Chapter 10 Fourier analysis of signals using discrete Fourier transform

10.1 Fourier analysis of signals using the DFT
10.2 DFT analysis of sinusoidal signals
10.3 the time-dependent Fourier transform

For finite-length signals, the DFT provides frequency-domain samples of the Discrete-time Fourier transform. In many cases, the signals do not inherently have finite length. The inconsistency between the finite-length requirement of the DFT and the reality of indefinitely long signals can be accommodated exactly or approximately through the concepts of windowing, block processing, and the time-dependent Fourier transform （短时傅立叶变换）.

10.1 Fourier analysis of signals using the DFT

One of the major applications of the DFT is in analyzing the frequency content of continuous-time signals.

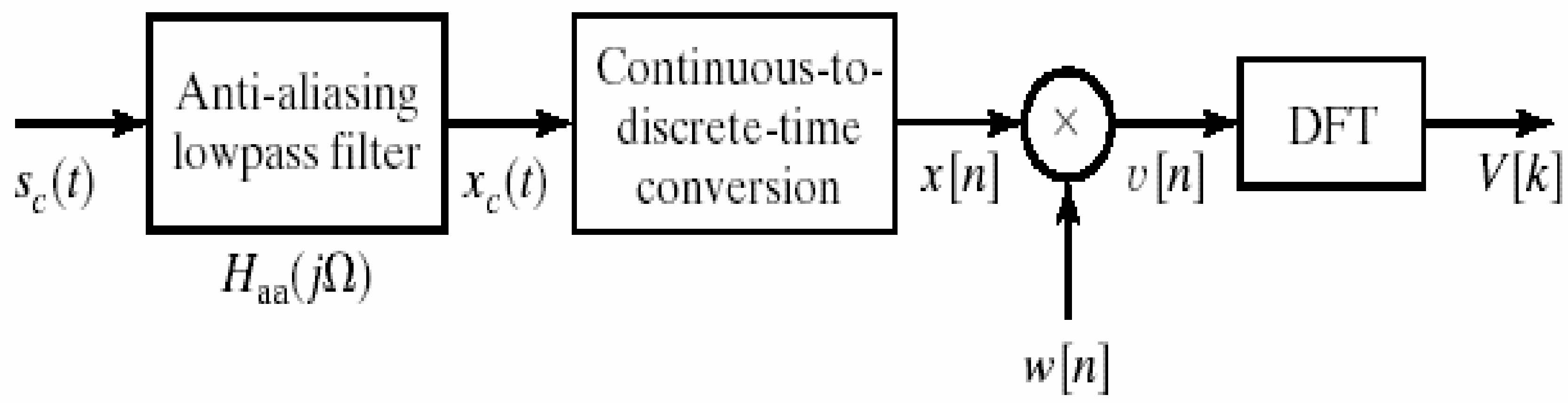
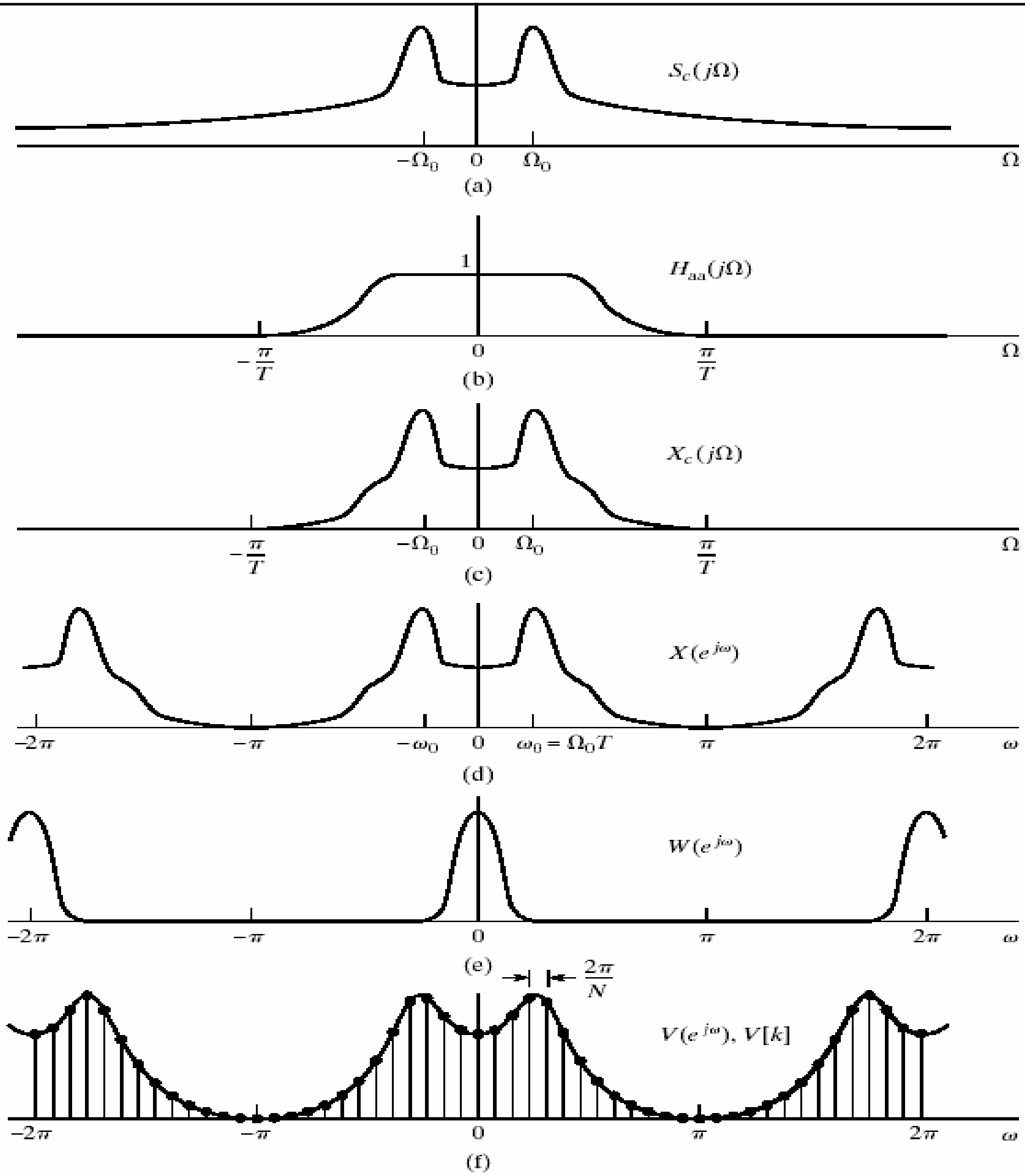


