kernel is the Perron-Frobenius operator of an expanding map T of $[0, 1]$ with a neutral fixed point. We use these coefficients to prove a central limit theorem for the partial sums of $f \circ T^i$, when f belongs to a large class of unbounded functions from $[0, 1]$ to $\mathbb{R}$. We also prove other limit theorems and moment inequalities.

1. Introduction and first results

For γ in $]0, 1[$, we consider the intermittent map T_γ from $[0, 1]$ to $[0, 1]$, studied for instance by Liverani et al. (1999), which is a modification of the Pomeau-Manneville map (1980):

$$T_\gamma(x) = \begin{cases} x(1 + 2^\gamma x^\gamma) & \text{if } x \in [0, 1/2[\\ 2x - 1 & \text{if } x \in [1/2, 1] \end{cases}$$

We denote by ν_γ the unique T_γ-invariant probability measure on $[0, 1]$ which is absolutely continuous with respect to the Lebesgue measure. We denote by K_γ the Perron-Frobenius operator of T_γ with respect to ν_γ: for any bounded measurable functions f, g,

$$\nu_\gamma(f \cdot g \circ T_\gamma) = \nu_\gamma(K_\gamma(f)g)\,.$$

Let $(X_i)_{i\geq 0}$ be a stationary Markov chain with invariant measure ν_γ and transition Kernel K_γ. It is well known (see for instance Lemma XI.3 in Hennion and Hervé (2001)) that

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on the probability space $([0,1], \nu_\gamma)$, the random variable $(T_\gamma, T_\gamma^2, \ldots, T_\gamma^n)$ is distributed as $(X_n, X_{n-1}, \ldots, X_1)$. Hence any information on the law of

$$S_n(f) = \sum_{i=1}^{n} f \circ T_\gamma^i$$

can be obtained by studying the law of $\sum_{i=1}^n f(X_i)$.

In 1999, Young proved that such systems (among many others) may be described by a Young tower with polynomial decay of the return time. From this construction, she was able to control the covariances $\nu_\gamma(f \circ T^n \cdot (g - \nu_\gamma(g)))$ for any bounded function f and any α-Hölder function g, and then to prove that $n^{-1/2}(S_n(f) - \nu_\gamma(f))$ converges in distribution to a normal law as soon as $\gamma < 1/2$ and f is any α-Hölder function. For $\gamma = 1/2$, Gouëzel (2004) proved that the central limit theorem remains true with the same normalization $\sqrt{n}$ if $f(0) = \nu_\gamma(f)$, and with the normalization $\sqrt{n \ln(n)}$ if $f(0) \neq \nu_\gamma(f)$. When $1/2 < \gamma < 1$, he proved that if f is α-Hölder and $f(0) \neq \nu_\gamma(f)$, $n^{-\gamma}(S_n(f) - \nu_\gamma(f))$ converges to a stable law.

At this point, two questions (at least) arise: 1) what happens if f is no longer continuous? 2) what happens if f is no longer bounded? More precisely, can we find a large class of such functions for which the central limit theorem holds? For instance, for the uniformly expanding map $T_0(x) = 2x - [2x]$, the central limit theorem holds with the normalization $\sqrt{n}$ as soon as f is monotonic and square integrable on $[0,1]$.

For the slightly different map $\theta_\gamma(x) = x(1-x^\gamma)^{-1/\gamma} - [x(1-x^\gamma)^{-1/\gamma}]$, with the same behavior around the indifferent fixed point, Raugi (2004) (following a work by Conze and Raugi (2003)) has given a precise criterion for the central limit theorem with the normalization $\sqrt{n}$ in the case where $0 < \gamma < 1/2$ (see his Corollary 1.7). In particular his result applies to a large class of non continuous functions. It also applies to the unbounded function $f(x) = x^{-a}$ with $0 < a < 1/2 - \gamma$. However, the function f is allowed to blow up near 0 only: if f tends to infinity when x tends to $x_0 \in]0,1]$, then the variation coefficient $v(fh_\gamma, k)$ defined in page 83 in Raugi (2004) is always infinite (here h_γ is the density of the θ_γ-invariant probability, and k is some positive integer).

We now go back to the map T_γ. In a short discussion after the proof of his Theorem 1.3, Gouëzel (2004) considers the case where $f(x) = x^{-a}$, with $0 < a < 1 - \gamma$. He shows that, if $0 < a < 1/2 - \gamma$ then the central limit theorem holds with the normalization $\sqrt{n}$, if $a = 1/2 - \gamma$ then the central limit theorem holds with the normalization $\sqrt{n \ln(n)}$, and if $0 < a < 1 - \gamma$ and $\gamma \geq 1/2$ then there is convergence to a stable law. Again, as for Raugi's result (2004) concerning the map θ_γ, the function f is allowed to blow up only near 0.

On another hand, we know that for stationary Harris recurrent Markov chains with invariant measure μ and β-mixing coefficients of order n^{-b}, $b > 1$, the central limit theorem holds with the normalization $\sqrt{n}$ as soon as the moment condition $\mu(|f|^p) < \infty$ holds for $p > 2b/(b-1)$. For T_γ, the covariances decay is of order $n^{(\gamma-1)/\gamma}$, so that one can expect the moment condition $\nu_\gamma(|f|^p) < \infty$ for $p > (2-2\gamma)/(1-2\gamma)$. For instance, if $f(x) = x^{-a}$, since the density of ν_γ is of order $x^{-\gamma}$ near 0, the moment condition is satisfied if $0 < a < 1/2 - \gamma$, which is coherent with Gouëzel's result (2004). However, since the chain (K_γ, ν_γ) is not β-mixing, the condition $\nu_\gamma(|f|^p) < \infty$ for $p > (2-2\gamma)/(1-2\gamma)$ alone is not sufficient to imply the central limit theorem, and one still needs some regularity on f.

Let us now define the class of functions of interest.

Definition 1.1. *For any probability measure μ on $\mathbb{R}$, any $M > 0$ and any $p \in]1, \infty]$, let $\mathrm{Mon}(M, p, \mu)$ be the class of functions g which are monotonic on some open interval of $\mathbb{R}$*

and null elsewhere, and such that $\mu(|g| > t) \leq M^p t^{-p}$ for $p < \infty$ and $\mu(|g| > M) = 0$ for $p = \infty$. Let $\mathcal{C}(M, p, \mu)$ be the closure in $\mathbb{L}^1(\mu)$ of the set of functions which can be written as $\sum_{i=1}^n a_i g_i$, where $\sum_{i=1}^n |a_i| \leq 1$ and g_i belongs to $\mathrm{Mon}(M, p, \mu)$.

Note that a function belonging to $\mathcal{C}(M, p, \mu)$ is allowed to blow up at an infinite number of points. Note also that any function f with bounded variation (BV) such that $|f| \leq M_1$ and $\|df\| \leq M_2$ belongs to the class $\mathcal{C}(M_1 + 2M_2, \infty, \mu)$ (here $\|df\|$ is the variation norm of the signed measure df). Hence, any BV function f belongs to $\mathcal{C}(M, \infty, \mu)$ for some M large enough. If g is monotonic on some open interval of $\mathbb{R}$ and null elsewhere, and if $\mu(|g|^p) \leq M^p$, then g belongs to $\mathrm{Mon}(M, p, \mu)$. Conversely, any function in $\mathcal{C}(M, p, \mu)$ belongs to $\mathbb{L}^q(\mu)$ for $1 \leq q < p$.

As a consequence of a general theorem for Markov chains (Theorem 4.1 of Section 4), we obtain the following corollary:

Corollary 1.1. *Let $\gamma \in]0, 1/2[$. If f belongs to the class $\mathcal{C}(M, p, \nu_\gamma)$ for some $M > 0$ and some $p > (2 - 2\gamma)/(1 - 2\gamma)$, then $n^{-1/2} S_n(f - \nu_\gamma(f))$ converges in distribution to $\mathcal{N}(0, \sigma^2(\nu_\gamma, K_\gamma, f))$, where the variance term $\sigma^2(\nu_\gamma, K_\gamma, f)$ is defined in Theorem 4.1.*

In particular, we infer from Corollary 1.1 that the central limit theorem holds for any BV function provided $\gamma < 1/2$. For the map $\theta_\gamma(x) = x(1 - x^\gamma)^{-1/\gamma} - [x(1 - x^\gamma)^{-1/\gamma}]$ and $\gamma < 1/2$, the central limit theorem for BV functions is a consequence of Corollary 1.7(i) in Raugi (2004). Here are some other applications of Corollary 1.1:

Two simple examples.

(1) Assume that f is positive and non increasing on $]0, 1[$, with $f(x) \leq C x^{-a}$ for some $a \geq 0$. Since the density g_{ν_γ} of ν_γ is such that $g_{\nu_\gamma}(x) \leq V(\gamma) x^{-\gamma}$, we infer that
$$\nu_\gamma(f > t) \leq \frac{C^{\frac{1-\gamma}{a}} V(\gamma)}{1 - \gamma} t^{-\frac{1-\gamma}{a}} .$$
Hence the central limit theorem holds as soon as $a < \frac{1}{2} - \gamma$.

