Alea 4, 185–203 (2008)

Convergence to the viscous porous medium equation and propagation of chaos

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Abstract. We study a sequence of nonlinear stochastic differential equations and show that the distributions of the solutions converge to the solution of the viscous porous medium equation with exponent $m > 1$, generalizing the results of Oelschläger (2001) and Philipowski (2006) which concern the case $m = 2$. Furthermore we explain how to apply this result to the study of interacting particle systems.

1. Introduction

Let $V \in \mathcal{C}_c^2(\mathbb{R}^d)$ (twice continuously differentiable with compact support) be an even non-negative function with $\int_{\mathbb{R}^d} V(x)dx = 1$. For $m > 1$, we consider the following sequence of nonlinear stochastic differential equations in $\mathbb{R}^d$ (which in the special case $m = 2$ coincides with the model studied in Philipowski (2006)):

$$\begin{cases} dY_t^\varepsilon &= -\left[\nabla V^\varepsilon * (V^\varepsilon * u^\varepsilon(t))^{m-1}\right](Y_t^\varepsilon)dt + dB_t \\ Y_0^\varepsilon &= \zeta \\ u^\varepsilon(t) &= \text{Law}(Y_t^\varepsilon). \end{cases} \tag{1.1}$$

Here V^ε is obtained from V by the scaling

$$V^\varepsilon(x) := \frac{1}{\varepsilon^d} V(x/\varepsilon),$$

$(B_t)_{t\geq 0}$ is a d-dimensional Brownian motion, and ζ is a random variable which is independent from $(B_t)_{t\geq 0}$ and whose distribution has a density

$$u_0 \in L^\infty(\mathbb{R}^d). \tag{1.2}$$

Received by the editors September 4, 2007; accepted May 16, 2008.
2000 *Mathematics Subject Classification.* 35K55, 45K05, 60H10, 60K35, 82C22.
Key words and phrases. Nonlinear stochastic differential equations; Viscous porous medium equation; Interacting particle systems.

Defining

$$g^\varepsilon(t) := V^\varepsilon * (V^\varepsilon * u^\varepsilon(t))^{m-1},$$

by Itô's formula u^ε is a distributional solution of

$$\begin{cases} \partial_t u^\varepsilon &= \frac{1}{2}\Delta u^\varepsilon + \operatorname{div}(\nabla g^\varepsilon u^\varepsilon) \\ u^\varepsilon(0,\cdot) &= u_0. \end{cases} \tag{1.3}$$

We will show that u^ε converges, as $\varepsilon \to 0$, to the solution u of the *viscous porous medium equation*

$$\begin{cases} \partial_t u &= \frac{1}{2}\Delta u + \frac{m-1}{m}\Delta(u^m) \\ u(0,\cdot) &= u_0. \end{cases}$$

In the case $m = 2$ the proof of existence and uniqueness of a strong solution of (1.1) can be found in Sznitman (1991, Chapter I, Theorem 1.1), and that proof can be easily generalized to arbitrary $m \geq 2$ thanks to the Lipschitz continuity of the function $s \mapsto s^{m-1}$. In the case $1 < m < 2$ we will prove existence of a strong solution in Section 2.2 using an approximation argument (see Proposition 2.8).

The importance of this convergence result comes from the fact that the nonlinear stochastic differential equation (1.1) arises in the study of large interacting particle systems, and by the above convergence one can prove a propagation of chaos result, see Section 2.

1.1. *Notations and statement of the main result.* Let $\mathcal{M}(\mathbb{R}^d)$ be the the space of probability measures on $\mathbb{R}^d$, equipped with the metric

$$d(\mu,\nu) := \sup_{f\in BL} \left| \int_{\mathbb{R}^d} f(x)\mu(dx) - \int_{\mathbb{R}^d} f(x)\nu(dx) \right|,$$

where BL is the set of all Lipschitz continuous functions on $\mathbb{R}^d$ which are bounded together with their Lipschitz constant by 1. It is well known (see for example Dudley (1989)) that d metrizes the weak convergence in $\mathcal{M}(\mathbb{R}^d)$ (that is, the convergence in the duality with bounded countinuous functions).

Definition 1.1. *A weak solution of the viscous porous medium equation*

$$\begin{cases} \partial_t u &= \frac{1}{2}\Delta u + \frac{m-1}{m}\Delta(u^m) \\ u(0,\cdot) &= u_0 \end{cases} \tag{1.4}$$

on the time interval $[0,T]$ with initial datum u_0 is a measure-valued function $u \in \mathcal{C}([0,T],\mathcal{M}(\mathbb{R}^d))$ with the following properties:

(1) *For almost every $t \in [0,T]$ the measure $u(t)$ has a density with respect to Lebesgue measure (which we still denote by $u(t)$), and $u \in L^m(\mathbb{R}^d \times [0,T])$.*
(2) *For all $f \in \mathcal{C}_b^2(\mathbb{R}^d)$ and all $t \in [0,T]$:*

$$\begin{aligned} \int_{\mathbb{R}^d} f(x)u(t,x)dx &= \int_{\mathbb{R}^d} f(x)u_0(x)dx + \frac{1}{2}\int_0^t\int_{\mathbb{R}^d}\Delta f(x)u(s,x)dxds \\ &\quad + \frac{m-1}{m}\int_0^t\int_{\mathbb{R}^d}\Delta f(x)u(s,x)^m dxds. \end{aligned}$$

As we will see in the sequel, thanks to the assumption $u_0 \in L^\infty(\mathbb{R}^d)$ all weak solutions of the viscous porous medium equation belong to $L^{m+1}(\mathbb{R}^d \times [0,T])$ (and not only to $L^m(\mathbb{R}^d \times [0,T])$), and therefore are unique (see Propositions 3.9 and 3.10). One so obtains the following convergence result (whose proof is given to Section 3):

Theorem 1.2. *The sequence* $(u^\varepsilon)_{\varepsilon>0}$ *converges in* $\mathcal{C}([0,T],\mathcal{M}(\mathbb{R}^d))$ *to the unique weak solution* u^∞ *of the viscous porous medium equation with initial datum* u_0.

Remark 1.3. *In the case* $m=2$ *a similar result was proved by Oelschläger (2001), but only under the very restrictive assumption* $u_0 \in \mathcal{C}_b^\infty(\mathbb{R}^d)$. *Philipowski (2006) generalized Oelschläger's result to the case* $u_0 \in L^2(\mathbb{R}^d)$, *but only in the case* $m=2$.

2. Application to interacting particle systems

Our result is of crucial importance in the study of systems of interacting diffusions related to the porous medium equation

$$\partial_t u = \frac{m-1}{m}\,\Delta(u^m).$$

The classical application of this equation concerns the density of an ideal gas flowing through a homogeneous porous medium (see Vázquez (1999, Chapter 1.9) or Vázquez (2007, Chapter 2.1)). Let u be the density of the gas, v its velocity and p the pressure. Then we have the following physical laws:

1. Conservation of mass: $\partial_t(\varepsilon u) + \operatorname{div}(uv) = 0$
2. Equation of state: $p \propto u^\gamma$
3. Darcy's law: $v \propto -\nabla p$

Here $\varepsilon \in (0,1)$ is the porosity of the medium (which is constant because we are dealing with a homogeneous medium), and γ the polytropic exponent. Combining these equations we see that (up to a positive constant factor that can be scaled away)

$$\partial_t u = \frac{\gamma}{\gamma+1}\,\Delta(u^{\gamma+1}),$$

so that the density of the gas satisfies the porous medium equation with $m=\gamma+1$. For an introduction to flows in porous media we refer to Vázquez (1999), and for the mathematical theory and other applications of the porous medium equation to Vázquez (2007).

We have given a physical derivation of the porous medium equation based on the hypotheses of continuum mechanics. But strictly speaking, a gas is not a continuum, but consists of atoms and molecules. It is therefore desirable to find rigorous connections between this microscale and the macroscale. Knowing that on the macroscale the behaviour of the gas is described by the porous medium equation, our goal is to find a microscopic model which allows us, when the number of particles tends to infinity, to derive the porous medium equation as limit equation.

In the special case $m=2$ this problem was solved by Philipowski (2007), and the question arose whether his approach could be adapted to treat more general values of m. As we will see in the sequel this is indeed possible.

We will distinguish two cases: the easier case $m \geq 2$ and the more complicated case $1 < m < 2$.

2.1. *The case* $m \geq 2$.

We consider the following system of interacting diffusions in $\mathbb{R}^d$ (which in the special case $m=2$ coincides with the model studied by

Philipowski (2007)):

$$\begin{cases} dX_t^{N,n,i,\varepsilon,\delta} &= -\Big[\int_{\mathbb{R}^d} \nabla V^\varepsilon(y) \Big\{ \frac{1}{N} \sum_{j=1}^N V^\varepsilon (X_t^{N,i,\varepsilon,\delta} - y - X_t^{N,j,\varepsilon,\delta}) \Big\}^{m-1} dy \Big] dt \\ & \quad + \delta dB_t^i \\ X_0^{N,i,\varepsilon,\delta} &= \zeta^i. \end{cases} \tag{2.1}$$

Here $(B^i)_{i\in\mathbb{N}}$ is a sequence of independent standard Brownian motions, and $(\zeta^i)_{i\in\mathbb{N}}$ is a sequence of independent and identically distributed random variables, independent of the Brownian motions and whose distributions have density u_0 with respect to Lebesgue measure.

