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Subclasses of Goldie-Steutel-Bondesson class of infinitely divisible distributions on $\mathbb{R}^d$ by Υ-mapping

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Abstract. Bondesson (1981) studied the class of generalized convolutions of mixtures of exponential distributions on $\mathbb{R}_+ = (0,\infty)$, which is the smallest class that contains all mixtures of exponential distributions and that is closed under convolution and weak convergence on $\mathbb{R}_+$. Barndorff-Nielsen, Maejima and Sato (2006) extended this class to $\mathbb{R}^d$, which they call Goldie-Steutel-Bondesson class $B(\mathbb{R}^d)$ for a historical reason. This class is characterized by the so-called Υ-mapping of infinitely divisible distributions in terms of stochastic integrals of Lévy processes. In this paper, we introduce nested subclasses of $B(\mathbb{R}^d)$ by the iteration of Υ-mapping, and characterize them in terms of stochastic integrals of Lévy processes as well as Lévy measures.

1. Introduction

Throughout this paper, for any $\mathbb{R}^d$-valued random variable X, we denote its law by $\mathcal{L}(X)$. The characteristic function and the cumulant function of a probability distribution μ on $\mathbb{R}^d$ are denoted by $\widehat{\mu}(z)$ and $C_\mu(z)$, $z \in \mathbb{R}^d$, respectively. Namely, $C_\mu(z)$ is a continuous function with $C_\mu(0) = 0$ such that $\widehat{\mu}(z) = \exp(C_\mu(z))$. $I(\mathbb{R}^d)$ denotes the class of all infinitely divisible distributions on $\mathbb{R}^d$. We use the Lévy-Khintchine triplet (A, ν, γ) of $\mu \in I(\mathbb{R}^d)$ in the sense that

$$\widehat{\mu}(z) = \exp\left\{-\frac{1}{2}\langle z, Az\rangle + \mathrm{i}\langle \gamma, z\rangle + \int_{\mathbb{R}^d}\left(e^{\mathrm{i}\langle z,x\rangle} - 1 - \frac{\mathrm{i}\langle z, x\rangle}{1+|x|^2}\right)\nu(dx)\right\},\ z \in \mathbb{R}^d,$$

where A is a symmetric nonnegative-definite $d\times d$ matrix, $\gamma \in \mathbb{R}^d$ and ν is a measure (called the Lévy measure) on $\mathbb{R}^d$ satisfying

$$\nu(\{0\}) = 0 \text{ and } \int_{\mathbb{R}^d}(|x|^2 \wedge 1)\nu(dx) < \infty.$$

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An $\mathbb{R}^d$-valued stochastic process $\{X_t, t \geq 0\}$ is called a Lévy process if (i) $X_0 = 0$, a.s., (ii) it has independent and stationary increments and (iii) it is stochastically continuous at each $t \geq 0$. Since $\mathcal{L}(X_1) \in I(\mathbb{R}^d)$ and the law of a Lévy process $\{X_t\}$ is determined by $\mathcal{L}(X_1)$, we denote by $\{X_t^{(\mu)}\}$ the Lévy process with $\mathcal{L}(X_1^{(\mu)}) = \mu$. As to the definition of stochastic integrals of nonrandom functions with respect to Lévy processes $\{X_t\}$ on $\mathbb{R}^d$, we follow the definition in (Sato, 2004, 2006), whose idea is to define the integrals with respect to $\mathbb{R}^d$-valued independently scattered random measure induced by a Lévy process on $\mathbb{R}^d$. This idea was used in Urbanik and Woyczyński (1967) and Rajput and Rosinski (1989) for the case $d = 1$. See also Barndorff-Nielsen et al. (2006).

Bondesson (1981) studied the class of generalized convolutions of mixtures of exponential distributions on $\mathbb{R}_+$, called $\mathcal{T}_2$ in his monograph (1992). It is the smallest class that contains all mixtures of exponential distributions and that is closed under convolution and weak convergence on $\mathbb{R}_+$. Goldie (1967) is the first person who proved the infinite divisibility of the mixtures of exponential distributions and Steutel (1967) found the form of their Lévy measures. The Lévy measure ν of a distribution in $\mathcal{T}_2$ has the following form:

$$\nu(dr) = l(r)dr, \quad r > 0, \tag{1.1}$$

where $l(r)$ is completely monotone. (See Bondesson (1992), Theorem 3.3.1.)