(2) Assume now that f is positive and non decreasing on $]0, 1[$ with $f(x) \leq C(1 - x)^{-a}$ for some $a \geq 0$. Here
$$\nu_\gamma(f > t) \leq \frac{V(\gamma)}{1 - \gamma} \Big(1 - \Big(1 - \Big(\frac{C}{t} \Big)^{1/a} \Big)^{1-\gamma} \Big) .$$
Hence the central limit theorem holds as soon as $a < \frac{1}{2} - \frac{\gamma}{2(1-\gamma)}$.

We shall also give some conditions on p to obtain rates of convergence in the central limit theorem (Corollary 5.1), as well as moment inequalities for $S_n(f - \nu_\gamma(f))$ (Corollary 6.1). A central limit theorem for the empirical distribution function of $(T_\gamma^i)_{1 \leq i \leq n}$ is given in the last section (Corollary 7.1).

Let us present some easy applications of the moment inequalities given in Corollary 6.1. For any $p > 2$ and any f in the class $\mathcal{C}(M, p, \nu_\gamma)$, we have:

(1) Let $\gamma < (p - 2)/(2p - 2)$. By Chebichev inequality applied with $2 \leq q < 2p(1 - \gamma)/(\gamma p + 2(1 - \gamma))$, we infer from Item (1) of Corollary 6.1 that, for any $\epsilon > 0$ and any $x > 0$,
$$\nu_\gamma\Big(\frac{1}{n} |S_n(f - \nu_\gamma(f))| > x \Big) \leq \frac{C}{(nx^2)^{p(1-\gamma)/(\gamma p + 2(1-\gamma)) - \epsilon}} .$$

(2) Let now $(p-2)/(2p-2) \le \gamma < 1$. By Chebichev inequality applied with $q=2$, we infer from Item (2) of Corollary 6.1 that, for any $\epsilon > 0$ and any $x > 0$,

$$\nu_\gamma\Big(\frac{1}{n}|S_n(f-\nu_\gamma(f))| > x\Big) \le \frac{C}{x^2 n^{(p-2)(1-\gamma)/\gamma p-\epsilon}} .$$

In particular, if f is BV (case $p=\infty$) and $\gamma < 1$, we obtain that, for any $\epsilon > 0$ and any $x > 0$,

$$\nu_\gamma\Big(\frac{1}{n}|S_n(f-\nu_\gamma(f))| > x\Big) \le \frac{H(x)}{n^{(1-\gamma)/\gamma-\epsilon}} ,$$

where $H(x) = O(x^{2(1-\gamma)/\gamma-2\epsilon})$ if $\gamma < 1/2$, and $H(x) = O(x^2)$ if $\gamma \ge 1/2$. Note that Melbourne and Nicol (2008) obtained the same bound when f is α-Hölder and $\gamma < 1/2$.

To prove these results, we compute the β-dependence coefficients (cf Dedecker and Prieur (2005, 2007)) of the Markov chain (K_γ, ν_γ). The main tool is a precise estimate of the Perron-Frobenius operator of the map F associated to T_γ on the Young tower, due to Maume-Deschamps (2001). Next, we apply some general results for β-dependent Markov chains (cf. Theorems 4.1, 5.1, 6.1 and 7.1).

For the sake of simplicity, we give all the computations in the case of the maps T_γ, but our arguments remain valid for many other one-dimensional systems modelled by Young towers. More precisely, all the arguments of Section 2, remain valid in any dimension, because they are only based on the results by Maume-Deschamps (2001) on abstract Young towers. In Section 3, we compute the (one-dimensional) coefficients $\beta_k(n)$ of the Markov chain with transition K_γ by approximating indicators of half line by Hölder functions. Since these coefficients may be defined in higher dimension through indicators of quadrant (see Dedecker and Prieur (2007)), the results of Section 3 can be also extended to higher dimension. However, the main results (Theorems 4.1, 5.1 and 6.1) are valid in the one-dimensional case only, because they are based on a covariance inequality for monotonic functions (see Lemma 4.1 and its proof).

2. The main inequality

For any Markov kernel K with invariant measure μ, any non-negative integers $n_1, \ldots, n_k$, and any bounded measurable functions $f_1, \ldots, f_k$, define

$$\begin{aligned} K^{(n_1,n_2,\ldots,n_k)}(f_1,f_2,\ldots,f_k) &= K^{n_1}(f_1 K^{n_2}(f_2 K^{n_3}(f_3 \cdots K^{n_{k-1}}(f_{k-1}K^{n_k}(f_k))\cdots))), \\ K^{(0)(n_1,n_2,\ldots,n_k)}(f_1,f_2,\ldots,f_k) &= K^{(n_1,n_2,\ldots,n_k)}(f_1,f_2,\ldots,f_k) \\ &\quad -\mu(K^{(n_1,n_2,\ldots,n_k)}(f_1,f_2,\ldots,f_k)) . \end{aligned}$$

For $\alpha \in]0,1]$ and $c > 0$, let $H_{\alpha,c}$ be the set of functions f such that $|f(x)-f(y)| \le c|x-y|^\alpha$.

Theorem 2.1. *Let $\gamma \in]0,1[$, and let $f^{(0)} = f - \nu_\gamma(f)$. For any $\alpha \in]0,1]$, the following inequality holds:*

$$\nu_\gamma\Big(\sup_{f_1,\ldots,f_k \in H_{\alpha,1}} \big|K_\gamma^{(0)(n_1,n_2,\ldots,n_k)}(f_1^{(0)}, f_2^{(0)}, \ldots, f_k^{(0)})\big|\Big) \le \frac{C(\alpha,k)(\ln(n_1+1))^2}{(n_1+1)^{(1-\gamma)/\gamma}} .$$

In particular,

$$\nu_\gamma\Big(\sup_{f\in H_{\alpha,1}} |K_\gamma^n f - \nu_\gamma(f)|\Big) \le \frac{C(\alpha,1)(\ln(n+1))^2}{(n+1)^{(1-\gamma)/\gamma}} .$$

Proof of Theorem 2.1. We refer to the paper by Young (1999) for the construction of the tower Δ associated to T_γ (with floors Λ_ℓ), and for the mappings π from Δ to $[0,1]$ and F from Δ to Δ such that $T_\gamma \circ \pi = \pi \circ F$. On Δ there is a probability measure m_0 and an unique F-invariant probability measure $\bar\nu$ with density h_0 with respect to m_0, and $\bar\nu(\Lambda_\ell) = O(\ell^{-1/\gamma})$. The unique T_γ-invariant probability measure ν_γ is then given by $\nu_\gamma = \bar\nu^\pi$. There exists a distance δ on Δ such that $\delta(x,y) \le 1$ and $|\pi(x) - \pi(y)| \le \kappa\delta(x,y)$ for some positive constant κ. For $\alpha \in]0,1]$, let $\delta_\alpha = \delta^\alpha$, let L_α be the space of Lipschitz functions with respect to δ_α, and let $L_\alpha(f) = \sup_{x,y\in\Delta} |f(x) - f(y)|/\delta_\alpha(x,y)$. Let $L_{\alpha,c}$ be the set of functions such that $L_\alpha(f) \le c$. For φ in $H_{\alpha,c}$, the function $\varphi \circ \pi$ belongs to $L_{\alpha,c\kappa^\alpha}$. Any function f in L_α is bounded and the space L_α is a Banach space with respect to the norm $\|f\|_\alpha = L_\alpha(f) + \|f\|_\infty$. The density h_0 belongs to any L_α and $1/h_0$ is bounded. As in Maume-Deschamps (2001), we denote by $\mathcal{L}_0$ the Perron-Frobenius operator of F with respect to m_0, and by P the Perron-Frobenius operator of F with respect to $\bar\nu$: for any bounded measurable functions φ, ψ,

$$m_0(\varphi \cdot \psi \circ F) = m_0(\mathcal{L}_0(\varphi)\psi) \quad \text{and} \quad \bar\nu(\varphi \cdot \psi \circ F) = \bar\nu(P(\varphi)\psi)\,.$$

We first state a useful lemma

Lemma 2.1. *For any positive $n_1, \ldots, n_k$ and any bounded measurable functions $f_1, \ldots, f_k$ from $[0,1]$ to $\mathbb{R}$, one has*

$$K_\gamma^{(n_1,n_2,\ldots,n_k)}(f_1, f_2, \ldots, f_k) \circ \pi = \mathbb{E}_{\bar\nu}\big(P^{(n_1,n_2,\ldots,n_k)}(f_1 \circ \pi, f_2 \circ \pi, \ldots, f_k \circ \pi)\big|\pi\big)\,.$$

We now complete the proof of Theorem 2.1 for $k = 2$, the general case being similar. Applying Lemma 2.1, it follows that

$$\sup_{f,g\in H_{\alpha,1}} |K_\gamma^n(f^{(0)}K_\gamma^m g^{(0)})(x) - \nu_\gamma(f^{(0)}K_\gamma^m g^{(0)})| \le \mathbb{E}_{\bar\nu}\Big(\sup_{\phi,\psi\in L_{\alpha,\kappa^\alpha}} |P^n(\phi^{(0)}P^m\psi^{(0)}) - \bar\nu(\phi^{(0)}P^m\psi^{(0)})|\Big|\pi = x\Big)\,.$$

Here, we need the following lemma, which is derived from Lemma 3.4 in Maume-Deschamps (2001).