The particle system (2.1) depends on three parameters: $N \in \mathbb{N}$, $\varepsilon > 0$ and $\delta > 0$. N is the number of particles, ε measures the range of interaction, and δ measures the strength of the additional diffusion caused by the Brownian motions.

Let M be a fixed natural number, and let $P_t^{N,M,\varepsilon,\delta}$ be the joint distribution on $\mathbb{R}^{Md}$ of the random variables $X_t^{N,i,\varepsilon,\delta}, i = 1, \ldots, M$. Moreover let $u \in \mathcal{C}([0,T], L^1(\mathbb{R}^d)) \cap L^\infty([0,T], L^\infty(\mathbb{R}^d))$ be the unique weak solution of the Cauchy problem for the porous medium equation

$$\begin{cases} \partial_t u &= \frac{m-1}{m} \Delta(u^m) \\ u(0,\cdot) &= u_0 \end{cases}$$

(see Brézis and Crandall (1979), Bénilan and Crandall (1981), Vázquez (2007)), and denote by P_t the measure on $\mathbb{R}^d$ with density $u(t,\cdot)$. Then we have the following theorem:

Theorem 2.1 (Propagation of chaos for $m \geq 2$)**.**

$$\lim_{\delta\to 0} \lim_{\varepsilon \to 0} \lim_{N\to\infty} P_t^{N,M,\varepsilon,\delta} = P_t^{\otimes M}, \tag{2.2}$$

locally uniformly in t.

This result has the following consequences:

Corollary 2.2.

(1) *The empirical measure $\mu_t^{N,\varepsilon,\delta} = \frac{1}{N}\sum_{i=1}^N \delta_{X_t^{N,i,\varepsilon,\delta}}$ of the particle system converges weakly to P_t.*

(2) *The distribution of the position of each particle also converges to P_t.*

(3) *Any fixed number of particles remains approximately independent in the course of time, in spite of the interaction.*

Proof. The second and the third statement follow immediately from Theorem 2.1. The first statement follows from Theorem 2.1 and the general fact (see Sznitman (1991, Chapter I.2, Proposition 2.2)) that propagation of chaos is equivalent to weak convergence of the empirical measure to a deterministic measure. □

Proof of Theorem 2.1. As intermediate objects between the particle system (2.1) and the porous medium equation we introduce *nonlinear processes* $Y^{i,\varepsilon,\delta}$ ($i \in \mathbb{N}$, $\varepsilon, \delta > 0$) defined as solutions of the following nonlinear stochastic differential equations:

$$\begin{cases} dY_t^{i,\varepsilon,\delta} &= -\left[\nabla V^\varepsilon * (V^\varepsilon * u^{\varepsilon,\delta}(t))^{m-1}\right](Y_t^{i,\varepsilon,\delta})dt + \delta dB_t^i, \\ Y_0^{i,\varepsilon,\delta} &= \zeta^i \\ u^{\varepsilon,\delta}(t) &= \mathrm{Law}(Y_t^{i,\varepsilon,\delta}). \end{cases} \tag{2.3}$$

These are (up to the factor δ in front of the Brownian motions) independent copies of our process Y_t^ε defined in (1.1). In a first step we show that $X_t^{N,i,\varepsilon,\delta}$ converges (for $N \to \infty$) to $Y_t^{i,\varepsilon,\delta}$ (the proof of the following result is postponed to the Appendix):

Proposition 2.3.

$$\mathbb{E}\left[\sup_{0\le t\le T}\left|X_t^{N,i,\varepsilon,\delta} - Y_t^{i,\varepsilon,\delta}\right|^2\right] \le \frac{C(\varepsilon)}{N},$$

where the dependence of $C(\varepsilon)$ on ε is made explicit in the proof.

By the above proposition, we easily have the convergence

$$\lim_{N\to\infty} P_t^{N,m,\varepsilon,\delta} = (P_t^{\varepsilon,\delta})^{\otimes m},$$

where $P_t^{\varepsilon,\delta} = u^{\varepsilon,\delta}$ is the law of $Y_t^{i,\varepsilon,\delta}$.

Then, Theorem 1.2 implies that the distribution of $Y_t^{i,\varepsilon,\delta}$ converges (for $\varepsilon \to 0$) to the solution u^δ of the viscous porous medium equation (with viscosity $\delta^2/2$)

$$\begin{cases} \partial_t u^\delta &= \frac{\delta^2}{2}\Delta u^\delta + \frac{m-1}{m}\Delta\big((u^\delta)^m\big) \\ u^\delta(0,\cdot) &= u_0, \end{cases}$$

so that

$$\lim_{\varepsilon\to 0}\lim_{N\to\infty} P_t^{N,M,\varepsilon,\delta} = (P_t^\delta)^{\otimes M},$$

P_t^δ being the measure with density $u^\delta(t,\cdot)$. Finally, a result of Bénilan and Crandall (1981) implies that u^δ converges in $L^\infty([0,T],L^1(\mathbb{R}^d))$ to the solution u of the porous medium equation as $\delta \to 0$, from which (2.2) follows. □

2.2. *The case* $1 < m < 2$. This case is more difficult since the function $s \mapsto s^{m-1}$ is not locally Lipschitz continuous. We therefore replace it with a Lipschitz continuous approximation φ_n (i.e. $(\varphi_n)_{n\in\mathbb{N}}$ is a sequence of non-negative Lipschitz continuous functions that converges uniformly to the function $s \mapsto s^{m-1}$) and study the following system:

$$\begin{cases} dX_t^{N,n,i,\varepsilon,\delta} = -\left[\int_{\mathbb{R}^d} \nabla V^\varepsilon(y)\,\varphi_n\Big(\frac{1}{N}\sum_{j=1}^N V^\varepsilon(X_t^{N,n,i,\varepsilon,\delta} - y - X_t^{N,n,j,\varepsilon,\delta})\Big)dy\right]dt \\ \qquad\qquad + \delta dB_t^i \\ X_0^{N,n,i,\varepsilon,\delta} = \zeta^i. \end{cases} \tag{2.4}$$

As before let M be a fixed natural number, and let $P_t^{N,M,n,\varepsilon,\delta}$ be the joint distribution of the random variables $X_t^{N,i,n,\varepsilon,\delta}, i = 1,\dots,M$. Moreover let P_t be the measure with density $u(t,\cdot)$ on $\mathbb{R}^d$, where $u \in \mathcal{C}([0,T],L^1(\mathbb{R}^d)) \cap L^\infty([0,T],L^\infty(\mathbb{R}^d))$ is the unique weak solution of the porous medium equation

$$\begin{cases} \partial_t u &= \frac{m-1}{m}\Delta(u^m) \\ u(0,\cdot) &= u_0. \end{cases}$$

Then we have the following:

Theorem 2.4 (Propagation of chaos for $1 < m < 2$)**.**

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\operatorname*{lim^*}_{n\to\infty}\lim_{N\to\infty} P_t^{N,M,n,\varepsilon,\delta} = P_t^{\otimes M}, \tag{2.5}$$

locally uniformly in t*. Here* $\lim^*_{n\to\infty}$ *denotes the limit of any converging subsequence of a precompact sequence.*

Remark 2.5. *This theorem means the following:*

(1) *For each* $\varepsilon, \delta > 0$ *the sequence* $(\lim_{N\to\infty} P_t^{N,M,n,\varepsilon,\delta})_{n\in\mathbb{N}}$ *is tight.*
(2) *Each accumulation point of this sequence (denoted by* $\lim^*_{n\to\infty} \lim_{N\to\infty} P_t^{N,M,n,\varepsilon,\delta}$*) satisfies* $\lim_{\delta\to 0} \lim_{\varepsilon\to 0} \lim^*_{n\to\infty} \lim_{N\to\infty} P_t^{N,M,n,\varepsilon,\delta} = P_t^{\otimes M}$*. This means that although the limit that we obtain after letting* $n \to \infty$ *might be non-unique, the whole limit is unique.*

Proof of Theorem 2.4. We define nonlinear processes $Y^{n,i,\varepsilon,\delta}$ ($i \in \mathbb{N}$, $\varepsilon, \delta > 0$) as solutions of the following nonlinear stochastic differential equations:

$$\begin{cases} dY_t^{n,i,\varepsilon,\delta} &= -\left[\nabla V^\varepsilon * \varphi_n(V^\varepsilon * u^{n,\varepsilon,\delta}(t))\right](Y_t^{n,i,\varepsilon,\delta})dt + \delta dB_t^i, \\ Y_0^{n,i,\varepsilon,\delta} &= \zeta^i \\ u^{n,\varepsilon,\delta}(t) &= \mathrm{Law}(Y_t^{n,i,\varepsilon,\delta}). \end{cases} \tag{2.6}$$

As in Proposition 2.3 (the proof is exactly the same, just write φ_n in place of the function $s \mapsto s^{m-1}$) one can show that $X_t^{N,n,i,\varepsilon,\delta}$ converges (for $N \to \infty$) to $Y_t^{n,i,\varepsilon,\delta}$:

Proposition 2.6.

$$\mathbb{E}\left[\sup_{0\le t\le T}\left|X_t^{N,n,i,\varepsilon,\delta} - Y_t^{n,i,\varepsilon,\delta}\right|^2\right] \le \frac{C(\varepsilon,n)}{N}.$$

As in the preceding subsection this proposition implies

$$\lim_{N\to\infty} P_t^{N,M,n,\varepsilon,\delta} = (P_t^{n,\varepsilon,\delta})^{\otimes M},$$

where $P_t^{n,\varepsilon,\delta} = u^{n,\varepsilon,\delta}(t)$ is the law of $Y_t^{n,i,\varepsilon,\delta}$. We now let $n \to \infty$. Since the processes $Y^{n,i,\varepsilon,\delta}$ are (for different i) independent copies of each other, we can omit the index i. Let $u^{n,\varepsilon,\delta}$ be the law of the process $Y^{n,\varepsilon,\delta}$, considered as an element of $\mathcal{M}(\mathcal{C}([0,T],\mathbb{R}^d))$ (so that $u^{n,\varepsilon,\delta}(t)$ is its marginal at time t).