In order to talk about infinitely divisible distributions on $\mathbb{R}^d$, it is useful to consider the polar decomposition of Lévy measures, because most interesting subclasses of $I(\mathbb{R}^d)$ can be determined only by the radial component ν_ξ of the Lévy measure defined below. The polar decomposition of Lévy measures on $\mathbb{R}^d$ is the following: Let ν be the Lévy measure of some $\mu \in I(\mathbb{R}^d)$ with $0 < \nu(\mathbb{R}^d) \leq \infty$. Then there exist a measure λ on $S = \{\xi \in \mathbb{R}^d : |\xi| = 1\}$ with $0 < \lambda(S) \leq \infty$ and a family $\{\nu_\xi : \xi \in S\}$ of measures on $(0, \infty)$ such that $\nu_\xi(B)$ is measurable in ξ for each $B \in \mathcal{B}((0,\infty))$, $0 < \nu_\xi((0,\infty)) \leq \infty$ for each $\xi \in S$ and that

$$\nu(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi)\nu_\xi(dr), \ B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}). \tag{1.2}$$

Here λ and $\{\nu_\xi\}$ are uniquely determined by ν up to multiplication of a measurable function $c(\xi)$ and $\frac{1}{c(\xi)}$ with $0 < c(\xi) < \infty$. We say that ν has the polar decomposition (λ, ν_ξ) and ν_ξ is called the radial component of ν. (See, e.g. Barndorff-Nielsen et al. (2006), Lemma 2.1.)

In Barndorff-Nielsen et al. (2006), the class $B(\mathbb{R}^d)$ is defined as the collection of $\mu \in I(\mathbb{R}^d)$ with Lévy measure ν such that $\nu = 0$ or $\nu \neq 0$, having the polar decomposition (λ, ν_ξ) satisfying

$$\nu_\xi(dr) = l_\xi(r)dr, \quad \text{for } \lambda\text{-a.e. } \xi \in S, \tag{1.3}$$

where $l_\xi(r)$ is measurable in ξ and completely monotone in r for λ-a.e. ξ. We call this l_ξ the l-function of the Lévy measure of $\mu \in B(\mathbb{R}^d)$. If we compare (1.1) and (1.3), we see that $B(\mathbb{R}^d)$ is an extension of $\mathcal{T}_2$. We call the class $B(\mathbb{R}^d)$ Goldie-Steutel-Bondesson class for the historical reason mentioned above.

On the other hand, Barndorff-Nielsen and Thorbjørnsen (2002a; 2002b; 2004; 2006) introduced a mapping Υ on $I(\mathbb{R})$ defined by a stochastic integral as follows.

Namely, for $\mu \in I(\mathbb{R})$,

$$\Upsilon(\mu) = \mathcal{L}\left(\int_0^1 (\log t^{-1}) dX_t^{(\mu)}\right) \in I(\mathbb{R}), \tag{1.4}$$

where $\{X_t^{(\mu)}\}$ is a Lévy process on $\mathbb{R}$ with $\mathcal{L}(X_1^{(\mu)}) = \mu$. They studied this mapping only in one dimension, but it can easily be extended to a mapping from $I(\mathbb{R}^d)$ into $I(\mathbb{R}^d)$ by the same definition, with the replacement of $\{X_t^{(\mu)}\}$ on $\mathbb{R}$ by $\{X_t^{(\mu)}\}$ on $\mathbb{R}^d$. The stochastic integral in (1.4) is definable for any for $\mu \in I(\mathbb{R}^d)$. (Barndorff-Nielsen et al., 2006, Proposition 2.3.) Then in Barndorff-Nielsen et al. (2006), we discussed Υ-mapping in $I(\mathbb{R}^d)$ and proved that the class $B(\mathbb{R}^d)$ is the image of the class of all infinitely divisible distributions on $\mathbb{R}^d$:

$$B(\mathbb{R}^d) = \Upsilon(I(\mathbb{R}^d)).$$

Note that the class $B(\mathbb{R}^d)$ is one of the subclasses of $I(\mathbb{R}^d)$ which can be determined only by the Lévy measures.

Recently, detailed studies of subclasses of infinitely divisible distributions have again been investigated by many authors. (See, e.g. Aoyama and Maejima, 2006.) In this paper, we will study nested subclasses of $B(\mathbb{R}^d)$ defined by the iteration of Υ-mapping. For $m = 1, 2, ...$, let $\Upsilon^{m+1}(\mu) = \Upsilon(\Upsilon^m(\mu))$, with $\Upsilon^0(\mu) = \mu$ and $\Upsilon^1 = \Upsilon$. Then we define nested subclasses of $B(\mathbb{R}^d)$ as follows.

Definition 1.1. (Class $B_m(\mathbb{R}^d)$) Let $B_0(\mathbb{R}^d) = B(\mathbb{R}^d)$. Define, for $m = 1, 2, ...$,

$$B_m(\mathbb{R}^d) = \Upsilon(B_{m-1}(\mathbb{R}^d)) = \Upsilon^{m+1}(I(\mathbb{R}^d))$$

and define

$$B_\infty(\mathbb{R}^d) = \cap_{m=0}^\infty B_m(\mathbb{R}^d).$$

By Lemma 4.1 of Barndorff-Nielsen et al. (2006), we see that $B_m(\mathbb{R}^d) \supset B_{m+1}(\mathbb{R}^d)$.