Lemma 2.2. *There exists $M_\alpha > 0$ such that, for any $\psi \in L_\alpha$,*

$$|P^m\psi(x) - P^m\psi(y)| \le M_\alpha\delta(x,y)\|\psi^{(0)}\|_\alpha \le 2M_\alpha\delta_\alpha(x,y)L_\alpha(\psi)\,.$$

Hence, if $\psi \in L_{\alpha,\kappa^\alpha}$, then $P^m(\psi^{(0)})$ belongs to $L_{\alpha,2M_\alpha\kappa^\alpha}$ and is centered, so that $\phi^{(0)}P^m\psi^{(0)}$ belongs to $L_{\alpha,4M_\alpha\kappa^{2\alpha}}$. It follows that

$$\sup_{f,g\in H_{\alpha,1}}|K_\gamma^n(f^{(0)}K_\gamma^m g^{(0)})(x) - \nu(f^{(0)}K_\gamma^m g^{(0)})| \le 4M_\alpha\kappa^{2\alpha}\mathbb{E}_{\bar\nu}\Big(\sup_{\varphi\in L_{\alpha,1}}|P^n(\varphi) - \bar\nu(\varphi)|\Big|\pi = x\Big).$$

Next, we apply the following Lemma, which is derived from Corollary 3.14 in Maume-Deschamps (2001).

Lemma 2.3. *Let $v_\ell = (\ell+1)^{(1-\gamma)/\gamma}(\ln(\ell+1))^{-2}$. There exists $C_\alpha > 0$ such that*

$$\mathbb{E}_{\bar\nu}\Big(\sup_{\varphi\in L_{\alpha,1}}|P^n(\varphi) - \bar\nu(\varphi)|\Big|\pi = x\Big) \le C_\alpha(\ln(n+1))^2(n+1)^{(\gamma-1)/\gamma}\sum_{\ell\ge 0} v_\ell\mathbb{E}_{\bar\nu}(\mathbf{1}_{\Lambda_\ell}|\pi = x)\,.$$

Hence

$$\nu_\gamma\Big(\sup_{f,g\in H_{\alpha,1}}|K_\gamma^n(f^{(0)}K_\gamma^m g^{(0)}) - \nu(f^{(0)}K_\gamma^m g^{(0)})|\Big) \le 4M_\alpha\kappa^{2\alpha}C_\alpha(\ln(n+1))^2(n+1)^{(\gamma-1)/\gamma}\sum_{\ell\ge 0} v_\ell\bar\nu(\Lambda_\ell)\,.$$

Since $\bar{\nu}(\Lambda_\ell) = O(\ell^{-1/\gamma})$, the result follows.

Proof of Lemma 2.1. We write the proof for $k = 2$ only, the general case being similar. Let φ, f and g be three bounded measurable functions. One has

$$\begin{aligned}
\nu_\gamma(\varphi K_\gamma^n (f K_\gamma^m g)) &= \nu_\gamma(\varphi \circ T_\gamma^{n+m} \cdot f \circ T_\gamma^m \cdot g) \\
&= \bar{\nu}(\varphi \circ \pi \circ F^{n+m} \cdot f \circ \pi \circ F^m \cdot g \circ \pi) \\
&= \bar{\nu}(\varphi \circ \pi P^n (f \circ \pi P^m (g \circ \pi))) \\
&= \bar{\nu}(\varphi \circ \pi \mathbb{E}_{\bar{\nu}}(P^n (f \circ \pi P^m (g \circ \pi))|\pi)) \\
&= \int \varphi(x) \mathbb{E}_{\bar{\nu}}(P^n (f \circ \pi P^m (g \circ \pi))|\pi = x) \nu_\gamma(dx)\,,
\end{aligned}$$

which proves Lemma 2.1 for $k = 2$.

Proof of Lemma 2.2. Applying Lemma 3.4 in Maume-Deschamps (2001) with $v_k = 1$, we see that there exists $D_\alpha > 0$ such that, for any ψ in L_α,

$$|\mathcal{L}_0^m \psi(x) - \mathcal{L}_0^m \psi(y)| \le D_\alpha \delta_\alpha(x, y) \|\psi\|_\alpha .$$

Now $P^m(\psi) = \mathcal{L}_0^m(\psi h_0)/h_0$. Since $1/h_0$ is bounded by $B(h_0)$, and since h_0 belongs to L_α, it follows that

$$|P^m \psi(x) - P^m \psi(y)| \le D_\alpha B(h_0) \|h_0\|_\alpha \delta_\alpha(x, y) \|\psi\|_\alpha .$$

Let $M_\alpha = D_\alpha B(h_0)\|h_0\|_\alpha$. Since $|P^m\psi(x) - P^m\psi(y)| = |P^m\psi^{(0)}(x) - P^m\psi^{(0)}(y)|$ and since $\|\psi^{(0)}\|_\infty \le L_\alpha(\psi)$, it follows that

$$|P^m \psi(x) - P^m \psi(y)| \le M_\alpha \delta_\alpha(x, y) \|\psi^{(0)}\|_\alpha \le 2 M_\alpha \delta_\alpha(x, y) L_\alpha(\psi)\,.$$

Proof of Lemma 2.3. Applying Corollary 3.14 in Maume-Deschamps (2001), there exists $B_\alpha > 0$ such that

$$|\mathcal{L}_0^n f - h_0 m_0(f)| \le B_\alpha \|f\|_\alpha (\ln(n+1))^2 (n+1)^{(\gamma-1)/\gamma} \sum_{\ell \ge 0} v_\ell \mathbf{1}_{\Delta_\ell}\,.$$

It follows that, with the notations of the proof of Lemma 2.2,

$$|P^n(f) - \bar{\nu}(f)| \le B_\alpha B(h_0) \|h_0\|_\alpha \|f\|_\alpha (\ln(n+1))^2 (n+1)^{(\gamma-1)/\gamma} \sum_{\ell \ge 0} v_\ell \mathbf{1}_{\Delta_\ell}\,.$$

Since $|P^n(f) - \bar{\nu}(f)| = |P^n(f^{(0)}) - \bar{\nu}(f^{(0)})|$ and since $\|f^{(0)}\|_\infty \le L_\alpha(f)$, it follows that

$$|P^n(f) - \bar{\nu}(f)| \le 2 B_\alpha B(h_0) \|h_0\|_\alpha L_\alpha(f) (\ln(n+1))^2 (n+1)^{(\gamma-1)/\gamma} \sum_{\ell \ge 0} v_\ell \mathbf{1}_{\Delta_\ell}\,,$$

and the result follows.

3. The dependence coefficients

Let $\mathbf{X} = (X_i)_{i \ge 0}$ be a stationary Markov chain with invariant measure μ and transition kernel K. Let $f_t(x) = \mathbf{1}_{x \le t}$. As in Dedecker and Prieur (2005, 2007), define the coefficients $\alpha_k(n)$ of the stationary Markov chain $(X_i)_{i \ge 0}$ by

$$\alpha_1(n) = \sup_{t \in \mathbb{R}} \mu(|K^n(f_t) - \mu(f_t)|)\,, \quad \text{and for } k \ge 2,$$

$$\alpha_k(n) = \alpha_1(n) \vee \sup_{2 \le l \le k} \sup_{n_2 \ge 1, \ldots n_l \ge 1} \sup_{t_1, \ldots, t_l \in \mathbb{R}} \mu\big(|K^{(0)(n, n_2, \ldots, n_l)}(f_{t_1}^{(0)}, f_{t_2}^{(0)}, \ldots, f_{t_l}^{(0)})|\big)\,.$$

In the same way, define the coefficients $\beta_k(n)$ by

$$\beta_1(n)=\mu\Big(\sup_{t\in\mathbb{R}}|K^n(f_t)-\mu(f_t)|\Big)\,,\qquad \text{and for } k\geq 2,$$

$$\beta_k(n)=\beta_1(n)\vee\sup_{2\leq l\leq k}\sup_{n_2\geq 1,\dots n_l\geq 1}\mu\Big(\sup_{t_1,\dots,t_l\in\mathbb{R}}|K^{(0)(n,n_2,\dots,n_l)}(f^{(0)}_{t_1},f^{(0)}_{t_2},\dots,f^{(0)}_{t_l})|\Big)\,.$$