Lemma 2.7. *The sequence* $(u^{n,\varepsilon,\delta})_{n\in\mathbb{N}}$ *is tight.*

Proof. In order to apply Theorem 1.4.6 of Stroock and Varadhan (1979) we have to show that for all non-negative $f \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ there is a constant $A_f \ge 0$ such that for all $n \in \mathbb{N}$ and all $x_0 \in \mathbb{R}^d$ the process $f(Y_t^{n,\varepsilon,\delta} + x_0) + A_f t$ is a non-negative submartingale. To do so we first observe (using Itô's formula) that, for all $f \in \mathcal{C}_b^2(\mathbb{R}^d)$, the process

$$f(Y_t^{n,\varepsilon,\delta} + x_0) - \int_0^t \Big\{ -\left[\nabla V^\varepsilon * \varphi_n(V^\varepsilon * u^{n,\varepsilon,\delta}(s))\right](Y_s^{n,i,\varepsilon,\delta}) \cdot \nabla f(Y_s^{n,i,\varepsilon,\delta} + x_0) + \frac{\delta^2}{2}\Delta f(Y_s^{n,i,\varepsilon,\delta} + x_0)\Big\} ds$$

is a martingale. Moreover, since $\varphi_n(s) \to s^{m-1}$ uniformly, we can assume $\varphi_n(s) \le 1 \vee s$ for all $n \in \mathbb{N}$. It is then clear that we can take $A_f := \left[\|\nabla V^\varepsilon\|_{L^1(\mathbb{R}^d)}(1 \vee \|V^\varepsilon\|_{L^\infty(\mathbb{R}^d)}) + \delta^2/2\right]\|f\|_{\mathcal{C}^2(\mathbb{R}^d)}$. □

Let $u^{\varepsilon,\delta}$ be an accumulation point of the sequence $(u^{n,\varepsilon,\delta})_{n\in\mathbb{N}}$ in $\mathcal{M}(\mathcal{C}([0,T],\mathbb{R}^d))$.

Proposition 2.8. *Up to a subsequence, the sequence of processes* $(Y^{n,\varepsilon,\delta})_{n\in\mathbb{N}}$ *converges almost surely to the strong solution* $Y^{\varepsilon,\delta}$ *of the equation*

$$\left\{\begin{array}{lcl} dY_t^{\varepsilon,\delta} & = & -\left[\nabla V^\varepsilon * (V^\varepsilon * u^{\varepsilon,\delta}(t))^{m-1}\right](Y_t^{\varepsilon,\delta})dt + \delta dB_t \\ Y_0^{\varepsilon,\delta} & = & \zeta, \end{array}\right.$$

i.e. $\sup_{0\leq t\leq T}\left|Y_t^{n,\varepsilon,\delta} - Y_t^{\varepsilon,\delta}\right| \to 0$ *a.s. for* $n\to\infty$. *Moreover* $u^{\varepsilon,\delta}(t)$ *equals the law of* $Y_t^{\varepsilon,\delta}$*, so that* $Y^{\varepsilon,\delta}$ *is in fact a strong solution of the nonlinear stochastic differential equation*

$$\left\{\begin{array}{lcl} dY_t^{\varepsilon,\delta} & = & -\left[\nabla V^\varepsilon * (V^\varepsilon * u^{\varepsilon,\delta}(t))^{m-1}\right](Y_t^{\varepsilon,\delta})dt + \delta dB_t \\ Y_0^{\varepsilon,\delta} & = & \zeta \\ u^{\varepsilon,\delta}(t) & = & \mathrm{Law}(Y_t^{\varepsilon,\delta}). \end{array}\right.$$

Proof. The weak convergence of $u^{n,\varepsilon,\delta}$ to $u^{\varepsilon,\delta}$ together with the uniform convergence of φ_n to the function $s\mapsto s^{m-1}$ implies uniform convergence of the drift coefficient $\nabla V^\varepsilon * \varphi_n(V^\varepsilon * u^{n,\varepsilon,\delta}(t))$ to $\nabla V^\varepsilon * (V^\varepsilon * u^{\varepsilon,\delta}(t))^{m-1}$, and the first statement follows immediately (use Gronwall's lemma and the Lipschitz continuity of $x\mapsto \nabla V^\varepsilon * (V^\varepsilon * u^{\varepsilon,\delta}(t))^{m-1}(x)$). The second statement is also clear because $u^{\varepsilon,\delta}(t)$ is the weak limit of $u^{n,\varepsilon,\delta}(t) = \mathrm{Law}(Y_t^{n,\varepsilon,\delta})$, and $Y_t^{\varepsilon,\delta}$ is the limit of $Y_t^{n,\varepsilon,\delta}$ for the almost sure convergence. □

Lemma 2.7 and Proposition 2.8 together imply that $\lim^*_{n\to\infty}\lim_{N\to\infty} P_t^{N,M,n,\varepsilon,\delta} = (P_t^{\varepsilon,\delta})^{\otimes M}$, where $P_t^{\varepsilon,\delta} = u^{\varepsilon,\delta}(t)$ is the law of $Y_t^{i,\varepsilon,\delta}$. Now we conclude, as in the preceding subsection, using Theorem 1.2 and the result of Bénilan and Crandall (1981). □

3. Proof of Theorem 1.2

We introduce the following smoothed version of u^ε

$$v^\varepsilon(t,x) := (u^\varepsilon(t) * V^\varepsilon)(x).$$

Observe that, since $u^\varepsilon(t)$ is a probability measure for all $t\in[0,T]$ and $V\in\mathcal{C}_c^2(\mathbb{R}^d)$, we have

$$v^\varepsilon, \nabla v^\varepsilon, D^2 v^\varepsilon \in L^\infty([0,T], L^1(\mathbb{R}^d)\cap\mathcal{C}(\mathbb{R}^d)). \tag{3.1}$$

Moreover v^ε solves

$$\left\{\begin{array}{lcl} \partial_t v^\varepsilon & = & \frac{1}{2}\Delta v^\varepsilon + \mathrm{div}(\nabla g^\varepsilon u^\varepsilon) * V^\varepsilon = \frac{1}{2}\Delta v^\varepsilon + (\nabla g^\varepsilon u^\varepsilon) * \nabla V^\varepsilon \\ v^\varepsilon(0,\cdot) & = & u_0 * V^\varepsilon. \end{array}\right. \tag{3.2}$$

Thus, since the right hand side of (3.2) belongs to $L^\infty([0,T], L^1(\mathbb{R}^d)\cap\mathcal{C}(\mathbb{R}^d))$, we also have

$$\partial_t v^\varepsilon \in L^\infty([0,T], L^1(\mathbb{R}^d)\cap\mathcal{C}(\mathbb{R}^d)). \tag{3.3}$$

We also remark that, with these notations,

$$g^\varepsilon = V^\varepsilon * (v^\varepsilon)^{m-1}. \tag{3.4}$$

The strategy of the proof is the following: first, in Lemma 3.1, we prove some a priori bounds on v^ε and g^ε which allow to show the tightness of both sequences $(u^\varepsilon)_{\varepsilon>0}$ and $(v^\varepsilon)_{\varepsilon>0}$, and that up to a subsequence they converge in $\mathcal{C}([0,T],\mathcal{M}(\mathbb{R}^d))$ to the same limit u^∞ (see Proposition 3.3 and Lemma 3.4). Then we take advance of the regularizing effect of the heat kernel to prove some stronger a priori estimates on v^ε (see Lemma 3.5), which give that v^ε converges to u^∞ strongly in $L^m([0,T]\times\mathbb{R}^d)$

(see Lemma 3.6). This fact allows to pass to the limit in the non-linear term of the equation and to show that u^∞ is a weak solution of the viscous porous medium equation (see Proposition 3.8). Finally we prove uniqueness of weak solutions (see Propositions 3.9 and 3.10), which implies that the whole sequence $(u^\varepsilon)_{\varepsilon>0}$ converges to u^∞.