Figure 10.1



真实频谱

抗混迭滤波器频响

由滤波器非理想引入误差

由量化和混迭引入误差

窗序列频谱

<u>时域加窗和频域取样引入误差</u>

Figure 10.2

采样率与抗混迭滤波器的截止频率的关系：

$$f_s = 2f_c$$

相临谱线间的频率间距与**DFT**点数的关系：

$$\Delta\omega = 2\pi / N$$

$$\Delta\Omega = \Delta\omega / T = 2\pi f_s / N$$

$$\Delta f = \Delta\Omega / 2\pi = f_s / N$$

每条谱线对应的频率：

$$\omega_k = 2\pi k / N$$

$$\Omega_k = 2\pi f_s k / N$$

$$f_k = f_s k / N$$

频率分辨率与窗形状和窗长的关系：

矩形窗： $\Delta_{ml} = 4\pi / N$

汉宁/明窗： $\Delta_{ml} = 8\pi / N$

布莱克曼窗： $\Delta_{ml} = 12\pi / N$

10.2 DFT analysis of sinusoidal signals

10.2.1 the effect of windowing

10.2.2 the effect of spectral sampling

We choose sinusoidal signals as the specific class of examples to discuss, but most of the issues raised apply more generally.

10.2.1 the effect of windowing

$$X(e^{j\omega}) \Rightarrow V(e^{j\omega})$$

Before windowing:

$$x[n] = A_0 \cos(\omega_0 n + \theta_0) + A_1 \cos(\omega_1 n + \theta_1)$$

$$= \frac{A_0}{2} e^{j\theta_0} e^{j\omega_0 n} + \frac{A_0}{2} e^{-j\theta_0} e^{-j\omega_0 n} + \frac{A_1}{2} e^{j\theta_1} e^{j\omega_1 n} + \frac{A_1}{2} e^{-j\theta_1} e^{-j\omega_1 n}$$

$$-\infty < n < \infty$$

$$X(e^{j\omega}) = 2\pi[\frac{A_0}{2} e^{j\theta_0} \delta(\omega - \omega_0) + \frac{A_0}{2} e^{j\theta_0} \delta(\omega + \omega_0)$$

$$+ \frac{A_1}{2} e^{j\theta_1} \delta(\omega - \omega_1) + \frac{A_1}{2} e^{j\theta_1} \delta(\omega + \omega_1)]$$

After windowing:

$$v[n] = x[n]w[n]$$

$$= \frac{A_0}{2} w[n] e^{j\theta_0} e^{j\omega_0 n} + \frac{A_0}{2} w[n] e^{-j\theta_0} e^{-j\omega_0 n}$$

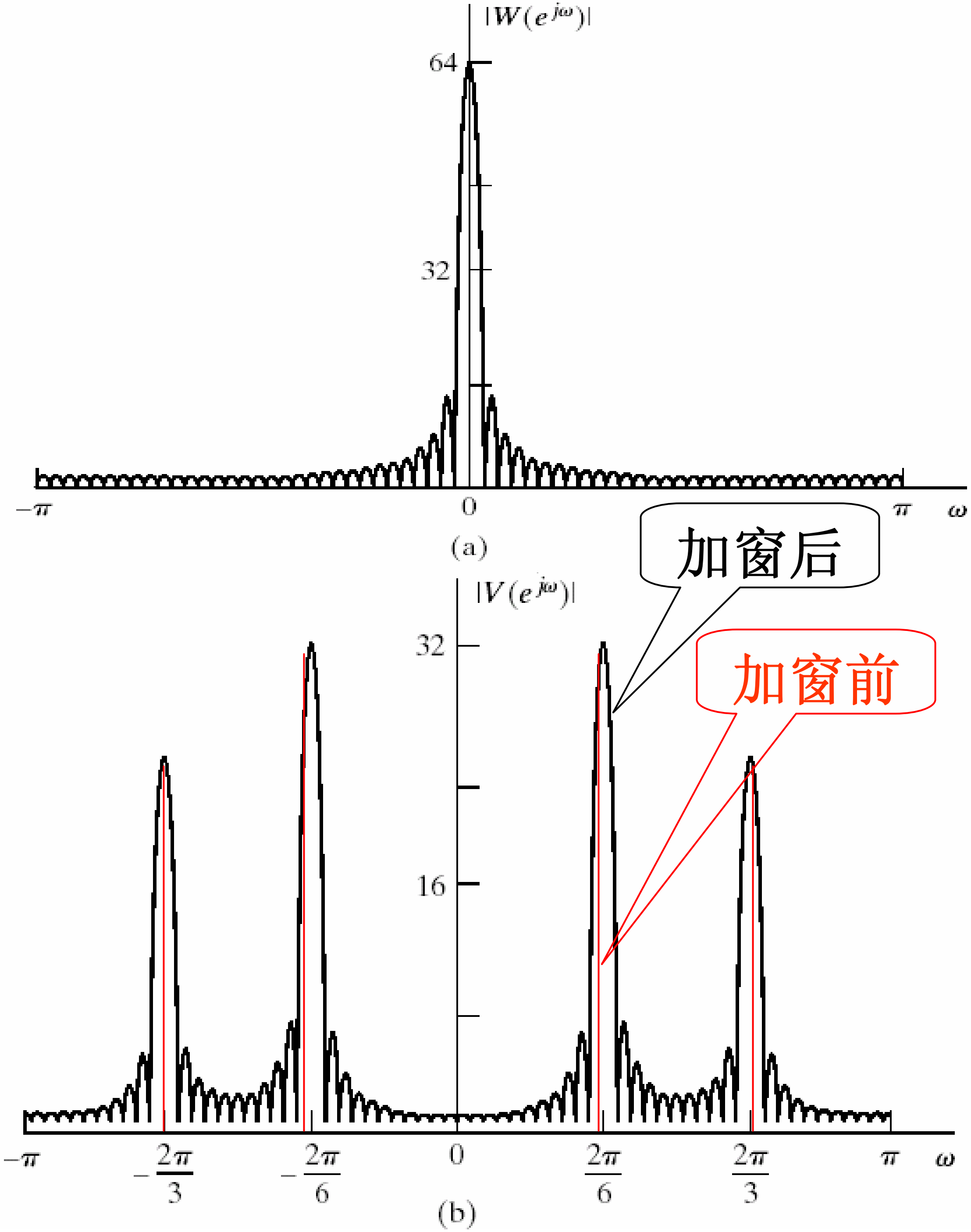
$$+ \frac{A_1}{2} w[n] e^{j\theta_1} e^{j\omega_1 n} + \frac{A_1}{2} w[n] e^{-j\theta_1} e^{-j\omega_1 n}$$