Theorem 3.1. *Let $0<\gamma<1$. Let $\mathbf{X}=(X_i)_{i\geq 0}$ be a stationary Markov chain with invariant measure ν_γ and transition kernel K_γ. There exist two positive constants $C_1(\gamma)$ and $C_2(\delta,\gamma,k)$ such that, for any δ in $]0,(1-\gamma)/\gamma[$ and any positive integer k,*

$$C_1(\gamma)(n+1)^{\frac{\gamma-1}{\gamma}}\leq\alpha_k(n)\leq\beta_k(n)\leq C_2(\delta,\gamma,k)(n+1)^{\frac{\gamma-1}{\gamma}+\delta}\,.$$

Proof of Theorem 3.1. Applying Proposition 2, Item 2, in Dedecker and Prieur (2005), we know that

$$\nu_\gamma\Big(\sup_{f\in H_{1,1}}|K^n_\gamma f-\nu_\gamma(f)|\Big)\leq 2\alpha_1(n)\,.$$

Hence, for any φ such that $|\varphi|\leq 1$ and any f in $H_{1,1}$,

$$\nu_\gamma(\varphi\cdot(K^n_\gamma f-\nu_\gamma(f)))=\nu_\gamma(\varphi\circ T^n\cdot(f-\nu_\gamma(f)))\leq 2\alpha_1(n)$$

The lower bound for $\alpha_k(n)$ follows from the lower bound for $\nu_\gamma(\varphi\circ T^n\cdot(f-\nu_\gamma(f)))$ given by Sarig (2002), Corollary 1.

It remains to prove the upper bound. The point is to approximate the indicator $f_t(x)=\mathbf{1}_{x\leq t}$ by some α-Hölder function. Let

$$f_{t,\epsilon,\alpha}(x)=f_t(x)+\Big(1-\Big(\frac{x-t}{\epsilon}\Big)^\alpha\Big)\mathbf{1}_{t<x\leq t+\epsilon}\,.$$

This function is α-Hölder with Hölder constant $\epsilon^{-\alpha}$. We now prove the upper bounds for $k=1$ and $k=2$ only, the general case being similar. For $k=1$, one has

$$\begin{aligned}K^n(f_{t-\epsilon,\epsilon,\alpha})-\nu_\gamma(f_{t-\epsilon,\epsilon,\alpha})-\nu_\gamma([t-\epsilon,t]) &\leq K^n_\gamma(f_t)-\nu_\gamma(f_t)\\ &\leq K^n_\gamma(f_{t,\epsilon,\alpha})-\nu_\gamma(f_{t,\epsilon,\alpha})+\nu_\gamma([t,t+\epsilon])\,.\end{aligned}$$

Since the density g_{ν_γ} of ν_γ is such that $g_{\nu_\gamma}(x)\leq V(\gamma)x^{-\gamma}$, we infer that for any real a, $\nu_\gamma([a,a+\epsilon])\leq V(\gamma)\varepsilon^{1-\gamma}(1-\gamma)^{-1}$. Consequently,

$$|K^n_\gamma(f_t)-\nu_\gamma(f_t)|\leq\epsilon^{-\alpha}\sup_{f\in H_{\alpha,1}}|K^n_\gamma(f)-\nu_\gamma(f)|+\frac{V(\gamma)}{1-\gamma}\epsilon^{1-\gamma}\,.$$

Applying Theorem 2.1 with $k=1$, we obtain that

$$\nu_\gamma\Big(\sup_{t\in[0,1]}|K^n_\gamma(f_t)-\nu_\gamma(f_t)|\Big)\leq C(\alpha,1)\epsilon^{-\alpha}(\ln(n+1))^2(n+1)^{\frac{\gamma-1}{\gamma}}+\frac{V(\gamma)}{1-\gamma}\epsilon^{1-\gamma}\,.$$

The optimal ϵ is equal to

$$\epsilon=\Big(\frac{\alpha C(\alpha,1)(\ln(n+1))^2(n+1)^{\frac{\gamma-1}{\gamma}}}{V(\gamma)}\Big)^{\frac{1}{\alpha+1-\gamma}}\,.$$

Consequently, for some positive constant $D(\gamma,\alpha)$, one has

$$\nu_\gamma\Big(\sup_{t\in[0,1]}|K^n_\gamma(f_t)-\nu_\gamma(f_t)|\Big)\leq D(\gamma,\alpha)\Big((\ln(n+1))^2(n+1)^{\frac{\gamma-1}{\gamma}}\Big)^{\frac{1-\gamma}{\alpha+1-\gamma}}\,..$$

Choosing $\alpha<\delta\gamma(1-\gamma)/(1-\gamma(1+\delta))$, the result follows for $k=1$.

We now prove the result for $k=2$. Clearly, the four following inequalities hold:

$$\begin{aligned}
K_\gamma^n(f_t^{(0)}K_\gamma^m f_s^{(0)}) &\le K_\gamma^n(f_{t,\epsilon,\alpha}^{(0)}K_\gamma^m f_{s,\epsilon,\alpha}^{(0)}) + \nu_\gamma([t,t+\epsilon]) + \nu_\gamma([s,s+\epsilon])\,,\\
K_\gamma^n(f_t^{(0)}K_\gamma^m f_s^{(0)}) &\ge K_\gamma^n(f_{t-\epsilon,\epsilon,\alpha}^{(0)}K_\gamma^m f_{s-\epsilon,\epsilon,\alpha}^{(0)}) - \nu_\gamma([t-\epsilon,t]) - \nu_\gamma([s-\epsilon,s])\,,\\
\nu_\gamma(f_t^{(0)}K_\gamma^m f_s^{(0)}) &\ge \nu_\gamma(f_{t,\epsilon,\alpha}^{(0)}K_\gamma^m f_{s,\epsilon,\alpha}^{(0)}) - 2\nu_\gamma([t,t+\epsilon]) - \nu_\gamma([s,s+\epsilon])\,,\\
\nu_\gamma(f_t^{(0)}K^m f_s^{(0)}) &\le \nu_\gamma(f_{t-\epsilon,\epsilon,\alpha}^{(0)}K_\gamma^m f_{s-\epsilon,\epsilon,\alpha}^{(0)}) + 2\nu_\gamma([t-\epsilon,t]) + \nu_\gamma([s-\epsilon,s])\,.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&|K_\gamma^n(f_t^{(0)}K_\gamma^m f_s^{(0)}) - \nu_\gamma(f_t^{(0)}K_\gamma^m f_s^{(0)})|\\
&\qquad\le \epsilon^{-\alpha}\sup_{f,g\in H_{\alpha,1}}|K_\gamma^n(f^{(0)}K_\gamma^m g^{(0)}) - \nu_\gamma(f^{(0)}K_\gamma^m g^{(0)})| + \frac{5V(\gamma)}{1-\gamma}\epsilon^{1-\gamma}\,.
\end{aligned}$$

Applying Theorem 2.1, we obtain that

$$\begin{aligned}
&\nu_\gamma\Big(\sup_{t\in[0,1]}|K_\gamma^n(f_t^{(0)}K_\gamma^m f_s^{(0)}) - \nu_\gamma(f_t^{(0)}K_\gamma^m f_s^{(0)})|\Big)\\
&\qquad\le C(\alpha,2)\epsilon^{-\alpha}(\ln(n+1))^2(n+1)^{\frac{\gamma-1}{\gamma}} + \frac{5V(\gamma)}{1-\gamma}\epsilon^{1-\gamma}\,,
\end{aligned}$$

and the proof can be completed as for $k=1$.

4. Central limit theorems

In this section we give a central limit theorem for $\sum_{i=1}^n f(X_i)$ where $(X_i)_{i\ge 0}$ is a stationary Markov chain, and f belongs to the class $\mathcal{C}(M,p,\mu)$ defined in the introduction. The condition are expressed in terms of the dependence coefficients $(\alpha_1(k))_{k\ge 0}$ of the chain, which have been defined in Section 3.

Theorem 4.1. *Let $\mathbf{X} = (X_i)_{i\ge 0}$ be a stationary and ergodic (in the ergodic theoretic sense) Markov chain with invariant measure μ and transition kernel K. Assume that f belongs to $\mathcal{C}(M,p,\mu)$ for some $M>0$ and some $p\in]2,\infty]$, and that*

$$\sum_{k>0}(\alpha_1(k))^{\frac{p-2}{p}} < \infty\,.$$

The following results hold:

(1) *The series*

$$\sigma^2(\mu,K,f) = \mu((f-\mu(f))^2) + 2\sum_{k>0}\mu((f-\mu(f))K^k(f))$$

converges to some non negative constant, and $n^{-1}\mathrm{Var}(\sum_{i=1}^n f(X_i))$ converges to $\sigma^2(\mu,K,f)$.