Lemma 3.1. *For each $t \geq 0$:*

$$\|v^\varepsilon(t,\cdot)\|^m_{L^m(\mathbb{R}^d)} + \frac{m(m-1)}{2}\int_0^t\int_{\mathbb{R}^d}|\nabla v^\varepsilon(s,x)|^2 v^\varepsilon(s,x)^{m-2}dx\,ds + m\int_0^t\int_{\mathbb{R}^d}|\nabla g^\varepsilon(s,x)|^2 u^\varepsilon(s,dx)\,ds = \|v^\varepsilon(0,\cdot)\|^m_{L^m(\mathbb{R}^d)} \tag{3.5}$$

Remark 3.2. *Since $v^\varepsilon(0,\cdot) = u_0 * V^\varepsilon$, $\|V^\varepsilon\|_{L^1(\mathbb{R}^d)} = 1$ and $u_0 \in L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d) \subset L^m(\mathbb{R}^d)$, we have*

$$\|v^\varepsilon(0,\cdot)\|_{L^m(\mathbb{R}^d)} \leq \|u_0\|_{L^m(\mathbb{R}^d)} < \infty.$$

Therefore Lemma 3.1 implies that each of the three terms on the left hand side of (3.5) is bounded uniformly in ε and t.

Proof of Lemma 3.1. Multiplying (3.2) by $(v^\varepsilon)^{m-1}$ and integrating this identity in space (which is admissible by (3.1) and (3.3)) we have

$$\begin{aligned}\frac{1}{m}\frac{d}{dt}\int_{\mathbb{R}^d} v^\varepsilon(t,x)^m dx &= \frac{1}{2}\int_{\mathbb{R}^d} v^\varepsilon(t,x)^{m-1}\Delta v^\varepsilon(t,x)dx \\ &+ \int_{\mathbb{R}^d} v^\varepsilon(t,x)^{m-1}[(\nabla g^\varepsilon u^\varepsilon) * \nabla V^\varepsilon](t,x)dx.\end{aligned}$$

Observe now that, since V^ε is even, ∇V^ε is odd, and thus

$$\int_{\mathbb{R}^d} b(x)[a * \nabla V^\varepsilon](x)dx = -\int_{\mathbb{R}^d}[\nabla V^\varepsilon * b](x)a(x)dx$$

for any a, b (provided everything is well-defined). Thus, by this fact and (3.4), we get

$$\begin{aligned}\frac{1}{m}\frac{d}{dt}\int_{\mathbb{R}^d} v^\varepsilon(t,x)^m dx &= \frac{1}{2}\int_{\mathbb{R}^d} v^\varepsilon(t,x)^{m-1}\Delta v^\varepsilon(t,x)dx \\ &- \int_{\mathbb{R}^d}[\nabla V^\varepsilon * (v^\varepsilon)^{m-1}](t,x)\nabla g^\varepsilon(t,x)u^\varepsilon(t,dx) \\ &= -\frac{m-1}{2}\int_{\mathbb{R}^d}|\nabla v^\varepsilon(t,x)|^2 v^\varepsilon(t,x)^{m-2}dx \\ &- \int_{\mathbb{R}^d}|\nabla g^\varepsilon(t,x)|^2 u^\varepsilon(t,dx).\end{aligned}$$

Integrating in time, the thesis follows. □

Proposition 3.3. *The set $(u^\varepsilon)_{\varepsilon>0}$ is relatively compact in $\mathcal{C}([0,T],\mathcal{M}(\mathbb{R}^d))$.*

Proof. In order to apply the Ascoli-Arzelà theorem we have to show:

(1) There is a compact set $\mathcal{K} \subset \mathcal{M}(\mathbb{R}^d)$ with $u^\varepsilon(t) \in \mathcal{K}$ for all $\varepsilon > 0$ and all $t \in [0,T]$.
(2) The set $\{u^\varepsilon \,|\, \varepsilon > 0\}$ is equicontinuous, i.e. for each $\eta > 0$ there exists $\delta 0$ such that, for all $\varepsilon > 0$ and all $s, t \in [0,T]$,

$$|s-t| \leq \delta \quad \Rightarrow \quad d(u^\varepsilon(s), u^\varepsilon(t)) \leq \eta.$$

We start with the first statement. Since a subset $\mathcal{K}$ of $\mathcal{M}(\mathbb{R}^d)$ is relatively compact if and only if it is tight, we have to show that for each $\eta > 0$ there exists a compact set $K \subset \mathbb{R}^d$ with $u^\varepsilon(t,K) \geq 1-\eta$ (or equivalently $\mathbb{P}\left[Y_t^\varepsilon \in K^c\right] \leq \eta$) for all $\varepsilon > 0$ and all $t \in [0,T]$.

Let $R > 0$. Then we have:

$$\begin{aligned}
\mathbb{P}\left[|Y_t^\varepsilon| > R\right] &= \mathbb{P}\left[\left|\zeta - \int_0^t \nabla g^\varepsilon(Y_s^\varepsilon, s)ds + B_t\right| > R\right] \\
&\leq \mathbb{P}\left[|\zeta| > \frac{R}{3}\right] + \mathbb{P}\left[\left|\int_0^t \nabla g^\varepsilon(Y_s^\varepsilon, s)ds\right| > \frac{R}{3}\right] + \mathbb{P}\left[|B_t| > \frac{R}{3}\right].
\end{aligned}$$

The first and the third term tend (for $R \to \infty$) to 0, uniformly in ε and $t \in [0,T]$. For the second term we obtain, using Chebyshev's inequality,

$$\begin{aligned}
\mathbb{P}\left[\left|\int_0^t \nabla g^\varepsilon(Y_s^\varepsilon, s)ds\right| > \frac{R}{3}\right] &\leq \frac{9}{R^2}\mathbb{E}\left[\left|\int_0^t \nabla g^\varepsilon(Y_s^\varepsilon, s)ds\right|^2\right] \\
&\leq \frac{9t}{R^2}\mathbb{E}\left[\int_0^t \left|\nabla g^\varepsilon(Y_s^\varepsilon, s)\right|^2 ds\right] \\
&= \frac{9t}{R^2}\int_0^t \int_{\mathbb{R}^d} \left|\nabla g^\varepsilon(x,s)\right|^2 u^\varepsilon(s,dx)ds,
\end{aligned}$$

and due to Lemma 3.1 this also tends (for $R \to \infty$) to 0, uniformly in ε and t. This completes the proof of the first statement.

We now prove the second statement. For $s, t \in [0,T]$ we obtain (using Lemma 3.1)

$$\begin{aligned}
d(u^\varepsilon(s), u^\varepsilon(t)) &= \sup_{f \in BL}\left|\int_{\mathbb{R}^d} f(x)u^\varepsilon(t,dx) - \int_{\mathbb{R}^d} f(x)u^\varepsilon(s,dx)\right| \\
&= \sup_{f \in BL}\left|\mathbb{E}\left[f(Y_t^\varepsilon)\right] - \mathbb{E}\left[f(Y_s^\varepsilon)\right]\right| \\
&\leq \mathbb{E}\left[|Y_t^\varepsilon - Y_s^\varepsilon|^2\right]^{1/2} \\
&= \mathbb{E}\left[\left|-\int_s^t \nabla g^\varepsilon(Y_r^\varepsilon, r)dr + B_t - B_s\right|^2\right]^{1/2} \\
&\leq \mathbb{E}\left[\left|-\int_s^t \nabla g^\varepsilon(Y_r^\varepsilon, r)dr\right|^2\right]^{1/2} + \mathbb{E}\left[\left|B_t - B_s\right|^2\right]^{1/2} \\
&\leq \mathbb{E}\left[|t-s|\int_s^t |\nabla g^\varepsilon(Y_r^\varepsilon, r)|^2 dr\right]^{1/2} + |t-s|^{1/2} \\
&= |t-s|^{1/2}\left(\left[\int_s^t \int_{\mathbb{R}^d} |\nabla g^\varepsilon(x,r)|^2 u^\varepsilon(r,dx)dr\right]^{1/2} + 1\right) \\
&\leq C|t-s|^{1/2}.
\end{aligned}$$

This means that $(u^\varepsilon)_{\varepsilon>0}$ is equicontinuous, so that the lemma is proved. □

We have shown that the sequence $(u^\varepsilon)_{\varepsilon>0}$ has a convergent subsequence. We now fix such a convergent subsequence, which we still denote by $(u^\varepsilon)_{\varepsilon>0}$. Let $u^\infty \in \mathcal{C}([0,T], \mathcal{M}(\mathbb{R}^d))$ be its limit.