$$V(e^{j\omega}) = \frac{1}{2\pi} X(e^{j\omega}) * W(e^{j\omega})$$

$$= \frac{A_0}{2} e^{j\theta_0} W(e^{j(\omega-\omega_0)n}) + \frac{A_0}{2} e^{j\theta_0} W(e^{j(\omega+\omega_0)n})$$

$$+ \frac{A_1}{2} e^{j\theta_1} W(e^{j(\omega-\omega_1)n}) + \frac{A_1}{2} e^{j\theta_1} W(e^{j(\omega+\omega_1)n})$$

EXAMPLE **Example 10.3**

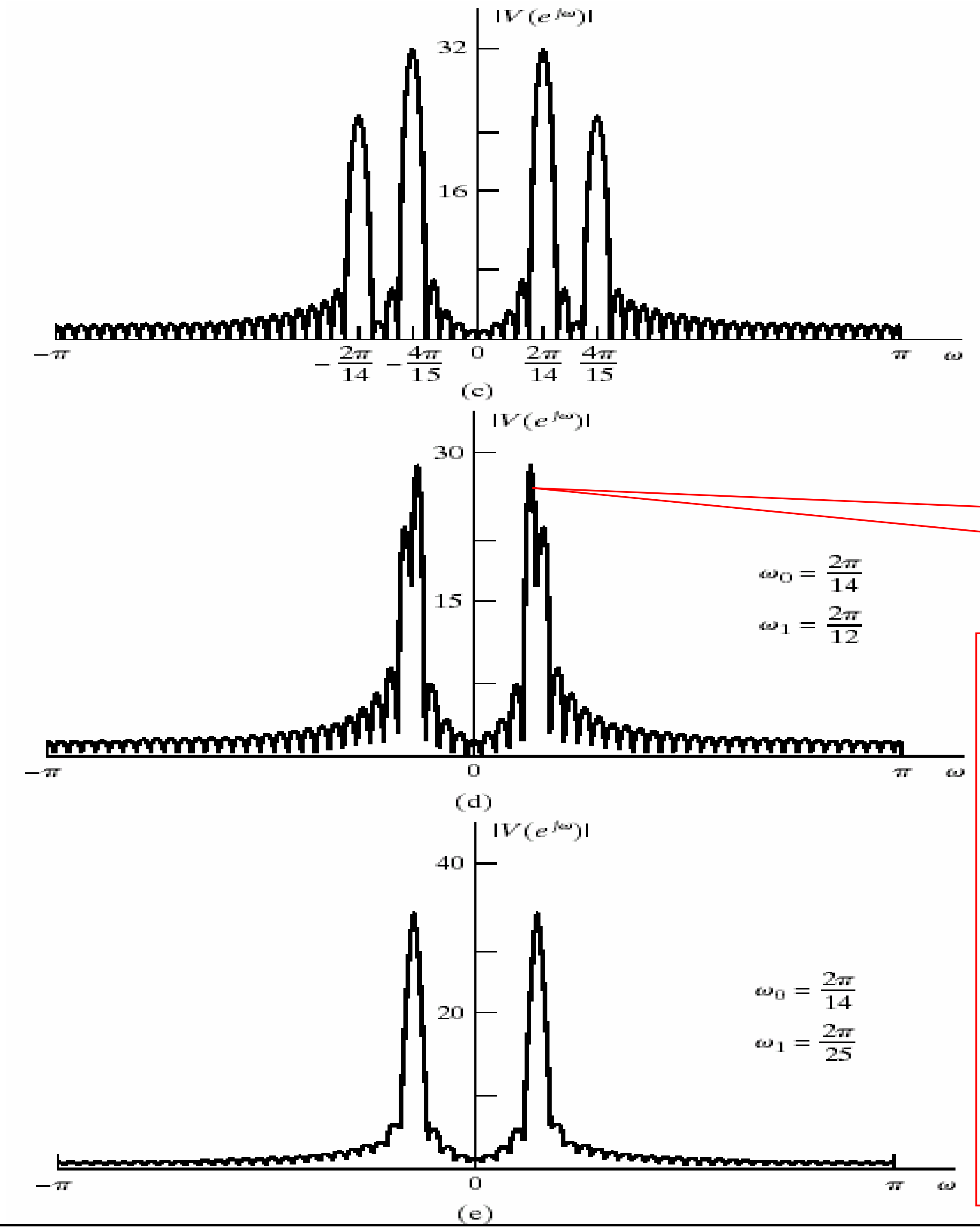


（1）谱线展宽成窗频谱的主瓣宽

（2）产生旁瓣，衰减等于窗频谱的旁瓣衰减

Figure 10.3(a)(b)

Figure 10.3(c)(d)(e)



（3）谱泄露

谱线展宽和谱泄露导致：

难以确定频率的位置和幅度；

降低频率分辨率。

产生旁瓣导致：

产生假信号；

淹没小信号。

We can find that windowing smears or broadens the impulse in theoretical Fourier representation, and thus reduces the ability to resolve sinusoidal signals that are closely spaced in frequency .

The amplitude of one spectrum is affected by the amplitude of another and vice versa when two components are closely spaced in frequency. This interaction is called leakage（泄露）. The component at one frequency leaks into the vicinity of another component due to the spectral smearing introduced by the window.

So reduced resolution and leakage are the two primary effects on the spectrum as a result of applying a window to the signal. The resolution is influenced primarily by the width of the main lobe of $W(e^{j\omega})$,while the degree of leakage depends on the relative amplitude of the main lobe and the side lobes of $W(e^{j\omega})$.

We define the frequency resolution（频率分辨率） is equal to the width of the main lobe of $W(e^{j\omega})$.