The organization of this paper is the following. In Section 2, we define stochastic integrals with respect to Lévy processes, which will appear in characterizations of the class $B_m(\mathbb{R}^d)$, and in Section 3, we characterize the class $B_m(\mathbb{R}^d), m < \infty$. In Section 4, we characterize distributions in $B_m(\mathbb{R}^d)$ in terms of Lévy measures, and discuss some properties of functions appearing in those Lévy measures. In the last Section 5, we discuss the class $B_\infty(\mathbb{R}^d)$ and show that all stable distributions belong to $B_\infty(\mathbb{R}^d)$.

We conclude this section with the following remark. Since the classes determined by Υ-mapping depend only on Lévy measures essentially, Υ-mapping can also be regarded as a mapping that sends a Lévy measure to another Lévy measure. Actually, if

$$\Upsilon(\mu) = \mathcal{L}\left(\int_0^1 (\log t^{-1}) dX_t^{(\mu)}\right) \in I(\mathbb{R}^d), \quad \mu \in I(\mathbb{R}^d),$$

then

$$\nu_{\Upsilon(\mu)}(B) = \int_0^\infty \nu_\mu(t^{-1}B) e^{-t} dt, \quad B \in \mathcal{B}(\mathbb{R}^d),$$

where ν_μ is the Lévy measure of $\mu \in I(\mathbb{R}^d)$. (See Barndorff-Nielsen et al., 2006, Theorem A (ii).) When we regard Υ as a mapping from the class of Lévy measures into itself, we may write

$$\Upsilon(\nu)(B) = \int_0^\infty \nu(t^{-1}B) e^{-t} dt, \quad B \in \mathcal{B}(\mathbb{R}^d). \tag{1.5}$$

Once we look at (1.5), we can easily extend Υ-mapping by replacing the probability density function e^{-t} in (1.5) by other probability densities on $(0, \infty)$, and we can investigate new subclasses of infinitely divisible distributions. For the recent studies in this direction, see, e.g. Barndorff-Nielsen et al. (2007). Also, Barndorff-Nielsen and Pérez-Abreu (2007) extended Υ-mapping to the class of Lévy measures on the cone of symmetric nonnegative-definite matrices and studied matrix subordinations. This is another new direction of the study of this topic.

2. Preliminaries

In this section, we define stochastic integrals with respect to Lévy processes, which are needed for characterizations of the class $B_m(\mathbb{R}^d)$. First we introduce a sequence of functions $\varepsilon_m(x), m = 0, 1, 2, ...$, whose inverse functions will appear as integrands of stochastic integrals in the class $B_m(\mathbb{R}^d)$ as follows: For $x \geq 0$,

$$\begin{aligned}\varepsilon_0(x) &= e^{-x},\\ \varepsilon_1(x) &= -\int_0^\infty e^{-x/u} d\varepsilon_0(u) > 0,\\ &\cdots\\ \varepsilon_m(x) &= -\int_0^\infty e^{-x/u} d\varepsilon_{m-1}(u) > 0. \qquad (2.1)\end{aligned}$$

We give some properties of $\varepsilon_m(x)$ for later use.

Proposition 2.1. *For $m = 1, 2, ...,$*
(1) *$\varepsilon_m(x)$ is definable on $[0, \infty)$,*
(2) *$\varepsilon_m(0) = 1$ and $\varepsilon_m(\infty) := \lim_{x\to\infty} \varepsilon_m(x) = 0$,*
(3) *$\varepsilon_m(x)$ is differentiable on $(0, \infty)$,*
(4)

$$\varepsilon'_m(x) = \int_0^\infty e^{-x/u} u^{-1} d\varepsilon_{m-1}(u) = \int_0^\infty e^{-x/u} u^{-1} \varepsilon'_{m-1}(u) du, \quad x > 0, \qquad (2.2)$$

(5) $|\varepsilon'_m(x)| \leq x^{-1},\ x > 0$
and
(6) *$\varepsilon_m(x)$ is strictly decreasing.*

Proof.
Proof of (1). It is trivial that $\varepsilon_1(x)$ is definable. Next suppose $\varepsilon_m(x)$ is definable for some $m \geq 1$. Then

$$\varepsilon_{m+1}(x) \leq -\int_0^\infty d\varepsilon_m(u) = -\varepsilon_m(\infty) + \varepsilon_m(0),$$

so that $\varepsilon_{m+1}(x)$ is definable.
Proof of (2). Note that $\varepsilon_0(0) = 1$ and $\varepsilon_0(\infty) = 0$. Suppose that $\varepsilon_m(0) = 1$ and $\varepsilon_m(\infty) = 0$ for some $m \geq 0$. Then

$$\varepsilon_{m+1}(0) = -\varepsilon_m(\infty) + \varepsilon_m(0) = 1.$$

Also, since $e^{-x/u} \leq 1$ and $\int_0^\infty (-d\varepsilon_m(u)) = 1$, we have by the dominated convergence theorem that

$$\lim_{x\to\infty} \varepsilon_{m+1}(x) = \lim_{x\to\infty} \int_0^\infty e^{-x/u}(-d\varepsilon_m(u)) = 0.$$