Lemma 3.4. *The sequence $(v^\varepsilon)_{\varepsilon>0}$ also converges in $\mathcal{C}([0,T], \mathcal{M}(\mathbb{R}^d))$ to u^∞.*

Proof. The result is a simple consequence of the fact that

$$\sup_{0\le t\le T} d(u^\varepsilon(t), v^\varepsilon(t)) \to 0$$

as $\varepsilon \to 0$. To prove this, observe that for any $t \in [0,T]$ and $f \in BL$ we have

$$\begin{aligned}
& \left| \int_{\mathbb{R}^d} f(x) v^\varepsilon(x)dx - \int_{\mathbb{R}^d} f(x) u^\varepsilon(t,dx) \right| \\
= & \left| \int_{\mathbb{R}^d} f(x) (u^\varepsilon(t) * V^\varepsilon)(x)dx - \int_{\mathbb{R}^d} f(x) u^\varepsilon(t,dx) \right| \\
= & \left| \int_{\mathbb{R}^d} [(f * V^\varepsilon)(x) - f(x)] u^\varepsilon(t,dx) \right| \\
\le & \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |f(x+y) - f(y)|\, V^\varepsilon(x)dx \right) u^\varepsilon(t,dy) \\
\le & \int_{\mathbb{R}^d} |x| V^\varepsilon(x)dx \int_{\mathbb{R}^d} u^\varepsilon(t,dy) \\
= & \ \varepsilon \int_{\mathbb{R}^d} |x| V(x)dx = C\varepsilon.
\end{aligned}$$

□

Since by Lemma 3.1 the sequence $(v^\varepsilon)_{\varepsilon>0}$ is bounded in $L^\infty([0,T], L^m(\mathbb{R}^d))$, up to a subsequence it weakly* converges in $L^\infty([0,T], L^m(\mathbb{R}^d))$. Therefore by the above lemma we get that $u^\infty \in L^\infty([0,T], L^m(\mathbb{R}^d))$. We now want to prove a strong convergence result.

To this aim, we introduce a fractional-type Sobolev space X_α, for $0 < \alpha < 1$:

$$X_\alpha := \left\{ w \in L^1(\mathbb{R}^d) \;\middle|\; \sup_{0<|h|\le 1} \frac{\|w(\cdot + h) - w(\cdot)\|_{L^1(\mathbb{R}^d)}}{|h|^\alpha} < +\infty \right\}$$

(this space coincides with the space $\Lambda_\alpha^{1,\infty}$, see Stein (1970, Paragraph V.5)). It is simple to check that this is a Banach space endowed with the norm

$$\|w\|_{X_\alpha} := \|w\|_{L^1(\mathbb{R}^d)} + \sup_{0<|h|\le 1} \frac{\|w(\cdot + h) - w(\cdot)\|_{L^1(\mathbb{R}^d)}}{|h|^\alpha}.$$

By the Riesz-Fréchet-Kolmogorov Theorem (see Brézis (1983, Theorem IV.25)), any bounded subset of X_α is compact in $L^1(\Omega)$ for any bounded domain $\Omega \subset \mathbb{R}^d$.

To prove our strong convergence result, we need a uniform bound on v^ε in $L^1([0,T], X_\alpha)$.

Lemma 3.5. *The sequence $(v^\varepsilon)_{\varepsilon>0}$ is uniformly bounded in $L^1([0,T], X_\alpha)$ for any $0 < \alpha < 1$.*

Proof. Observe that by (3.2), (3.1) and (3.3), v^ε is a smooth bounded solution of the parabolic equation

$$\begin{cases} \partial_t v^\varepsilon &= \frac{1}{2}\Delta v^\varepsilon + \operatorname{div} f^\varepsilon \\ v^\varepsilon(0,\cdot) &= u_0 * V^\varepsilon, \end{cases}$$

with $f^\varepsilon := (\nabla g^\varepsilon u^\varepsilon) * V^\varepsilon$. Therefore it is well-known that v^ε is given by

$$\begin{aligned} v^\varepsilon(t) &= \Gamma(t) * v^\varepsilon(0) + \int_0^t (\Gamma(t-s) * \operatorname{div} f^\varepsilon(s))ds \\ &= \Gamma(t) * v^\varepsilon(0) + \int_0^t (\nabla\Gamma(t-s) * f^\varepsilon(s))ds, \end{aligned} \tag{3.6}$$

where $\Gamma(t,x)$ is the heat kernel given by

$$\Gamma(t,x) := \begin{cases} \frac{1}{(2\pi t)^{d/2}} e^{-\frac{|x|^2}{2t}} & \text{for } t > 0, \\ \delta_x & \text{for } t = 0 \end{cases}$$

(see for instance Ladyženskaja et al. (1967)). Moreover, the following estimates are true:

$$\Gamma, \nabla\Gamma \in L^1([0,T], X_\alpha) \quad \forall\ 0 < \alpha < 1.$$

Indeed, by a direct computation one has

$$\|\Gamma(t,\cdot)\|_{L^1(\mathbb{R}^d)} = 1, \quad \|\nabla\Gamma(t,\cdot)\|_{L^1(\mathbb{R}^d)} \le \frac{C}{\sqrt{t}}, \quad \|D^2\Gamma(t,\cdot)\|_{L^1(\mathbb{R}^d)} \le \frac{C}{t}.$$

These estimates immediately give $\Gamma \in L^1([0,T], X_\alpha)$, $\nabla\Gamma \in L^1([0,T] \times \mathbb{R}^d)$. Moreover one has

$$\begin{aligned} & \|\nabla\Gamma(t,\cdot+h) - \nabla\Gamma(t,\cdot)\|_{L^1(\mathbb{R}^d)} \\ = \ & \|\nabla\Gamma(t,\cdot+h) - \nabla\Gamma(t,\cdot)\|^\alpha_{L^1(\mathbb{R}^d)} \|\nabla\Gamma(t,\cdot+h) - \nabla\Gamma(t,\cdot)\|^{1-\alpha}_{L^1(\mathbb{R}^d)} \\ \le \ & |h|^\alpha \|D^2\Gamma(t,\cdot)\|^\alpha_{L^1(\mathbb{R}^d)} \left(2\|\nabla\Gamma(t,\cdot)\|_{L^1(\mathbb{R}^d)}\right)^{1-\alpha} \\ \le \ & \frac{C}{t^{\frac{1+\alpha}{2}}} |h|^\alpha, \end{aligned}$$

so that $\nabla\Gamma \in L^1([0,T], X_\alpha)$ for any $0 < \alpha < 1$. We now remark that, since

$$\begin{aligned} \|f^\varepsilon\|_{L^1([0,T]\times\mathbb{R}^d)} &\le \int_0^T \int_{\mathbb{R}^d} |\nabla g^\varepsilon(t,x)| u^\varepsilon(t,dx)\, dt \\ &\le \left(T \int_0^T \int_{\mathbb{R}^d} |\nabla g^\varepsilon(t,x)|^2 u^\varepsilon(t,dx)\, dt\right)^{1/2}, \end{aligned}$$

by Lemma 3.1 f^ε is uniformly bounded in $L^1([0,T] \times \mathbb{R}^d)$. We therefore easily deduce from (3.6) that v^ε is uniformly bounded in $L^1([0,T], X_\alpha)$, as wanted. □

Lemma 3.6. *We have $v^\varepsilon \to u^\infty$ in $L^m([0,T] \times \mathbb{R}^d)$.*

Remark 3.7. *Observe that, since v^ε is bounded in $L^\infty([0,T], L^m(\mathbb{R}^d))$, the lemma implies also that $v^\varepsilon \to u^\infty$ in $L^p([0,T], L^m(\mathbb{R}^d))$ for all $p < \infty$. Indeed, if $v^\varepsilon \to u^\infty$ in $L^m([0,T] \times \mathbb{R}^d)$, then up to a subsequence $\|v^\varepsilon(t,\cdot) - u^\infty(t,\cdot)\|_{L^m(\mathbb{R}^d)} \to 0$ for almost every $t \in [0,T]$. This fact and Lebesgue's dominated convergence theorem give the strong convergence in $L^p([0,T], L^m(\mathbb{R}^d))$ for all $p < \infty$.*

Proof. We first remark that

$$\frac{m^2}{4} \int_0^t \int_{\mathbb{R}^d} |\nabla v^\varepsilon(t,x)|^2 v^\varepsilon(t,x)^{m-2} dx\, ds = \int_0^t \int_{\mathbb{R}^d} |\nabla (v^\varepsilon)^{m/2}(t,x)|^2 dx\, ds,$$

so that by Lemma 3.1 the sequence $(v^\varepsilon)^{m/2}$ is bounded in $L^2([0,T], H^1(\mathbb{R}^d))$.