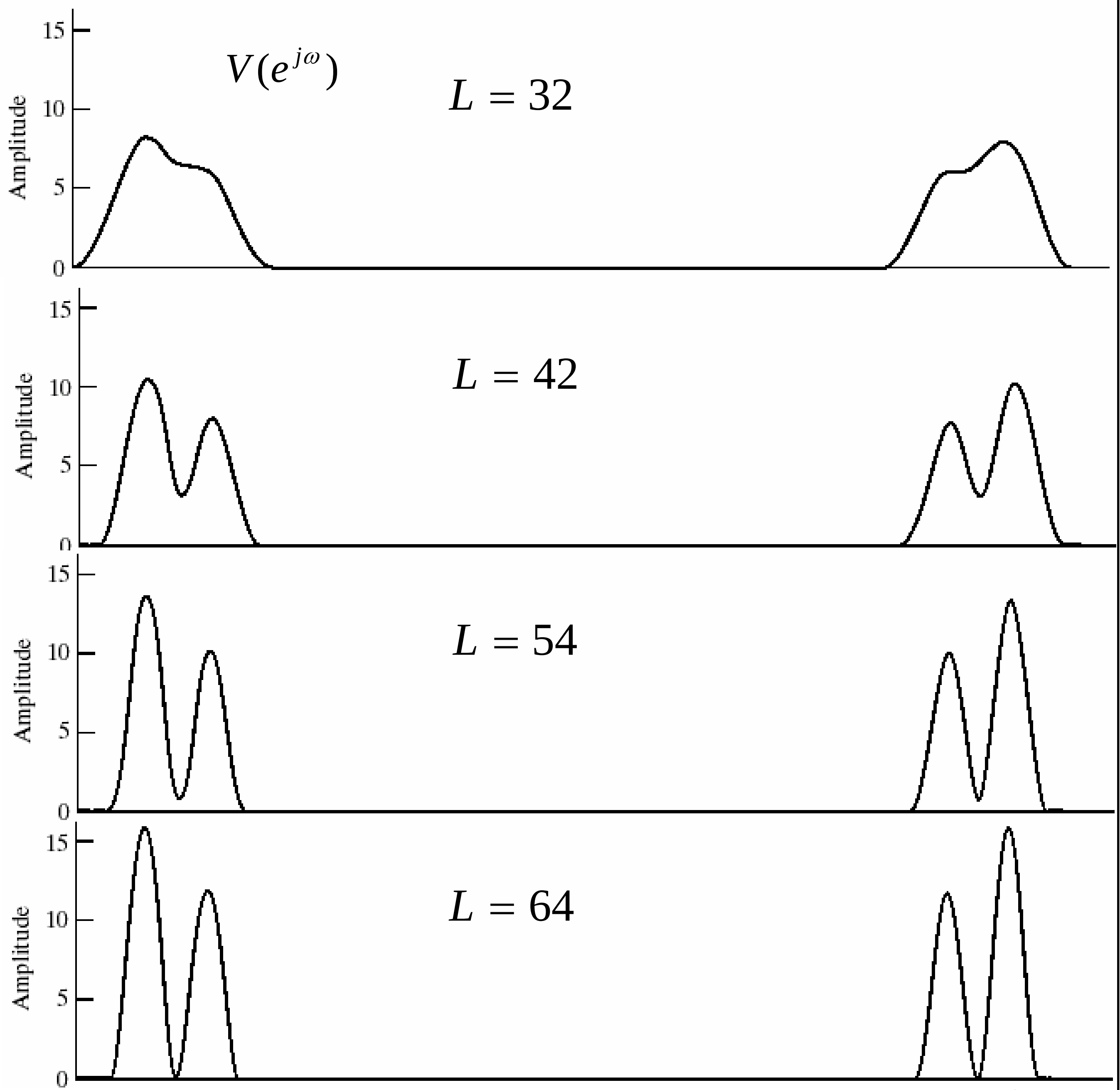
注意：频率分辨率=主瓣宽>DFT谱线间距

EXAMPLE

example 10.8 :

$$x[n] = (\cos(\frac{2\pi}{14}n) + 0.75\cos(\frac{4\pi}{15}n))$$

$w[n] = kaiser\ (L = 32 \sim 64, \beta = 5.48)$