Proofs of (3), (4) and (5). We have

$$\frac{1}{h}\left(\varepsilon_1(x+h)-\varepsilon_1(x)\right)=\int_0^\infty e^{-x/u}\frac{1-e^{-h/u}}{h}e^{-u}du.$$

The absolute value of the integrand here is dominated by $e^{-x/u}u^{-1}e^{-u}\le e^{-x/u}u^{-2}$, which is integrable over $(0,\infty)$ for each $x>0$, because

$$\int_0^\infty e^{-x/u}u^{-2}du=x^{-1}\int_0^\infty e^{-y}dy=x^{-1}.$$

Thus by the dominated convergence theorem,

$$\varepsilon_1'(x)=\lim_{h\to 0}\frac{1}{h}\left(\varepsilon_1(x+h)-\varepsilon_1(x)\right)=-\int_0^\infty e^{-x/u}u^{-1}e^{-u}du=\int_0^\infty e^{-x/u}u^{-1}d\varepsilon_0(u),$$

which shows that (3) and (4) are true for $m=1$. Now, note that $|\varepsilon_0'(x)|=e^{-x}\le x^{-1}, x>0$. Then

$$|\varepsilon_1'(x)|\le\int_0^\infty e^{-x/u}u^{-2}du=x^{-1},\quad x>0.$$

Hence, (5) holds for $m=1$.

Next suppose that $\varepsilon_m'(x)$ exists, (2.2) holds, and $|\varepsilon_m'(x)|\le x^{-1}, x>0$. We have

$$\frac{1}{h}\left(\varepsilon_{m+1}(x+h)-\varepsilon_{m+1}(x)\right)=\int_0^\infty e^{-x/u}\frac{1-e^{-h/u}}{h}u^{-1}\varepsilon_m'(u)du.$$

The absolute value of the integrand above is dominated by $e^{-x/u}u^{-2}$, which is integrable over $(0,\infty)$ for each $x>0$. Thus $\varepsilon_{m+1}'(x)$ exists, by the dominate convergence theorem. As to the orders of $\varepsilon_{m+1}'(x)$, we have

$$|\varepsilon_{m+1}'(x)|\le\int_0^\infty e^{-x/u}u^{-2}du=x^{-1},$$

Proof of (6). Since $\varepsilon_0'(x)<0$ for all $x>0$, (2.2) implies $\varepsilon_1'(x)<0$ and thus $\varepsilon_1(x)$ is strictly decreasing. If we suppose, for some $m\ge 1$, $\varepsilon_m'(x)<0$ for all $x>0$, then $\varepsilon_{m+1}(x)$ is strictly increasing by (2.2) again. □

Since $\varepsilon_m(x)$ is strictly decreasing, we can define the inverse function $x=\varepsilon_m^*(t)$ by $t=\varepsilon_m(x)$. We are now ready to define stochastic integrals for our purpose.

Proposition 2.2. *Let $m=0,1,2,\ldots$. For any $\mu\in I(\mathbb{R}^d)$,*

$$\int_0^1\varepsilon_m^*(t)dX_t^{(\mu)}$$

exists and finite a.s.

Proof. The case $m=0$ is proved in Proposition 2.3 in Barndorff-Nielsen et al. (2006). For general m, it is enough to apply the following lemmas, the first of which is from parts of Propositions 2.17 and 3.4 of Sato (2006).

Lemma 2.3 (Sato, 2006)**.** *Let $\mu\in I(\mathbb{R}^d)$. Let $\{X_t^{(\mu)}\}$ be the Lévy process with $\mathcal{L}(X_1^{(\mu)})=\mu$ on $\mathbb{R}^d$ and $f(t)$ a real-valued measurable function on $[0,1]$. If $\int_0^1 f(t)^2dt<\infty$, then $Y=\int_0^1 f(t)dX_t^{(\mu)}$ is definable, $\int_0^1|C_\mu(f(t)z)|dt<\infty$ and $C_{\mathcal{L}(Y)}(z)=\int_0^1 C_\mu(f(t)z)dt$.*

Lemma 2.4. *Let* $p = 1, 2, \ldots.$

$$\int_0^1 \varepsilon_m^*(t)^p dt = \Gamma(p+1)^m \int_0^1 (\log t^{-1})^p dt < \infty.$$