(2) *Let $(D([0,1],d)$ be the space of cadlag functions from $[0,1]$ to $\mathbb{R}$ equipped with the Skorohod metric d. The process $\{n^{-1/2}\sum_{i=1}^{[nt]}(f(X_i)-\mu(f)), t\in[0,1]\}$ converges in distribution in $(D([0,1],d)$ to $\sigma(\mu,K,f)W$, where W is a standard Wiener process.*

(3) *One has the representation*

$$f(X_1)-\mu(f) = m(X_1,X_0) + g(X_1) - g(X_0)$$

with $\mu(|g|^{p/(p-1)})<\infty$, $\mathbb{E}(m(X_1,X_0)|X_0)=0$ and $\mathbb{E}(m^2(X_1,X_0))=\sigma^2(\mu,K,f)$.

Proof of Theorem 4.1. Let f in $\mathcal{C}(M,p,\mu)$. From Dedecker and Rio (2000), Items (1) and (2) of Theorem 4.1 hold as soon as

$$\sum_{n>0} \|(f(X_0)-\mu(f))(\mathbb{E}(f(X_n)|X_0)-\mu(f))\|_1 < \infty\,.$$

Assume first that $f=\sum_{i=1}^k a_i g_i$, where $\sum_{i=1}^k |a_i| \leq 1$, and g_i belongs to $\mathrm{Mon}(M,p,\mu)$. Clearly, the series on left side is bounded by

$$\sum_{i=1}^k \sum_{j=1}^k |a_i a_j| \sum_{n>0} \|(g_i(X_0)-\mu(g_i))(\mathbb{E}(g_j(X_n)|X_0)-\mu(g_j))\|_1\,.$$

Here, we use the following lemma

Lemma 4.1. *Let g_i and g_j be two functions in $\mathrm{Mon}(M,p,\mu)$ for some $p\in]2,\infty]$. For any $1\leq q\leq p$ one has*

$$\|\mathbb{E}(g_j(X_n)|X_0)-\mu(g_j)\|_q \leq 2M\Big(\frac{p}{p-q}\Big)^{1/q}(2\alpha_1(n))^{\frac{p-q}{pq}}\,.$$

For any $1\leq q<p/2$, one has

$$\|(g_i(X_0)-\mu(g_i))(\mathbb{E}(g_j(X_n)|X_0)-\mu(g_j))\|_q \leq 4M^2\Big(\frac{p}{p-2q}\Big)^{1/q}(2\alpha_1(n))^{\frac{p-2q}{pq}}\,..$$

From Lemma 4.1 with $q=1$, we conclude that

$$\sum_{n>0} \|(f(X_0)-\mu(f))(\mathbb{E}(f(X_n)|X_0)-\mu(f))\|_1 \leq \frac{4pM^2}{p-2}\sum_{n>0}(2\alpha_1(n))^{\frac{p-2}{p}}\,. \qquad (4.1)$$

Since the bound (4.1) is true for any function $f=\sum_{i=1}^k a_i g_i$, it is true also for any f in $\mathcal{C}(M,p,\mu)$, and Items (1) and (2) follow.

The last assertion is rather standard. From the first inequality of Lemma 4.1 with $q=p/(p-1)$, we infer that if $\sum_{n>0}(\alpha_1(n))^{(p-2)/p}<\infty$, then $\sum_{n>0}\|\mathbb{E}(f(X_n)|X_0)-\mu(f)\|_{p/(p-1)}<\infty$ for any f in $\mathcal{C}(M,p,\mu)$. It follows that $g(x)=\sum_{k=1}^{\infty}\mathbb{E}(f(X_k)-\mu(f)|X_0=x)$ belongs to $\mathbb{L}^{p/(p-1)}(\mu)$ and that $m(X_1,X_0)=\sum_{k\geq 1}(\mathbb{E}(f(X_k)|X_0)-\mathbb{E}(f(X_k)|X_1))$ belongs to $\mathbb{L}^{p/(p-1)}$. Clearly

$$f(X_1)-\mu(f)=m(X_1,X_0)+g(X_0)-g(X_1)\,,$$

with $\mathbb{E}(m(X_1,X_0)|X_0)=0$. Moreover, it follows from the preceding result that

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}}\Big\|\sum_{k=1}^n m(X_k,X_{k-1})\Big\|_1 = \lim_{n\to\infty}\frac{1}{\sqrt{n}}\Big\|\sum_{k=1}^n (f(X_k)-\mu(f))\Big\|_1 \leq \sigma(\mu,K,f)\,.$$

By Theorem 1 in Esseen and Janson (1985), it follows that $\mathbb{E}(m^2(X_1,X_0))=\sigma^2(\mu,K,f)$.

Proof of Lemma 4.1. We only prove the second inequality (the proof of the first one is easier). Let $r=q/(q-1)$ and let $B_r(\sigma(X_0))$ be the set of $\sigma(X_0)$-measurable random variables such that $\|Y\|_r\leq 1$. By duality,

$$\begin{aligned}&\|(g_i(X_0)-\mu(g_i))(\mathbb{E}(g_j(X_n)|X_0)-\mu(g_j))\|_q\\ &\qquad = \sup_{Y\in B_r(\sigma(X_0))}\mathbb{E}(Y(g_i(X_0)-\mu(g_i))(g_j(X_n)-\mu(g_j)))\\ &\qquad = \sup_{Y\in B_r(\sigma(X_0))}\mathrm{Cov}(Y(g_i(X_0)-\mu(g_i),g_j(X_n))\,.\end{aligned}$$

Define the coefficients $\alpha_{k,g}(n)$ of the sequence $(g(X_i))_{i\geq 0}$ as in Section 3 with $g \circ f_t$ instead of f_t. If g is monotonic on some open interval of $\mathbb{R}$ and null elsewhere, the set $\{x : g(x) \leq t\}$ is either some interval or the complement of some interval, so that $\alpha_{k,g}(n) \leq 2^k \alpha_k(n)$. Let Q_Y be the generalized inverse of the tail function $t \to \mathbb{P}(|Y| > t)$. From Theorem 1.1 and Lemma 2.1 in Rio (2000), one has that

$$\begin{aligned}\mathrm{Cov}(Y g_i(X_0), g_j(X_n)) &\leq 2\int_0^{\alpha_{1,g_i}(n)} Q_Y(u) Q_{g_i(X_0)}(u) Q_{g_j(X_0)}(u)du \\ &\leq 2\int_0^{2\alpha_1(n)} Q_Y(u) Q_{g_i(X_0)}(u) Q_{g_j(X_0)}(u)du\,.\end{aligned}$$

In the same way, applying first Theorem 1.1 in Rio (2000) and next Fréchet's inequality (1957) (see also Inequality (1.11b) in Rio (2000)),

$$\begin{aligned}\mathrm{Cov}(Y \mu(g_i), g_j(X_n)) &\leq 2\mu(|g_i|)\int_0^{2\alpha_1(n)} Q_Y(u) Q_{g_j(X_0)}(u)du \\ &\leq 2\int_0^{2\alpha_1(n)} Q_Y(u) Q_{g_i(X_0)}(u) Q_{g_j(X_0)}(u)du\,.\end{aligned}$$

Since $\int_0^1 Q_Y^r(u)du \leq 1$, it follows that

$$\|(g_i(X_0)-\mu(g_i))(\mathbb{E}(g_j(X_n)|X_0)-\mu(g_j))\|_q \leq 4\Big(\int_0^{2\alpha_1(n)} Q^q_{g_i(X_0)}(u) Q^q_{g_j(X_0)}(u)du\Big)^{1/q}.$$

Since g_i and g_j belong to $\mathrm{Mon}(M,p,\mu)$ for some $p > 2q$, we have that $Q_{g_i(X_0)}(u)$ and $Q_{g_j(X_0)}(u)$ are smaller than $Mu^{-1/p}$, and the result follows.

Proof of Corollary 1.1. We have seen that $(T_\gamma^1, \ldots, T_\gamma^n)$ is distributed as $(X_n, \ldots, X_1)$ where $(X_i)_{i\geq 0}$ is the stationary Markov chain with invariant measure ν_γ and transition kernel K_γ. Consequently, on the probability space $([0,1], \nu_\gamma)$, the sum $S_n(f - \nu_\gamma(f))$ is distributed as $\sum_{i=1}^n (f(X_i) - \nu_\gamma(f))$, so that $n^{-1/2} S_n(f - \nu_\gamma(f))$ satisfies the central limit theorem if and only if $n^{-1/2}\sum_{i=1}^n (f(X_i) - \nu_\gamma(f))$ does. Moreover, we infer from Theorem 3.1 that

$$\alpha_1(n) = O(n^{\frac{\gamma-1}{\gamma}+\epsilon})$$

for any $\epsilon > 0$. Consequently, if $p > (2-2\gamma)/(1-2\gamma)$, one has that $\sum_{k>0}(\alpha_1(n))^{\frac{p-2}{p}} < \infty$ so that Theorem 4.1 applies: the central limit theorem holds provided that f belongs to $\mathcal{C}(M, p, \nu_\gamma)$.