Conclusion: increase L can increase resolution

Figure 10.10

EXAMPLE

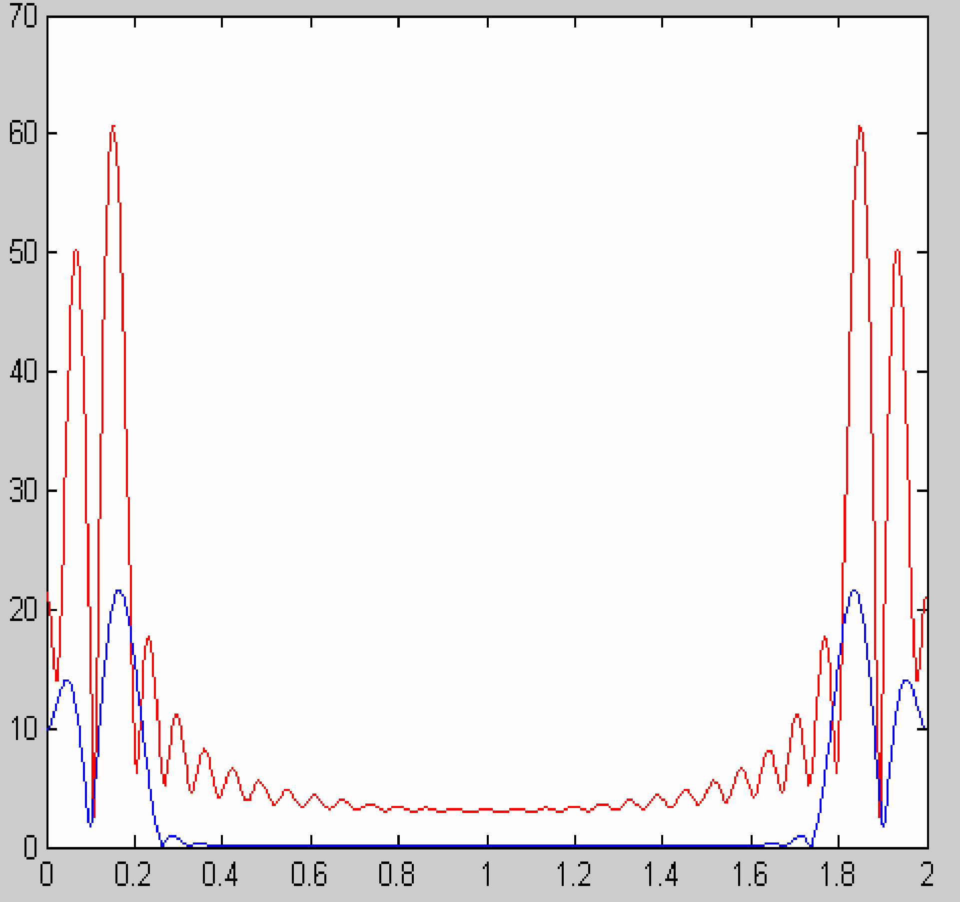
$$v[n] = (3.5 * \cos(\frac{2\pi}{14} n)$$