We now claim that it suffices to prove the convergence result in $L^1([0,T]\times B_R)$ for any fixed $R>0$. Indeed, by Lemma 3.4 the measures $v^\varepsilon(t,\cdot)$ are uniformly tight in t and ε. Thus, for any $\eta>0$ there exists R_η such that

$$\int_0^T\int_{B^c_{R_\eta}} v^\varepsilon(t,x)dx\,dt \le \eta. \tag{3.7}$$

Moreover we observe that, if $w\in H^1(\mathbb{R}^d)$, then $w^2\in W^{1,1}(\mathbb{R}^d)\subset L^{d/(d-1)}(R^d)$, with the convention $d/(d-1)=\infty$ if $d=1$ (see Adams (1975)). Therefore, since $(v^\varepsilon)^{m/2}$ is bounded in $L^2([0,T],H^1(\mathbb{R}^d))$, we obtain that $(v^\varepsilon)^m$ is bounded in $L^2([0,T],L^{d/(d-1)}(\mathbb{R}^d))$. Thus, if $d\ge 2$, by the inclusion $L^2([0,T],L^{d/(d-1)}(\mathbb{R}^d))\subset L^{d/(d-1)}([0,T]\times\mathbb{R}^d)$ we have

$$\int_0^T\int_{\mathbb{R}^d} v^\varepsilon(t,x)^{\frac{md}{d-1}}dx\,dt \le C \tag{3.8}$$

for a certain C independent of ε. So by (3.7), (3.8) and Hölder's inequality, we get

$$\begin{aligned}
&\int_0^T\int_{B^c_{R_\eta}} v^\varepsilon(t,x)^m dx\,dt\\
\le\ &\left(\int_0^T\int_{B^c_{R_\eta}} v^\varepsilon(t,x)dx\,dt\right)^{\frac{m}{1-d+md}}\left(\int_0^T\int_{B^c_{R_\eta}} v^\varepsilon(t,x)^{\frac{md}{d-1}}dx\,dt\right)^{\frac{(m-1)(d-1)}{1-d+md}}\\
\le\ &C\eta^{\frac{m}{1-d+md}}
\end{aligned} \tag{3.9}$$

(the case $d=1$ is also simpler thanks to the boundness of v^ε in $L^2([0,T],L^\infty(\mathbb{R}^d))$. From (3.9) and the uniform integrability of $(v^\varepsilon)^m$ (which is a simple consequence of the uniform bound of $(v^\varepsilon)^m$ in $L^2([0,T],L^{d/(d-1)}(\mathbb{R}^d))$), the claim easily follows.

To conclude the proof, we now show the strong convergence of v^ε to u^∞ in $L^1([0,T]\times B_R)$. Fix $0<\alpha<1$, and take $s>0$ big enough so that $\mathcal{M}(B_R)$ (the space of probability measures on B_R endowed with the weak topology) continuously embeds into $H^{-s}(B_R)$ ($H^{-s}(B_R)$ being the dual space of $H^s(B_R)$). We claim that for any $\delta>0$ there exists a constant C_δ such that, for any smooth function f on $\mathbb{R}^d$, we have

$$\|f\|_{L^1(B_R)}\le\delta\|f\|_{X_\alpha}+C_\delta\|f\|_{H^{-s}(B_R)}. \tag{3.10}$$

Indeed, if not, there would exist a $\delta>0$ and a sequence of functions $(f_k)_{k\in\mathbb{N}}$, such that

$$\|f_k\|_{X_\alpha}=1,\qquad \|f_k\|_{L^1(B_R)}\ge\delta+k\|f_k\|_{H^{-s}(B_R)}. \tag{3.11}$$

As we remarked before, X_α compactly embeds into $L^1(B_R)$, thus there exist a subsequence (still denoted by f_k) such that $f_k\to g$ in $L^1(B_R)$. Since

$$\|f_k\|_{L^1(B_R)}\le\|f_k\|_{X_\alpha}=1,$$

by (3.11) we get that $f_k\to 0$ in $H^{-s}(B_R)$, so that $g=0$. But, on the other hand,

$$\|g\|_{L^1(B_R)}=\lim_k\|f_k\|_{L^1(B_R)}\ge\delta>0,$$

a contradiction.

Applying (3.10) to $v^\varepsilon(t,\cdot) - v^{\tilde{\varepsilon}}(t,\cdot)$ and integrating in time, we get

$$\begin{aligned}
\|v^\varepsilon - v^{\tilde{\varepsilon}}\|_{L^1([0,T]\times B_R)} &\leq \delta\|v^\varepsilon - v^{\tilde{\varepsilon}}\|_{L^1([0,T],X_\alpha)} \\
&+ C_\delta\|v^\varepsilon - v^{\tilde{\varepsilon}}\|_{L^1([0,T],H^{-s}(B_R))} \\
&\leq \delta\left(\|v^\varepsilon\|_{L^1([0,T],X_\alpha)} + \|v^{\tilde{\varepsilon}}\|_{L^1([0,T],X_\alpha)}\right) \\
&+ C_\delta\|v^\varepsilon - v^{\tilde{\varepsilon}}\|_{L^1([0,T],H^{-s}(B_R))} \\
&\leq C\left(\delta + C_\delta \int_0^T d(v^\varepsilon(t), v^{\tilde{\varepsilon}}(t)\,dt\right),
\end{aligned}$$

where in the last step we used that the sequence $(v^\varepsilon)_{\varepsilon>0}$ is bounded in $L^1([0,T],X_\alpha)$ and that $\mathcal{M}(B_R)) \hookrightarrow H^{-s}(B_R)$ continuously. Since, by Lemma 3.4, $(v^\varepsilon)_{\varepsilon>0}$ is a Cauchy sequence in $\mathcal{C}([0,T],\mathcal{M}(\mathbb{R}^d))$, we finally obtain

$$\limsup_{\varepsilon,\tilde{\varepsilon}\to 0} \|v^\varepsilon - v^{\tilde{\varepsilon}}\|_{L^1([0,T]\times B_R)} \leq C\delta,$$

which implies that $(v^\varepsilon)_{\varepsilon>0}$ is a Cauchy sequence in $L^1([0,T]\times B_R)$ by the arbitrariness of δ. □

Proposition 3.8. *For all $t \in [0,T]$ and all $f \in \mathcal{C}_b^2(\mathbb{R}^d)$ we have:*

$$\begin{aligned}
\int_{\mathbb{R}^d} f(x)u^\infty(t,x)dx &= \int_{\mathbb{R}^d} f(x)u_0(x)dx + \frac{1}{2}\int_0^t\int_{\mathbb{R}^d}\Delta f(x)u^\infty(s,x)dxds \\
&+ \frac{m-1}{m}\int_0^t\int_{\mathbb{R}^d}\Delta f(x)u^\infty(s,x)^m dxds, \qquad (3.12)
\end{aligned}$$

that is u^∞ is a weak solution of the viscous porous medium equation with initial datum u_0.

Proof. According to (1.3) we have

$$\begin{aligned}
\int_{\mathbb{R}^d} f(x)u^\varepsilon(t,dx) &= \int_{\mathbb{R}^d} f(x)u_0(x)dx + \frac{1}{2}\int_0^t\int_{\mathbb{R}^d}\Delta f(x)u^\varepsilon(s,dx)ds \\
&- \int_0^t\int_{\mathbb{R}^d}\nabla f(x)\cdot\nabla g^\varepsilon(s,x)u^\varepsilon(s,dx)ds. \qquad (3.13)
\end{aligned}$$

Since $u^\varepsilon \to u^\infty$ in $\mathcal{C}([0,T],\mathcal{M}(\mathbb{R}^d))$, the convergence of all the terms in (3.13) is trivial except for the third term in the right hand side. We have

$$\begin{aligned}
&\left|\int_0^t\int_{\mathbb{R}^d}\nabla f(x)\cdot\nabla g^\varepsilon(s,x)u^\varepsilon(s,dx)ds + \frac{m-1}{m}\int_0^t\int_{\mathbb{R}^d}\Delta f(x)u^\infty(s,x)^m dxds\right| \\
\leq\ &\left|\int_0^t\int_{\mathbb{R}^d}\nabla f(x)\cdot\nabla g^\varepsilon(s,x)u^\varepsilon(s,dx)ds\right. \\
&\left. - \int_0^t\int_{\mathbb{R}^d}\nabla f(x)\cdot\nabla\left(v^\varepsilon(s,x)^{m-1}\right)v^\varepsilon(s,x)dxds\right| \\
+\ &\left|\frac{m-1}{m}\int_0^t\int_{\mathbb{R}^d}\Delta f(x)v^\varepsilon(s,x)^m dxds - \frac{m-1}{m}\int_0^t\int_{\mathbb{R}^d}\Delta f(x)u^\infty(s,x)^m dxds\right|.
\end{aligned}$$

By the convergence $v^\varepsilon \to u^\infty$ in $L^m([0,T]\times\mathbb{R}^d)$, the second term goes to 0.