5. Rates of convergence in the CLT

Let c be some concave function from $\mathbb{R}^+$ to $\mathbb{R}^+$, with $c(0) = 0$. Denote by Lip_c the set of functions g such that

$$|g(x) - g(y)| \leq c(|x-y|)\,.$$

When $c(x) = x^\alpha$ for $\alpha \in]0,1]$, we have $\mathrm{Lip}_c = H_{\alpha,1}$. For two probability measures P, Q with finite first moment, let

$$d_c(P,Q) = \sup_{f\in \mathrm{Lip}_c} |P(f) - Q(f)|\,.$$

When $c =$Id, we write $d_c = d_1$. Note that $d_1(P,Q)$ is the so-called Kantorovič distance between P and Q.

Theorem 5.1. *Let $\mathbf{X} = (X_i)_{i\geq 0}$ be a stationary Markov chain with invariant measure μ and transition kernel K. Let $\sigma^2(f) = \sigma^2(\mu, K, f)$ be the non-negative number defined in Theorem 4.1, and let $G_{\sigma^2(f)}$ be the Gaussian distribution with mean 0 and variance $\sigma^2(f)$. Let $P_n(f)$ be the distribution of the normalized sum $n^{-1/2}\sum_{i=1}^n (f(X_i) - \mu(f))$.*

(1) *Assume that f belongs to $\mathcal{C}(M, p, \mu)$ for some $M > 0$ and some $p \in]2, \infty]$, and that*

$$\sum_{k>0} (\alpha_1(k))^{\frac{p-2}{p}} < \infty\,.$$

If $\sigma^2(f) = 0$, then $d_c(P_n(f), \delta_{\{0\}}) = O(c(n^{-1/2}))$.

(2) *If f belongs to $\mathcal{C}(M, p, \mu)$ for some $M > 0$ and some $p \in]3, \infty]$, and if*

$$\sum_{k>0} k(\alpha_3(k))^{\frac{p-3}{p}} < \infty\,,$$

then $d_c(P_n(f), G_{\sigma^2(f)}) = O(c(n^{-1/2}))$.

(3) *If f belongs to $\mathcal{C}(M, p, \mu)$ for some $M > 0$ and some $p \in]3, \infty]$, and if*

$$\alpha_2(k) = O(k^{-(1+\delta)p/(p-3)}) \quad \text{for some } \delta \in]0, 1[,$$

then $d_c(P_n(f), G_{\sigma^2(f)}) = O(c(n^{-\delta/2}))$.

Corollary 5.1. *Let $\delta \in]0, 1]$ and $\gamma < 1/(2 + \delta)$, and let $\mu_n(f)$ be the distribution of $n^{-1/2}S_n(f - \nu_\gamma(f))$. If f belongs to the class $\mathcal{C}(M, p, \nu_\gamma)$ for some $M > 0$ and some $p > (3 - 3\gamma)/(1 - (2 + \delta)\gamma)$, then $d_c(\mu_n(f), G_{\sigma^2(f)}) = O(c(n^{-\delta/2}))$, where $\sigma^2(f) = \sigma^2(\nu_\gamma, K_\gamma, f)$.*

Remark 5.1. *We infer from Corollary 5.1 that if f is BV, then $d_1(\mu_n(f), G_{\sigma^2(f)}) = O(n^{-1/2})$ if $\gamma < 1/3$, and $d_1(\mu_n(f), G_{\sigma^2(f)}) = O(n^{-\delta/2})$ if $\gamma < 1/(2 + \delta)$. Denote by $d_{BV}(P, Q)$ the uniform distance between the distribution functions of P and Q. If f is α-Hölder (Gouëzel, 2005, Theorem 1.5) has proved that $d_{BV}(\mu_n(f), G_{\sigma^2}(f)) = O(n^{-1/2})$ if $\gamma < 1/3$, and $d_{BV}(\mu_n(f), G_{\sigma^2(f)}) = O(n^{-\delta/2})$ if $\gamma = 1/(2 + \delta)$. In fact, from a general result of Bolthausen (1982) for Harris recurrent Markov chains, we conjecture that the results of Corollary 5.1 are true with d_{BV} instead of d_1.*

Two simple examples (continued).

(1) Assume that f is positive and non increasing on $[0, 1]$, with $f(x) \leq Cx^{-a}$ for some $a \geq 0$. Let $\delta \in]0, 1]$ and $\gamma < 1/(2 + \delta)$. If $a < \frac{1}{3} - \frac{(2+\delta)\gamma}{3}$, then $d_c(\mu_n(f), G_{\sigma^2(f)}) = O(c(n^{-\delta/2}))$.

(2) Assume that f is positive and non increasing on $[0, 1]$, with $f(x) \leq C(1 - x)^{-a}$ for some $a \geq 0$. Let $\delta \in]0, 1]$ and $\gamma < 1/(2 + \delta)$. If $a < \frac{1}{3} - \frac{(1+\delta)\gamma}{3(1-\gamma)}$, then $d_c(\mu_n(f), G_{\sigma^2(f)}) = O(c(n^{-\delta/2}))$.

Proof of Theorem 5.1. From the Kantorovič-Rubinšteĭn theorem (1957), there exists a probability measure π with margins P and Q, such that $d_1(P, Q) = \int |x - y|\pi(dx, dy)$. Since c is concave, we then have

$$d_c(P, Q) = \sup_{f \in H_c} \Big| \int (f(x) - f(y))\pi(dx, dy) \Big| \leq \int c(|x - y|)\pi(dx, dy) \leq c(d_1(P, Q))\,.$$

Hence, it is enough to prove the theorem for d_1 only.

If $\sum_{k>0}(\alpha_1(k))^{(p-2)/p} < \infty$, f belongs to $\mathcal{C}(M,p,\mu)$ for some $M>0$ and some $p \in]2,\infty]$, and $\sigma^2(f)=0$, it follows from Theorem 4.1 that $f(X_1)=g(X_0)-g(X_1)$ with $\mu(|g|)<\infty$. Hence

$$d_1(P_n(f),\delta_{\{0\}}) \leq \frac{2\mu(|g|)}{\sqrt{n}}\,,$$

and Item (1) is proved.

From now, we assume that $\sigma^2(f)>0$ (otherwise, the result follows from Item (1)). If $f=g_1-g_2$, where g_1, g_2 belong to $\mathrm{Mon}(M,p,\mu)$ for some $M>0$ and some $p\in]3,\infty]$, Item (2) of Theorem 5.1 follows from Theorem 3.1(b) in Dedecker and Rio (2008). In fact the proof remains unchanged if f belongs to $\mathcal{C}(M,p,\mu)$ for some $M>0$ and some $p\in]3,\infty]$.

It remains to prove Item (3). Let $Y_k=f(X_k)-\mu(f)$, $\sigma^2(f)=\sigma^2$, and $s_m=\sum_{i=1}^m Y_i$. Define

$$W_m = A_m + B_m, \quad \text{with } A_m=\mathbb{E}(s_m^2|X_0)-m\sigma^2 \quad \text{and } B_m = 2\sum_{k=1}^m \mathbb{E}\Big(Y_k\sum_{i>m}Y_i\Big|X_0\Big)\,.$$

From Theorem 2.2 in Dedecker and Rio (2008), we have that, if $\sum_{k>0}\|Y_0\mathbb{E}(Y_k|X_0)\|_1<\infty$,

$$\sqrt{n}d_1(P_n(f),G_{\sigma^2}) \leq C\ln(n) + \sum_{m=1}^{[\sqrt{2n}]}\frac{\|(|Y_0|+2\sigma)W_m\|_1}{m\sigma^2} + D_{1,n}+D_{2,n}\,, \tag{5.1}$$

where

$$D_{1,n}=\sum_{m=1}^n \frac{1}{\sigma\sqrt{m}}\sum_{i\geq m}\|Y_0\mathbb{E}(Y_i|X_0)\|_1 \text{ and } D_{2,n}=\sum_{m=1}^n\frac{1}{2\sigma^2 m}\sum_{k=1}^m\|(\sigma^2+Y_0^2)\mathbb{E}(Y_k|X_0)\|_1.$$

From Lemma 4.1 with $q=1$, the bound (4.1) holds for any f in $\mathcal{C}(M,p,\mu)$ for $p>2$. Consequently, if $\alpha_2(k)=O(k^{-(1+\delta)p/(p-3)})$ for some $\delta\in]0,1[$ and $p>3$, then $\sum_{k>0}\|Y_0\mathbb{E}(Y_k|X_0)\|_1<\infty$, so that the bound (5.1) holds. Moreover $n^{-1/2}D_{1,n}=O(n^{-1/2}\ln(n)\vee n^{-\delta})$. Arguing as in Lemma 4.1, one can prove that

$$\|Y_0^2\mathbb{E}(Y_k|X_0)\|_1 \leq C(M,p)(\alpha_1(k))^{\frac{p-3}{p}}\,,$$

so that $n^{-1/2}D_{2,n}=O(n^{-1/2}\ln(n))$.