$$+ 3.5 * 0.75 \cos(\frac{2\pi}{25} n))$$

$$w_R[n] : red$$

$$w_{hanning}\ [n] : blue$$

$$L = 32$$



Conclusion: **shape of window has effect on frequency resolution**

Determine window's shape and length

（1） for Kaiser windows:

$$\beta = \begin{cases} 0.12438(A_{sl} + 6.3) & 60 < A_{sl} < 120 \\ 0.76609(A_{sl} - 13.26)^{0.4} + 0.09834(A_{sl} - 13.26) & 13.26 \le A_{sl} \le 60 \\ 0 & A_{sl} < 13.26 \end{cases}$$

$$L = \frac{24\pi(A_{sl} + 12)}{155\Delta_{ml}} + 1$$

Δ_{ml} : $main-lobe \quad width$

A_{sl} : $relative \quad side-lobe \quad level$

（2）for Blackman window:

look up the table

10.2.2 the effect of spectral sampling

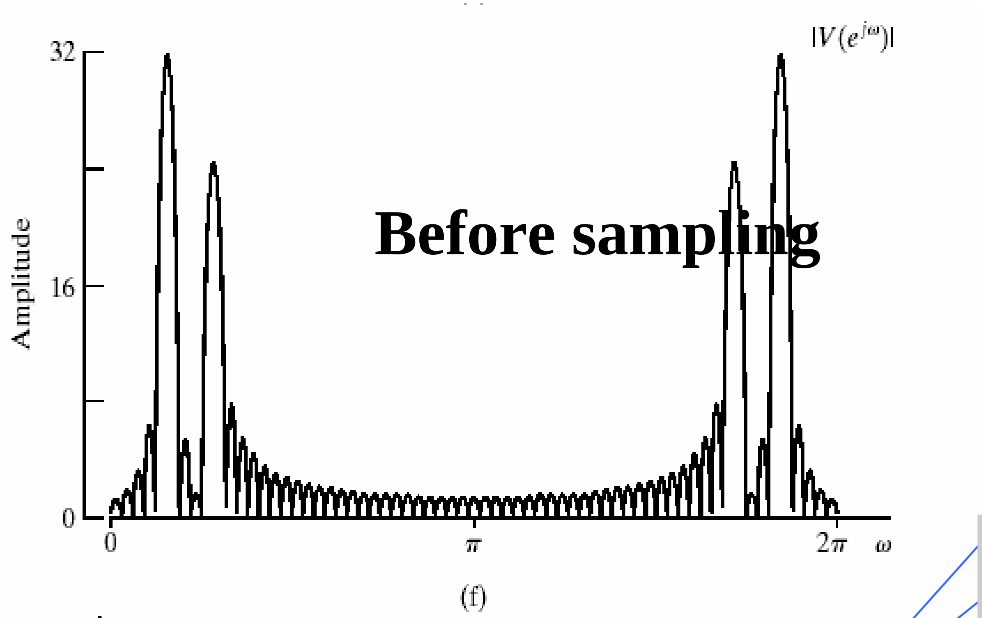
$$V(e^{j\omega}) \Rightarrow V[k]$$

The DFT of the windowed sequence provides samples of $V(e^{j\omega})$ **.**
Spectral sampling can sometimes produce misleading results.