Regarding the first term, we observe that

$$\begin{aligned}
& \left|\int_0^t \int_{\mathbb{R}^d} \nabla f(x) \cdot \nabla g^\varepsilon(s,x) u^\varepsilon(s,dx) ds \right. \\
& \left. - \int_0^t \int_{\mathbb{R}^d} \nabla f(x) \cdot \nabla \left(v^\varepsilon(s,x)^{m-1}\right) v^\varepsilon(s,x) dx ds \right| \\
= & \left|\int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \nabla f(x) \cdot \nabla \left(v^\varepsilon(s,y)^{m-1}\right) V^\varepsilon(x-y) u^\varepsilon(s,dx) dy ds \right. \\
& \left. - \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \nabla f(y) \cdot \nabla \left(v^\varepsilon(s,y)^{m-1}\right) V^\varepsilon(x-y) u^\varepsilon(s,dx) dy ds \right| \\
\leq & \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\nabla f(x) - \nabla f(y)| \left|\nabla \left(v^\varepsilon(y,s)^{m-1}\right)\right| V^\varepsilon(x-y) u^\varepsilon(s,dx) dy ds \\
\leq & \|D^2 f\|_\infty \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left|\nabla \left(v^\varepsilon(s,y)^{m-1}\right)\right| |x-y| V^\varepsilon(x-y) u^\varepsilon(s,dx) dy ds.
\end{aligned}$$

Since $\operatorname{diam}(\operatorname{supp} V^\varepsilon) = \varepsilon \operatorname{diam}(\operatorname{supp} V)$ (recall that V has compact support) we see that the last term is bounded by

$$\begin{aligned}
& \varepsilon \operatorname{diam}(\operatorname{supp} V) \|D^2 f\|_\infty \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left|\nabla \left(v^\varepsilon(s,y)^{m-1}\right)\right| V^\varepsilon(x-y) u^\varepsilon(s,dx) dy ds \\
= & \varepsilon \operatorname{diam}(\operatorname{supp} V) \|D^2 f\|_\infty (m-1) \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\nabla v^\varepsilon(s,y)| \, v^\varepsilon(s,y)^{m-1} dy ds.
\end{aligned}$$

To conclude, we observe that by Hölder inequality

$$\begin{aligned}
\|\nabla v^\varepsilon (v^\varepsilon)^{m-1}\|_{L^1([0,T]\times\mathbb{R}^d)} & \leq \|\nabla v^\varepsilon (v^\varepsilon)^{m/2-1}\|_{L^2[0,T]\times\mathbb{R}^d)} \|(v^\varepsilon)^{m/2}\|_{L^2[0,T]\times\mathbb{R}^d)} \\
& = \|(v^\varepsilon)^{m-2} |\nabla v^\varepsilon|^2\|_{L^1[0,T]\times\mathbb{R}^d)}^{1/2} \|v^\varepsilon\|_{L^m[0,T]\times\mathbb{R}^d)}^{m/2},
\end{aligned}$$

and the right hand side is uniformly bounded thanks to Lemma 3.1. □

Up to now, we have proved that the sequence $(u^\varepsilon)_{\varepsilon>0}$ is relatively compact (Proposition 3.3) and that any limit point u^∞ of a subsequence is a weak solution of the viscous porous medium equation (Proposition 3.8).

It remains to show uniqueness of weak solutions u of this equation. To do so, we first prove that, thanks to the assumption $u_0 \in L^\infty(\mathbb{R}^d)$, any weak solution of (1.4) belongs to $L^{m+1}([0,T]\times\mathbb{R}^d)$ (and not only to $L^m([0,T]\times\mathbb{R}^d)$), and then we conclude using Proposition 3.10.

Proposition 3.9. *Let $v \in L^1([0,T]\times\mathbb{R}^d) \cap L^m([0,T]\times\mathbb{R}^d)$ be a weak solution of the viscous porous medium equation such that $v(0,\cdot) \in L^\infty(\mathbb{R}^d)$. Then $v \in L^{m+1}([0,T]\times\mathbb{R}^d)$.*

Proof. Let us consider the convex function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ given by

$$\Phi(s) := \frac{1}{2}s + \frac{m-1}{m}s^m.$$

Then v is a weak solution of

$$\partial_t v = \Delta(\Phi(v)).$$

Fix $\varphi(x)$ a smooth convolution kernel on $\mathbb{R}^d$, and define

$$v_\eta(t,\cdot) := v(t,\cdot) * \varphi_\eta$$

with $\varphi_\eta(x) := \frac{1}{\eta^d}\varphi(x/\eta)$. Then v_η is smooth and integrable in x with all its derivatives. Moreover, since

$$\partial_t v_\eta = \Delta(\Phi(v) * \varphi_\eta),$$

v_η is also smooth as a function of t. We can therefore multiply the above equation by $\int_t^T \Phi(v) * \varphi_\eta(s,\cdot)ds$ and integrate in space-time, obtaining

$$\begin{aligned}
& \int_0^T \int_{\mathbb{R}^d} \partial_t v_\eta(t,x)\left(\int_t^T \Phi(v) * \varphi_\eta(s,x)ds\right) dx\, dt \\
= \;& \int_0^T \int_{\mathbb{R}^d} \Delta(\Phi(v) * \varphi_\eta)(t,x)\left(\int_t^T \Phi(v) * \varphi_\eta(s,x)ds\right) dx\, dt \\
= \;& -\int_0^T \int_{\mathbb{R}^d} \nabla(\Phi(v) * \varphi_\eta)(t,x)\left(\int_t^T \nabla(\Phi(v) * \varphi_\eta)(s,x)ds\right) dx\, dt \\
= \;& -\frac{1}{2}\int_{\mathbb{R}^d}\left|\int_0^T \nabla\Phi(v) * \varphi_\eta(t,x)dt\right|^2 dx \le 0.
\end{aligned}$$

Therefore, integrating by parts in the first line of the above equation, we get

$$\begin{aligned}
& \int_0^T \int_{\mathbb{R}^d} v_\eta(t,x)\Phi(v) * \varphi_\eta(t,x)\, dx\, dt \\
\le \;& \int_{\mathbb{R}^d} v_\eta(0,x) \int_0^T \Phi(v) * \varphi_\eta(t,x)\, dx\, dt \\
\le \;& \|v_\eta(0)\|_\infty \|\Phi(v)\|_{L^1([0,T]\times\mathbb{R}^d)} \\
= \;& \|v_\eta(0)\|_\infty \left(\frac{1}{2}\|v\|_{L^1([0,T]\times\mathbb{R}^d)} + \frac{m-1}{m}\|v\|^m_{L^m([0,T]\times\mathbb{R}^d)}\right).
\end{aligned}$$

Since Φ is convex, by Jensen's inequality we have $\Phi(v_\eta)(t,x) \le \Phi(v) * \varphi_\eta(t,x)$, and we finally obtain

$$\begin{aligned}
\int_0^T \int_{\mathbb{R}^d} \frac{1}{2}v_\eta(t,x)^2 \quad + \quad & \frac{m-1}{m}v_\eta(t,x)^{m+1}\, dx\, dt = \int_0^T \int_{\mathbb{R}^d} v_\eta(t,x)\Phi(v_\eta)\, dx\, dt \\
& \le \|v_\eta(0)\|_\infty \left(\frac{1}{2}\|v\|_{L^1([0,T]\times\mathbb{R}^d)} + \frac{m-1}{m}\|v\|^m_{L^m([0,T]\times\mathbb{R}^d)}\right).
\end{aligned}$$

Taking the limit as $\eta \to 0$, we conclude that $v \in L^{m+1}([0,T]\times\mathbb{R}^d)$. □

Proposition 3.10. *Let v and $\tilde{v}$ be two weak solutions of the viscous porous medium equation on $[0,T]$ such that $v, \tilde{v} \in L^{m+1}(\mathbb{R}^d \times [0,T])$. Then $v = \tilde{v}$.*

Proof. Using the same notations of the proof of the above proposition, v and $\tilde{v}$ are weak solutions of

$$\partial_t w = \Delta(\Phi(w)).$$

As before we regularize the solutions with a smooth convolution kernel φ, and we get

$$\partial_t(v_\eta - \tilde{v}_\eta) = \Delta(\Phi(v) * \varphi_\eta - \Phi(\tilde{v}) * \varphi_\eta).$$

Multiplying the equation by $\int_t^T [\Phi(v) * \varphi_\eta(s,\cdot) - \Phi(\tilde v) * \varphi_\eta(s,\cdot)]ds$ and integrating in space-time, as in the proof of the above proposition we obtain

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^d} \partial_t(v_\eta - \tilde v_\eta)(t,x)\left(\int_t^T [\Phi(v) * \varphi_\eta(s,x) - \Phi(\tilde v) * \varphi_\eta(s,x)]ds\right) dx\, dt \\
&= -\int_0^T \int_{\mathbb{R}^d} \nabla(\Phi(v) * \varphi_\eta - \Phi(\tilde v) * \varphi_\eta)(t,x)\left(\int_t^T \nabla(\Phi(v) * \varphi_\eta - \Phi(\tilde v) * \varphi_\eta)(s,x) ds\right) dx\, dt \\
&= -\frac{1}{2}\int_{\mathbb{R}^d} \left|\int_0^T \nabla(\Phi(v) * \varphi_\eta - \Phi(\tilde v) * \varphi_\eta)(t,x) dt\right|^2 dx \le 0.
\end{aligned}$$

Integrating by parts in the first line of the above equation and using that u and $\tilde u$ coincide at time 0, we have

$$\int_0^T \int_{\mathbb{R}^d} [v_\eta(t,x) - \tilde v_\eta(t,x)][\Phi(v) * \varphi_\eta(t,x) - \Phi(\tilde v) * \varphi_\eta(t,x)]\, dx\, dt \le 0.$$

Since $v, \tilde v \in L^{m+1}(\mathbb{R}^d \times [0,T])$, we can take the limit as $\eta \to 0$, and we get

$$\int_0^T \int_{\mathbb{R}^d} [v(t,x) - \tilde v(t,x)][\Phi(v)(t,x) - \Phi(\tilde v)(t,x)]\, dx\, dt \le 0.$$