Proof. We have

$$\begin{aligned}\int_0^1 \varepsilon_m^*(t)^p dt &= -\int_0^\infty x^p d\varepsilon_m(x) = -\int_0^\infty x^p dx \int_0^\infty e^{-x/u} u^{-1} d\varepsilon_{m-1}(u) \\ &= -\int_0^\infty u^{-1} d\varepsilon_{m-1}(u) \int_0^\infty x^p e^{-x/u} dx = -\Gamma(p+1) \int_0^\infty u^p d\varepsilon_{m-1}(u) \\ &= \Gamma(p+1) \int_0^1 \varepsilon_{m-1}^*(t)^p dt = \Gamma(p+1)^m \int_0^1 \varepsilon_0^*(t)^p dt \\ &= \Gamma(p+1)^m \int_0^1 (\log t^{-1})^p dt < \infty.\end{aligned}$$

□

3. Stochastic integral characterizations of $B_m(\mathbb{R}^d), m < \infty$

We are now going to show that the elements of $B_m(\mathbb{R}^d)$ have the representation $\mathcal{L}\left(\int_0^1 \varepsilon_m^*(t) dX_t^{(\mu)}\right)$. Actually, we have the following.

Theorem 3.1. *Let* $m = 0, 1, 2, \ldots.$ *Then for* $\mu \in I(\mathbb{R}^d)$,

$$\Upsilon^{m+1}(\mu) = \mathcal{L}\left(\int_0^1 \varepsilon_m^*(t) dX_t^{(\mu)}\right).$$

Then, we can characterize $B_m(\mathbb{R}^d)$ as follows.

Corollary 3.2.

$$B_m(\mathbb{R}^d) = \left\{\mathcal{L}\left(\int_0^1 \varepsilon_m^*(t) dX_t^{(\mu)}\right), \quad \mu \in I(\mathbb{R}^d)\right\}.$$

Proof of Theorem 3.1. Let $\mu \in I(\mathbb{R}^d)$. We first note that

$$\int_0^1 |C_\mu(\varepsilon_m^*(t)z)| dt < \infty \tag{3.1}$$

and show that

$$C_{\Upsilon^{m+1}(\mu)}(z) = \int_0^1 C_\mu(\varepsilon_m^*(t)z) dt. \tag{3.2}$$

(3.1) follows from Proposition 2.3 and Lemma 2.4. We are going to prove (3.2). For $m = 0$,

$$C_{\Upsilon(\mu)}(z) = \int_0^1 C_\mu\left((\log t^{-1})z\right) dt = \int_0^\infty C_\mu(uz) e^{-u} du = -\int_0^\infty C_\mu(uz) d\varepsilon_0(u).$$

Next suppose, for some $m \geq 1$,

$$C_{\Upsilon^m(\mu)}(z) = -\int_0^\infty C_\mu(uz) d\varepsilon_{m-1}(u).$$

We claim that

$$\int_0^\infty e^{-w}dw \int_0^\infty |C_\mu(zwu)|(-d\varepsilon_{m-1}(u)) < \infty \tag{3.3}$$

for the use of Fubini theorem in calculation of cumulants below.

The proof of (3.3) is as follows. The idea is from Barndorff-Nielsen et al. (2006). If the Lévy-Khintchine triplet of μ is (A, ν, γ), then

$$|C_\mu(z)| \le 2^{-1}(\mathrm{tr}A)|z|^2 + |\gamma||z| + \int_{\mathbb{R}^d} |g(z,x)|\nu(dx),$$

where

$$g(z,x) = e^{i\langle z,x\rangle} - 1 - i\langle z,x\rangle(1+|x|^2)^{-1}.$$

Hence

$$\begin{aligned} |C_\mu(wuz)| &\le 2^{-1}(\mathrm{tr}A)w^2u^2|z|^2 + |\gamma||w||u||z| + \int_{\mathbb{R}^d} |g(z,wux)|\nu(dx) \\ &\quad + \int_{\mathbb{R}^d} |g(wuz,x) - g(z,wux)|\nu(dx) =: J_1 + J_2 + J_3 + J_4, \end{aligned}$$

say. The finiteness of $\int_0^\infty e^{-w}dw \int_0^\infty (J_1+J_2)(-d\varepsilon_{m-1}(u))$ follows from Lemma 2.4 with $p = 1,2$. Noting that $|g(z,x)| \le C_z|x|^2(1+|x|^2)^{-1}$ with a positive constant C_z depending on z, we have

$$\begin{aligned} &\int_0^\infty e^{-w}dw \int_0^\infty J_3(-d\varepsilon_{m-1}(u)) \\ &\quad \le C_z \int_{\mathbb{R}^d} \nu(dx) \int_0^\infty e^{-w}dw \int_0^\infty \frac{(wu|x|)^2}{1+(wu|x|)^2}(-d\varepsilon_{m-1}(u)) \\ &\quad = C_z \left(\int_{|x|\le 1} \nu(dx) + \int_{|x|>1} \nu(dx) \right) \int_0^\infty e^{-w}dw \int_0^\infty \frac{(wu|x|)^2}{1+(wu|x|)^2}(-d\varepsilon_{m-1}(u)) \\ &\quad =: J_{31} + J_{32}, \end{aligned}$$

say, and

$$\begin{aligned} J_{31} &\le C_z \int_{|x|\le 1} |x|^2\nu(dx) \int_0^\infty w^2e^{-w}dw \int_0^\infty u^2(-d\varepsilon_{m-1}(u)) < \infty, \\ J_{32} &\le C_z \int_{|x|>1} \nu(dx) \int_0^\infty e^{-w}dw \int_0^\infty (-d\varepsilon_{m-1}(u)) < \infty. \end{aligned}$$