Arguing as in Lemma 4.1, one can prove that, for $0<k<i$,

$$\|(|Y_0|+2\sigma)\mathbb{E}(Y_kY_i|X_0)\|_1 \leq \|(|Y_0|+2\sigma)Y_k\mathbb{E}(Y_i|X_k)\|_1 \leq C(M,p,\sigma)(\alpha_1(i-k))^{\frac{p-3}{p}}\,. \tag{5.2}$$

Consequently,

$$\frac{1}{\sqrt{n}}\sum_{m=1}^{[\sqrt{2n}]}\frac{\|(|Y_0|+2\sigma)B_m\|_1}{m\sigma^2} = O\Big(\frac{1}{\sqrt{n}}\sum_{m=1}^{[\sqrt{2n}]}\frac{1}{m\sigma^2}\sum_{k=1}^m\sum_{i>m}\frac{1}{(i-k)^{1+\delta}}\Big) = O(n^{-\delta/2})\,.$$

Now,

$$\begin{aligned}\frac{\|(|Y_0|+2\sigma)A_m\|_1}{m} \leq & \frac{2}{m}\sum_{i=1}^m\sum_{j=i}^m\|(|Y_0|+2\sigma)(\mathbb{E}(Y_iY_j|X_0)-\mathbb{E}(Y_iY_j))\|_1 \\ & + (\|Y_0\|_1+2\sigma)\Big|\frac{1}{m}\mathbb{E}(s_m^2)-\sigma^2\Big|\,.\end{aligned}$$

For the second term on right hand, we have

$$\Big|\frac{1}{m}\mathbb{E}(s_m^2)-\sigma^2\Big| \le 2\sum_{k=1}^{\infty}\frac{k\wedge m}{m}|\mathbb{E}(Y_0Y_k)| = O\Big(\sum_{k>0}\frac{k\wedge m}{m}(\alpha_1(k))^{\frac{p-2}{p}}\Big) = O(m^{-\delta})\,,$$

so that

$$\frac{1}{\sqrt{n}}\sum_{m=1}^{[\sqrt{2n}]}\Big|\frac{1}{m}\mathbb{E}(s_m^2)-\sigma^2\Big| = O(n^{-\delta/2})\,.$$

To complete the proof of the theorem, it remains to prove that

$$\frac{1}{\sqrt{n}}\sum_{m=1}^{[\sqrt{2n}]}\frac{2}{m}\sum_{i=1}^{m}\sum_{j=i}^{m}\|(|Y_0|+2\sigma)(\mathbb{E}(Y_iY_j|X_0)-\mathbb{E}(Y_iY_j))\|_1 = O(n^{-\delta/2})\,. \tag{5.3}$$

Applying first (5.2), we have for $j>i$,

$$\|(|Y_0|+2\sigma)(\mathbb{E}(Y_iY_j|X_0)-\mathbb{E}(Y_iY_j))\|_1 \le 2C(M,p,\sigma)(\alpha_1(j-i))^{\frac{p-3}{p}}\,. \tag{5.4}$$

We need a second bound for this quantity. Assume first that $f=\sum_{i=1}^k a_i g_i$, where $\sum_{i=1}^k |a_i|\le 1$ and g_i belongs to $\mathrm{Mon}(M,p,\mu)$. Let $g_i^{(0)} = g_i-\mu(g_i)$. We have that

$$\begin{aligned}&\|Y_0(\mathbb{E}(Y_iY_j|X_0)-\mathbb{E}(Y_iY_j))\|_1\\ &\quad\le \sum_{l=1}^k\sum_{q=1}^k\sum_{r=1}^k |a_la_qa_r|\|g_l^{(0)}(X_0)(\mathbb{E}(g_q^{(0)}(X_i)g_r^{(0)}(X_j)|X_0)-\mathbb{E}(g_q^{(0)}(X_i)g_r^{(0)}(X_j)))\|_1\,.\end{aligned}$$

For 3 real-valued random variables A,B,C, define the numbers $\bar\alpha(A,B)$ and $\bar\alpha(A,B,C)$ by

$$\begin{aligned}\bar\alpha(A,B) &= \sup_{s,t\in\mathbb{R}}|\mathrm{Cov}(\mathbf{1}_{A\le s},\mathbf{1}_{B\le t})|\\ \bar\alpha(A,B,C) &= \sup_{s,t,u\in\mathbb{R}}|\mathbb{E}((\mathbf{1}_{A\le s}-\mathbb{P}(A\le s))(\mathbf{1}_{B\le t}-\mathbb{P}(B\le t))(\mathbf{1}_{C\le u}-\mathbb{P}(C\le u)))|\end{aligned}$$

(note that $\bar\alpha(A,B,B)\le\bar\alpha(A,B)$). Let

$$A = |g_l^{(0)}(X_0)|\mathrm{sign}\{\mathbb{E}(g_q^{(0)}(X_i)g_r^{(0)}(X_j)|X_0)-\mathbb{E}(g_q^{(0)}(X_i)g_r^{(0)}(X_j))\}\,,$$

and note that $Q_A = Q_{g_l^{(0)}(X_0)}$. From Proposition 6.1 and Lemma 6.1 in Dedecker and Rio (2008), we have that

$$\begin{aligned}&\|g_l^{(0)}(X_0)(\mathbb{E}(g_q^{(0)}(X_i)g_r^{(0)}(X_j)|X_0)-\mathbb{E}(g_q^{(0)}(X_i)g_r^{(0)}(X_j)))\|_1\\ &\qquad= \mathbb{E}((A-\mathbb{E}(A))g_q^{(0)}(X_i)g_r^{(0)}(X_j))\\ &\qquad\le 16\int_0^{\bar\alpha(A,g_q(X_i),g_r(X_j))/2} Q_{g_l^{(0)}(X_0)}(u)Q_{g_q(X_0)}(u)Q_{g_r(X_0)}(u)du\,.\end{aligned}$$

Note that $Q_{g_l^{(0)}(X_0)} \le Q_{g_l(X_0)} + \|g_l(X_0)\|_1$. Hence, by Fréchet's inequality (1957),

$$\begin{aligned}&\int_0^{\bar\alpha(A,g_q(X_i),g_r(X_j))/2} Q_{g_l^{(0)}(X_0)}(u)Q_{g_q(X_0)}(u)Q_{g_r(X_0)}(u)du\\ &\qquad\le 2\int_0^{\bar\alpha(A,g_q(X_i),g_r(X_j))/2} Q_{g_l(X_0)}(u)Q_{g_q(X_0)}(u)Q_{g_r(X_0)}(u)du\,.\end{aligned}$$

Since $\{x: g_i(x)\le t\}$ is either some interval or the complement of some interval, we have that for $j>i\ge 1$

$$\bar\alpha(A,g_q(X_i),g_r(X_j)) \le 4\bar\alpha(A,X_i,X_j)\le 4\alpha_2(i)\,,$$

and for $i = j$,

$$\bar{\alpha}(A, g_q(X_i), g_r(X_i)) \le 4\bar{\alpha}(A, X_i, X_i) \le 4\bar{\alpha}(X_0, X_i) \le 4\alpha_1(i) \le 4\alpha_2(i)\,.$$

Since $Q_{g_i(X_0)}(u) \le Mu^{-1/p}$, it follows that, for $1 \le i \le j$,

$$\|g_l(X_0)(\mathbb{E}(g_q(X_i)g_r(X_j)|X_0) - \mathbb{E}(g_q(X_i)g_r(X_j)))\|_1 \le \frac{32M^3p}{p-3}(2\alpha_2(i))^{\frac{p-3}{p}}\,.$$

Consequently, for any f in $\mathcal{C}(M,p,\mu)$ with $p > 3$,

$$\|Y_0(\mathbb{E}(Y_iY_j|X_0) - \mathbb{E}(Y_iY_j))\|_1 \le \frac{32M^3p}{p-3}(2\alpha_2(i))^{\frac{p-3}{p}}\,.$$

In the same way,

$$2\sigma\|\mathbb{E}(Y_iY_j|X_0) - \mathbb{E}(Y_iY_j)\|_1 \le \frac{32\sigma M^2p}{p-2}(2\alpha_2(i))^{\frac{p-2}{p}}\,.$$

It follows that, for any $1 \le i \le j$,

$$\|(|Y_0| + 2\sigma)(\mathbb{E}(Y_iY_j|X_0) - \mathbb{E}(Y_iY_j))\|_1 \le D(M,p,\sigma)(\alpha_2(i))^{\frac{p-3}{p}}\,. \tag{5.5}$$

Combining (5.4) and (5.5), we infer that

$$\sum_{i=1}^{m}\sum_{j=i}^{m}\|(|Y_0| + 2\sigma)(\mathbb{E}(Y_iY_j|X_0) - \mathbb{E}(Y_iY_j))\|_1 = O(m^{1-\delta})\,,$$

and (5.3) easily follows. This completes the proof.