As the integrand is non-negative everywhere, it follows that $v = \tilde v$ almost everywhere. □

Appendix: Proof of Proposition 2.3

Let $K := \|V\|_{L^\infty(\mathbb{R})}$, L a Lipschitz constant for V and $I := \int_{\mathbb{R}^d} |\nabla V(y)|\, dy$, so that V^ε is bounded by $K_\varepsilon := K/\varepsilon^d$ and Lipschitz-continuous with Lipschitz constant $L_\varepsilon := L/\varepsilon^{d+1}$, and $\int_{\mathbb{R}^d} |\nabla V^\varepsilon(y)|\, dy = I/\varepsilon$. In order to simplify the notation we omit the indices N, ε and δ. For $t \in [0,T]$ we set

$$\Phi(t) := \mathbb{E}\left[\sup_{0\le s\le t} \left|X_s^i - Y_s^i\right|^2\right].$$

Because of the symmetry of the particle system and the system of the nonlinear processes, $\Phi(t)$ does not depend on i. By (2.1) and (2.3) we have

$$\begin{aligned}
&\left|X_t^i - Y_t^i\right|^2 \\
=&\left|\int_0^t \int_{\mathbb{R}^d} \nabla V^\varepsilon(y)\left[\left\{(V^\varepsilon * u_s^{\varepsilon,\delta})(Y_s^i - y)\right\}^{m-1} - \left\{\frac{1}{N}\sum_{j=1}^N V^\varepsilon(X_s^i - y - X_s^j)\right\}^{m-1}\right] dyds\right|^2 \\
\le& t\int_0^t \left(\int_{\mathbb{R}^d} |\nabla V^\varepsilon(y)| \left|\left\{(V^\varepsilon * u_s^{\varepsilon,\delta})(Y_s^i - y)\right\}^{m-1} - \left\{\frac{1}{N}\sum_{j=1}^N V^\varepsilon(X_s^i - y - X_s^j)\right\}^{m-1}\right| dy\right)^2 ds \\
\le& t\int_0^t\left(\int_{\mathbb{R}^d}|\nabla V^\varepsilon(y)|\, dy \sup_{y\in\mathbb{R}^d}\left|\left\{(V^\varepsilon * u_s^{\varepsilon,\delta})(Y_s^i - y)\right\}^{m-1} - \left\{\frac{1}{N}\sum_{j=1}^N V^\varepsilon(X_s^i - y - X_s^j)\right\}^{m-1}\right|\right)^2 ds \\
=&\frac{I^2}{\varepsilon^2} t\int_0^t \sup_{y\in\mathbb{R}^d}\left|\left\{(V^\varepsilon * u_s^{\varepsilon,\delta})(Y_s^i - y)\right\}^{m-1} - \left\{\frac{1}{N}\sum_{j=1}^N V^\varepsilon(X_s^i - y - X_s^j)\right\}^{m-1}\right|^2 ds.
\end{aligned}$$

Since the right-hand side is non-decreasing in t, the same estimate also holds for $\sup_{0\le s\le t}\left|X_s^i - Y_s^i\right|^2$ in place of $\left|X_t^i - Y_t^i\right|^2$. We now estimate

$$\left|\left\{(V^\varepsilon * u_s^{\varepsilon,\delta})(Y_s^i - y)\right\}^{m-1} - \left\{\frac{1}{N}\sum_{j=1}^N V^\varepsilon(X_s^i - y - X_s^j)\right\}^{m-1}\right|^2.$$

Since $m \ge 2$ the Lipschitz continuity of the function $s \mapsto s^{m-1}$ implies that this is bounded by

$$(m-1)^2 K_\varepsilon^{2(m-2)}\left|(V^\varepsilon * u_s^{\varepsilon,\delta})(Y_s^i - y) - \frac{1}{N}\sum_{j=1}^N V^\varepsilon(X_s^i - y - X_s^j)\right|^2.$$

Using the triangle inequality we obtain

$$\begin{aligned}
&\left|(V^\varepsilon * u_s^{\varepsilon,\delta})(Y_s^i - y) - \frac{1}{N}\sum_{j=1}^N V^\varepsilon(X_s^i - y - X_s^j)\right|^2 \\
\le\;& 3\left|(V^\varepsilon * u_s^{\varepsilon,\delta})(Y_s^i - y) - \frac{1}{N}\sum_{j=1}^N V^\varepsilon(Y_s^i - y - Y_s^j)\right|^2 \\
+\;& 3\left|\frac{1}{N}\sum_{j=1}^N V^\varepsilon(Y_s^i - y - Y_s^j) - \frac{1}{N}\sum_{j=1}^N V^\varepsilon(X_s^i - y - Y_s^j)\right|^2 \\
+\;& 3\left|\frac{1}{N}\sum_{j=1}^N V^\varepsilon(X_s^i - y - Y_s^j) - \frac{1}{N}\sum_{j=1}^N V^\varepsilon(X_s^i - y - X_s^j)\right|^2
\end{aligned}$$

so that, combining all these estimates,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le s\le t}\left|X_s^i - Y_s^i\right|^2\right] \\
\le\;& C'(\varepsilon)t\Bigg\{\int_0^t \sup_{y\in\mathbb{R}^d}\mathbb{E}\left[\left|(V^\varepsilon * u_s^{\varepsilon,\delta})(Y_s^i - y) - \frac{1}{N}\sum_{j=1}^N V^\varepsilon(Y_s^i - y - Y_s^j)\right|^2\right]ds \\
&+\int_0^t \sup_{y\in\mathbb{R}^d}\mathbb{E}\left[\left|\frac{1}{N}\sum_{j=1}^N V^\varepsilon(Y_s^i - y - Y_s^j) - \frac{1}{N}\sum_{j=1}^N V^\varepsilon(X_s^i - y - Y_s^j)\right|^2\right]ds \\
&+\int_0^t \sup_{y\in\mathbb{R}^d}\mathbb{E}\left[\left|\frac{1}{N}\sum_{j=1}^N V^\varepsilon(X_s^i - y - Y_s^j) - \frac{1}{N}\sum_{j=1}^N V^\varepsilon(X_s^i - y - X_s^j)\right|^2\right]ds\Bigg\},
\end{aligned}$$

where $C'(\varepsilon) := 3(m-1)^2 K_\varepsilon^{2(m-2)} I^2 \varepsilon^{-2}$.

Thanks to the Lipschitz continuity of V^ε the second and the third term are both bounded by

$$L_\varepsilon^2 \int_0^t \mathbb{E}\left[\left|X_s^i - Y_s^i\right|^2\right] ds,$$

while for the first term we have

$$\begin{aligned}
&\mathbb{E}\left[\left|(V^\varepsilon * u_s^{\varepsilon,\delta})(Y_s^i - y) - \frac{1}{N}\sum_{j=1}^N V^\varepsilon(Y_s^i - y - Y_s^j)\right|^2\right] \\
= \quad & \frac{1}{N^2}\sum_{j,k=1}^N \mathbb{E}\Bigg[\Big((V^\varepsilon * u_s^{\varepsilon,\delta})(Y_s^i - y) - V^\varepsilon(Y_s^i - y - Y_s^j)\Big) \times \\
& \qquad \Big((V^\varepsilon * u_s^{\varepsilon,\delta})(Y_s^i - y) - V^\varepsilon(Y_s^i - y - Y_s^k)\Big)\Bigg].
\end{aligned}$$

If $j \neq k$ the expectation vanishes, and otherwise it is bounded by K_ε^2. Therefore

$$\begin{aligned}
\Phi(t) &= \mathbb{E}\left[\sup_{0\le s\le t}\left|X_s^i - Y_s^i\right|^2\right] \\
&\le C'(\varepsilon)t\left\{\frac{K_\varepsilon^2 t}{N} + 2L_\varepsilon^2\int_0^t \mathbb{E}\left[\left|X_s^i - Y_s^i\right|^2\right] ds\right\} \\
&\le 2TC'(\varepsilon)L_\varepsilon^2\int_0^t \Phi(s)ds + C'(\varepsilon)\frac{K_\varepsilon^2}{N}t^2 \\
&= \alpha\int_0^t \Phi(s)ds + \beta t^2,
\end{aligned}$$

where

$$\alpha := 2TC'(\varepsilon)L_\varepsilon^2, \qquad \beta := C'(\varepsilon)\frac{K_\varepsilon^2}{N}.$$

Gronwall's lemma now implies

$$\Phi(t) \le 2\beta e^{\alpha t}\int_0^t s e^{-\alpha s} ds \le 2\frac{\beta}{\alpha^2}e^{\alpha t}\int_0^\infty s e^{-s} ds = 2\frac{\beta}{\alpha^2}e^{\alpha t},$$

that is

$$\Phi(t) \le \frac{1}{N}\frac{K_\varepsilon^2}{2T^2C'(\varepsilon)L_\varepsilon^4}e^{2TC'(\varepsilon)L_\varepsilon^2 t} \sim \frac{1}{N}\varepsilon^{6+2d(m-1)}e^{t/\varepsilon^{4+2d(m-1)}}.$$

□

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