As to J_4, note that for $a \in \mathbb{R}$,

$$|g(az,x) - g(z,ax)| = \frac{|\langle az,x\rangle||x|^2|1-a^2|}{(1+|x|^2)(1+|ax|^2)} \le \frac{|z||x|^3(|a|+|a|^3)}{(1+|x|^2)(1+|ax|^2)} \le \frac{|z||x|^2(1+|a|^2)}{2(1+|x|^2)},$$

since $|b|(1+b^2)^{-1} \le 2^{-1}$. Then

$$\begin{aligned} &\int_0^\infty e^{-w}dw \int_0^\infty J_4(-d\varepsilon_{m-1}(u)) \\ &\quad \le |z| \int_{\mathbb{R}^d} \frac{|x|^2}{1+|x|^2}\nu(dx) \int_0^\infty e^{-w}dw \int_0^\infty (1+w^2u^2)(-d\varepsilon_{m-1}(u)) < \infty. \end{aligned}$$

This completes the proof of (3.3).

Then

$$\begin{aligned}
C_{\Upsilon^{m+1}(\mu)}(z) &= \int_0^1 C_{\Upsilon^m(\mu)}\left((\log t^{-1})z\right)dt \\
&= -\int_0^1 dt \int_0^\infty C_\mu\left((\log t^{-1})uz\right)d\varepsilon_{m-1}(u) \\
&\qquad \left(\text{by the change of variables } \log t^{-1} = w\right) \\
&= -\int_0^\infty e^{-w}dw \int_0^\infty C_\mu(wuz)d\varepsilon_{m-1}(u) \\
&= -\int_0^\infty d\varepsilon_{m-1}(u) \int_0^\infty e^{-w}C_\mu(wuz)dw \\
&= -\int_0^\infty d\varepsilon_{m-1}(u)u^{-1} \int_0^\infty e^{-s/u}C_\mu(sz)ds \\
&= -\int_0^\infty C_\mu(sz)ds \int_0^\infty e^{-s/u}u^{-1}d\varepsilon_{m-1}(u) \\
&= -\int_0^\infty C_\mu(sz)d\varepsilon_m(s) = \int_0^1 C_\mu(\varepsilon_m^*(t)z)dt,
\end{aligned}$$

which is (3.2). Here we have used $\varepsilon_m(0) = 1$. Hence

$$\Upsilon^{m+1}(\mu) = \mathcal{L}\left(\int_0^1 \varepsilon_m^*(t)dX_t^{(\mu)}\right).$$

This concluses the proof. $\square$

4. Lévy measures

The first theorem of this section is characterizations of distributions in $B_m(\mathbb{R}^d)$, $m < \infty$, in terms of their Lévy measures.

Theorem 4.1. *Let $m = 1, 2, \ldots$.*
(i) Let ν_m be a Lévy measure. It is the Lévy measure of some distribution in $B_m(\mathbb{R}^d)$ if and only if ν_m can be represented as

$$\nu_m(B) = -\int_0^\infty \nu_\mu(t^{-1}B)d\varepsilon_m(t)$$

for the Lévy measure ν_μ of some $\mu \in I(\mathbb{R}^d)$.
(ii) Let ν_m be a Lévy measure. It is the Lévy measure of some distribution in $B_m(\mathbb{R}^d)$ if and only if ν_m can be represented as

$$\nu_m(B) = -\int_0^\infty \nu_0(t^{-1}B)d\varepsilon_{m-1}(t) \tag{4.1}$$

for the Lévy measure ν_0 of some distribution in $B_0(\mathbb{R}^d)$.

Proof. The proof is almost the same as for that of Theorem 3.1. So, we omit it. $\square$

Next we consider some detailed properties of the Lévy measures of distributions in $B_m(\mathbb{R}^d), m < \infty$. Since $B_m(\mathbb{R}^d) \subset B_0(\mathbb{R}^d)$, the Lévy measure ν_m of some distribution in $B_m(\mathbb{R}^d)$ has the following property in terms of the polar decomposition in (1.2);

$$\nu_m(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi) l_{m,\xi}(r) dr,$$

where $l_{m,\xi}(r)$ is measurable in ξ and completely monotone in r for λ-a.e. ξ. In the following, we give a representation of the l-function $l_{m.\xi}$.