6. Moment inequalities

Theorem 6.1. *Let $\mathbf{X} = (X_i)_{i\ge 0}$ be a stationary Markov chain with invariant measure μ and transition kernel K. If f belong to $\mathcal{C}(M,p,\mu)$ for some $M > 0$ and some $p > 2$, then, for any $2 \le q < p$*

$$\Big\|\sum_{i=1}^{n}(f(X_i) - \mu(f))\Big\|_q \le \sqrt{2q}\Big(n\|f(X_0) - \mu(f)\|_q^2 + 4M^2\Big(\frac{p}{p-q}\Big)^{\frac{2}{q}}\sum_{k=1}^{n-1}(n-k)(2\alpha_1(k))^{\frac{2(p-q)}{pq}}\Big)^{\frac{1}{2}}.$$

Corollary 6.1. *Let $0 < \gamma < 1$. Let f belong to $\mathcal{C}(M,p,\nu_\gamma)$ for some $M > 0$ and some $p > 2$, and let $2 \le q < p$.*

(1) *If $\gamma < 2(p-q)/(2(p-q)+pq)$, then $\|S_n(f - \nu_\gamma(f))\|_q = O(\sqrt{n})$.*
(2) *If $2(p-q)/(2(p-q)+pq) \le \gamma < 1$, then, for any $\epsilon > 0$,*

$$\|S_n(f - \nu_\gamma(f))\|_q = O\Big(n^{1+\epsilon-\frac{(1-\gamma)(p-q)}{\gamma pq}}\Big)\,.$$

Two simple examples (continued).

(1) Assume that f is positive and non increasing on $[0,1]$, with $f(x) \le Cx^{-a}$ for some $a > 0$. If $a < \frac{1}{2} - \gamma$ and $2 \le q < \frac{2(1-\gamma)}{\gamma+2a}$, then $\|S_n(f - \nu_\gamma(f))\|_q = O(\sqrt{n})$. If now $a < \frac{1-\gamma}{2}$ and $2 \vee \frac{2(1-\gamma)}{\gamma+2a} \le q < \frac{1-\gamma}{a}$, then, for any $\epsilon > 0$,

$$\|S_n(f - \nu_\gamma(f))\|_q = O\Big(n^{1+\epsilon-\frac{(1-\gamma-aq)}{\gamma q}}\Big)\,.$$

(2) Assume that f is positive and non increasing on $[0,1]$, with $f(x) \le C(1-x)^{-a}$ for some $a \ge 0$. If $a < \frac{1-2\gamma}{2(1-\gamma)}$ and $2 \le q < \frac{2(1-\gamma)}{\gamma+(1-\gamma)2a}$, then $\|S_n(f-\nu_\gamma(f))\|_q = O(\sqrt{n})$. If $a < \frac{1}{2}$ and $2 \vee \frac{2(1-\gamma)}{\gamma+(1-\gamma)2a} \le q < \frac{1}{a}$, then, for any $\epsilon > 0$,

$$\|S_n(f-\nu_\gamma(f))\|_q = O\Big(n^{1+\epsilon-\frac{(1-\gamma)(1-aq)}{\gamma q}}\Big).$$

Proof of Theorem 6.1. From Proposition 4 in Dedecker and Doukhan (2003) (see also Theorem 2.5 in Rio (2000)), we have that, for any $q \ge 2$,

$$\Big\|\sum_{i=1}^{n}(f(X_i)-\mu(f))\Big\|_q \le \sqrt{2q}\Big(n\|f(X_0)-\mu(f)\|_q^2 + \sum_{k=1}^{n-1}(n-k)\|(f(X_0)-\mu(f))(\mathbb{E}(f(X_k)|X_0)-\mu(f))\|_{\frac{q}{2}}\Big)^{\frac{1}{2}}.$$

Assume first that $f = \sum_{i=1}^k a_i g_i$, where $\sum_{i=1}^k |a_i| \le 1$, and g_i belongs to $\mathrm{Mon}(M,p,\mu)$. Clearly

$$\|(f(X_0)-\mu(f))(\mathbb{E}(f(X_n)|X_0)-\mu(f))\|_{q/2} \le \sum_{i=1}^{k}\sum_{j=1}^{k}|a_i a_j|\|(g_i(X_0)-\mu(g_i))(\mathbb{E}(g_j(X_n)|X_0)-\mu(g_j))\|_{q/2}.$$

Applying Lemma 4.1, we obtain that

$$\|(f(X_0)-\mu(f))(\mathbb{E}(f(X_n)|X_0)-\mu(f))\|_{q/2} \le 4M^2\Big(\frac{p}{p-q}\Big)^{2/q}(2\alpha_1(n))^{\frac{2(p-q)}{pq}}..$$

Clearly, this inequality remains valid for any f in $\mathcal{C}(M,p,\mu)$, and the result follows.

7. The empirical distribution function

Theorem 7.1. *Let $\mathbf{X} = (X_i)_{i\ge 0}$ be a stationary Markov chain with invariant measure μ and transition kernel K. Let $F_n(t) = n^{-1}\sum_{i=1}^n \mathbf{1}_{X_i \le t}$ and $F_\mu(t) = \mu(]-\infty,t])$.*

(1) *If $\mathbf{X}$ is ergodic (in the ergodic theoretic sense) and if $\sum_{k>0}\beta_1(k) < \infty$, then, for any probability π on $\mathbb{R}$, the process $\{\sqrt{n}(F_n(t)-F_\mu(t)), t \in \mathbb{R}\}$ converges in distribution in $\mathbb{L}^2(\pi)$ to a tight Gaussian process G with covariance function*

$$\begin{aligned}\mathrm{Cov}(G(s),G(t)) &= C_{\mu,K}(s,t)\\ &= \mu(f_t^{(0)}f_s^{(0)}) + \sum_{k>0}\mu(f_t^{(0)}K^k f_s^{(0)}) + \sum_{k>0}\mu(f_s^{(0)}K^k f_t^{(0)}).\end{aligned}$$

(2) *Let $(D(\mathbb{R}),d)$ be the space of cadlag functions equipped with the Skorohod metric d. If $\beta_2(k) = O(k^{-2-\epsilon})$ for some $\epsilon > 0$, then the process $\{\sqrt{n}(F_n(t)-F_\mu(t)), t \in \mathbb{R}\}$ converges in distribution in $(D(\mathbb{R}),d)$ to a tight Gaussian process G with covariance function $C_{\mu,K}$.*

Corollary 7.1. *Let $F_{n,\gamma}(t) = n^{-1}\sum_{i=1}^n \mathbf{1}_{T_\gamma^i \le t}$.*

(1) *If $0 < \gamma < 1/2$, then, for any probability π on $[0,1]$, the process $\{\sqrt{n}(F_{n,\gamma}(t) - F_{\nu_\gamma}(t)), t \in [0,1]\}$ converges in distribution in $\mathbb{L}^2(\pi)$ to a tight Gaussian process G_γ with covariance function C_{ν_γ,K_γ}.*

(2) *If $0 < \gamma < 1/3$, the process $\{\sqrt{n}(F_{n,\gamma}(t) - F_{\nu_\gamma}(t)), t \in [0,1]\}$ converges in distribution in $(D([0,1]), d)$ to a tight Gaussian process G_γ with covariance function C_{ν_γ, K_γ}.*

Remark 7.1. *Denote by $\|\cdot\|_{p,\pi}$ the $\mathbb{L}^p(\pi)$-norm. If $\gamma < 1/2$, we have that, for any $1 \le p \le 2$,*

$$\sqrt{n}\|F_{n,\gamma} - F_{\nu_\gamma}\|_{p,\pi} \quad \text{converges in distribution to} \quad \|G_\gamma\|_{p,\pi}\,. \tag{7.1}$$

In particular, if $\pi = \lambda$ is the Lebesgue measure on $[0,1]$ and $q = p/(p-1)$, we obtain that

$$\frac{1}{\sqrt{n}} \sup_{\|f'\|_q \le 1} |S_n(f - \nu_\gamma(f))| \quad \text{converges in distribution to} \quad \|G_\gamma\|_{p,\lambda}\,.$$

For $p = 1$ and $q = \infty$, we obtain the limit distribution of the Kantorovič distance $d_1(F_{n,\gamma}, F_{\nu_\gamma})$:

$$\sqrt{n} d_1(F_{n,\gamma}, F_{\nu_\gamma}) = \frac{1}{\sqrt{n}} \sup_{f \in H_{1,1}} |S_n(f - \nu_\gamma(f))| \text{ converges in distribution to } \int_0^1 |G_\gamma(t)| dt\,.$$

Now if $\gamma < 1/3$, the limit in (7.1) holds for any $p \ge 1$.

Note that, for Harris recurrent Markov chains, Item (2) of Theorem 7.1 holds as soon as the sum of the β-mixing coefficients of the chain is finite. Hence, we conjecture that Item (2) of Corollary 7.1 remains true for $\gamma < 1/2$.

Proof of Theorem 7.1. Item (1) has been proved in Dedecker and Merlevède (2006, Theorem 2, Item 2) and Item (2) in Dedecker and Prieur (2007, Proposition 2).

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