Theorem 4.2. *Let ν_m be a Lévy measure. It is the Lévy measure of some distribution in $B_m(\mathbb{R}^d)$ if and only if the l-function of ν_m can be expressed as*

$$l_{m,\xi}(r) = -\int_0^\infty t^{-1} l_{0,\xi}\left(t^{-1}r\right) d\varepsilon_{m-1}(t), \tag{4.2}$$

where $l_{0,\xi}$ is the l-function of the Lévy measure of some distribution in $\mu_0 \in B_0(\mathbb{R}^d)$.

Proof. We first show the "only if" part. Let ν_m be the Lévy measure of some distribution in $B_m(\mathbb{R}^d)$. Then by (4.1),

$$\nu_m(B) = -\int_0^\infty \nu_0(t^{-1}B) d\varepsilon_{m-1}(t)$$

for the Lévy measure ν_0 of some distribution in $B_0(\mathbb{R}^d)$. Here, ν_0 has the following polar decomposition:

$$\nu_0(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi) l_{0,\xi}(r) dr,$$

where $l_{0,\xi}$ is the l–function of ν_0. Thus,

$$\begin{aligned}
\nu_m(B) &= -\int_0^\infty d\varepsilon_{m-1}(t) \int_S \lambda(d\xi) \int_0^\infty 1_{t^{-1}B}(r\xi) l_{0,\xi}(r) dr \\
&= -\int_0^\infty d\varepsilon_{m-1}(t) \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi) t^{-1} l_{0,\xi}\left(t^{-1}r\right) dr \\
&= \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi) dr \left(-\int_0^\infty t^{-1} l_{0,\xi}\left(t^{-1}r\right) d\varepsilon_{m-1}(t) \right).
\end{aligned}$$

Thus the l-function $l_{m,\xi}$ of ν_m can be expressed as (4.2). This shows the "only if" part of the theorem, but the argument above and (4.1) also show the "if" part. The proof is complete □

Another view about $l_{m.\xi}$ is the following. The function $l_{m,\xi}(r)$ is completely monotone, namely, Laplace transform of some measure. But, the measure depends on m. We are interested in how. We are going to show it in some sense below. (The following is based on a discussion with K. Sato.)

For any measurable function f, write Laplace transform depending on f as

$$L(f)(x) = \int_0^\infty e^{-xy} f(y) dy$$

and define

$$\mathrm{Inv}(f)(x) = x^{-1} f(x^{-1}).$$

Then

$$(L \circ \mathrm{Inv})(f)(r) := L(\mathrm{Inv}(f))(x) = \int_0^\infty e^{-rx} x^{-1} f(x^{-1}) dx.$$

Theorem 4.3. *The l-function $l_{m,\xi}(r)$ of the Lévy measure of some distribution in $B_m(\mathbb{R}^d)$ can be expressed as*

$$l_{m,\xi}(r) = \underbrace{(L \circ \mathrm{Inv}) \circ \cdots \circ (L \circ \mathrm{Inv})}_{m \text{ times}}(l_{0,\xi})(r),$$

where $l_{0,\xi}$ is the l-function of the Lévy measure of some distribution in $B_0(\mathbb{R}^d)$.

Proof. By Theorem 2 with $m = 1$, we have

$$l_{1,\xi}(r) = \int_0^\infty t^{-1} l_{0,\xi}(t^{-1}r)e^{-t}dt = \int_0^\infty s^{-1} l_{0,\xi}(s^{-1})e^{-sr}ds = (L \circ \mathrm{Inv})(l_{0,\xi})(r).$$

Thus, the assertion is true for $m = 1$. Suppose that the assertion is true for some $m \geq 1$. Then, by Theorem 4.2 and Proposition 2.1, we have, by using Fubini theorem twice,

$$\begin{aligned}
l_{m+1,\xi}(r) &= -\int_0^\infty t^{-1} l_{0,\xi}(t^{-1}r)d\varepsilon_m(t) \\
&= \int_0^\infty t^{-1} l_{0,\xi}(t^{-1}r)dt \int_0^\infty e^{-t/u}u^{-1}(-\varepsilon'_{m-1}(u))du \\
&= \int_0^\infty u^{-1}(-\varepsilon'_{m-1}(u))du \int_0^\infty t^{-1} l_{0,\xi}(t^{-1}r)e^{-t/u}dt \\
&= \int_0^\infty u^{-1}(-\varepsilon'_{m-1}(u))du \int_0^\infty s^{-1} l_{0,\xi}(u^{-1}s^{-1}r)e^{-s}ds \\
&= \int_0^\infty s^{-1}e^{-s}ds \int_0^\infty u^{-1} l_{0,\xi}(u^{-1}s^{-1}r)(-\varepsilon'_{m-1}(u))du \\
&= \int_0^\infty s^{-1}e^{-s}ds \int_0^\infty u^{-1} l_{0,\xi}(u^{-1}s^{-1}r)d\varepsilon_{m-1}(u) \\
&= \int_0^\infty s^{-1}e^{-s} l_{m,\xi}(s^{-1}r)ds \\
&= (L \circ \mathrm{Inv})(l_{m,\xi})(r).
\end{aligned}$$

Thus, by the induction hypothesis, the assertion is also true for $m + 1$. This completes the proof. □

Remark 4.4. Theorem 4.3 shows us that $l_{m,\xi}(r)$ is Laplace transform of a special function

$$(\mathrm{Inv}) \circ \underbrace{(L \circ \mathrm{Inv}) \circ \cdots \circ (L \circ \mathrm{Inv})}_{(m-1) \text{ times}}(l_{0,\xi})(r),$$

depending on m in this way.

5. The class $B_\infty(\mathbb{R}^d)$

Finally, we study the class $B_\infty(\mathbb{R}^d)$. The main result is the statement (iii) in the following theorem that all stable distributions belong to $B_\infty(\mathbb{R}^d)$.

Theorem 5.1. (*i*) *The class $B_\infty(\mathbb{R}^d)$ is invariant under Υ-mapping.*
(*ii*) *The class $B_\infty(\mathbb{R}^d)$ is the largest class among the classes which are invariant*

under Υ*-mapping.*
(iii) All stable distributions belong to $B_\infty(\mathbb{R}^d)$.

Proof. (i) By Lemma 4.1 of Barndorff-Nielsen et al. (2006), $\Upsilon(B_\infty(\mathbb{R}^d)) \subset B_\infty(\mathbb{R}^d)$. Conversely, let $\widetilde{\mu} \in B_\infty(\mathbb{R}^d)$. Then $\widetilde{\mu} \in B_m(\mathbb{R}^d)$ for any $m \geq 0$. Fix m and suppose $\widetilde{\mu} \in B_m(\mathbb{R}^d)$. Then there exists uniquely $\mu_m \in B_{m-1}(\mathbb{R}^d)$ such that $\widetilde{\mu} = \Upsilon(\mu_m)$. However, the μ_m is the same for all m. Thus we can say that there exists a $\mu \in B_m(\mathbb{R}^d)$ for all m such that $\widetilde{\mu} = \Upsilon(\mu)$. Hence $\widetilde{\mu} \in \Upsilon(B_\infty(\mathbb{R}^d))$. This completes the proof of (i).

(ii) Let H be a class of infinitely divisible distributions such that $\Upsilon(H) = H$. Then for any m,

$$H = \Upsilon^m(H) \subset \Upsilon^m(I(\mathbb{R}^d)) = B_{m-1}(\mathbb{R}^d).$$

Hence

$$H \subset \cap_{m=1}^{\infty} B_m(\mathbb{R}^d) = B_\infty(\mathbb{R}^d).$$

This completes the proof of (ii).

(iii) Let $m \geq 1$. When μ_A is Gaussian with zero mean and covariance matrix A, suppose $\{X_t\}$ is a Gaussian Lévy process such that the covariance matrix of X_1 is $c_m^{-1}A$, where $c_m = \left(\int_0^1 \varepsilon_m^*(t)^2 dt\right)$. Then we have

$$\mu_A = \mathcal{L}\left(\int_0^1 \varepsilon_m^*(t) dX_t\right) \in B_m(\mathbb{R}^d)$$

for any $m \geq 1$. Hence $\mu \in B_\infty(\mathbb{R}^d)$. If μ is Gaussian with mean γ and covariance matrix A, it is trivial that μ also belongs to $B_\infty(\mathbb{R}^d)$, because $\mu = \mu_A * \delta_\gamma$.

When μ is non-Gaussian α-stable with the Lévy measure ν, we have

$$\nu(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi) \frac{1}{r^{1+\alpha}} dr = \int_S \lambda_m(d\xi) \int_0^\infty 1_B(r\xi) \frac{c_m}{r^{1+\alpha}} dr,$$

where

$$c_m = \int_0^1 \varepsilon_{m-1}^*(t)^\alpha dt \quad \text{and} \quad \lambda_m(d\xi) = c_m^{-1} \lambda(d\xi).$$

We also have

$$\begin{aligned} c_m r^{-(1+\alpha)} &= -r^{-(1+\alpha)} \int_0^\infty u^\alpha d\varepsilon_{m-1}(u) \\ &= -\int_0^\infty u^{-1} \left(ur^{-1}\right)^{1+\alpha} d\varepsilon_{m-1}(u) \\ &= -\int_0^\infty u^{-1} l_{0,\xi}\left(u^{-1}r\right) d\varepsilon_{m-1}(u), \end{aligned}$$

where

$$l_{0,\xi}(x) = x^{-(1+\alpha)},$$

which is completely monotone. Thus, by Theorem 4.2, $c_m r^{-(1+\alpha)}$ can be regarded as $l_{m,\xi}(r)$, implying that ν is the Lévy measure of a distribution in $B_m(\mathbb{R}^d)$. This is true for all m, and thus $\mu \in B_\infty(\mathbb{R}^d)$. This completes the proof. $\square$

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