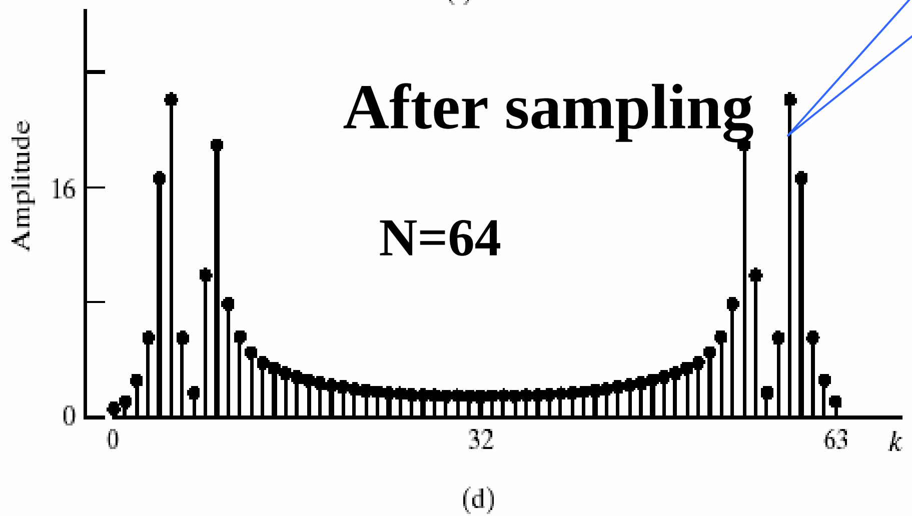
EXAMPLE

$$example\ 10.4: x[n] = (\cos(\frac{2\pi}{14}n) + 0.75\cos(\frac{4\pi}{15}n))R_{64}[n]$$

$$w[n]'s \quad length \quad L = 64$$

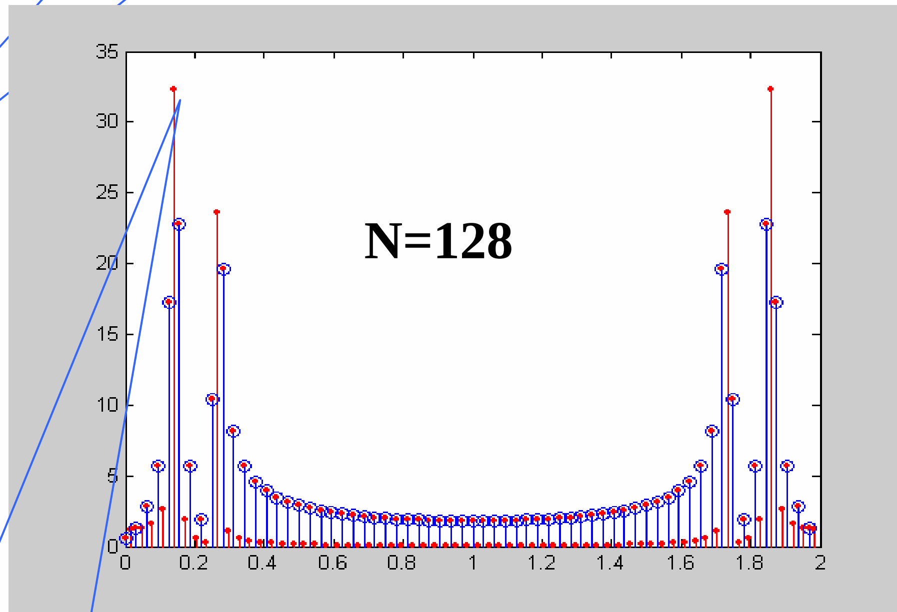


(f)



(d)

峰值未取到



增加N峰值取到

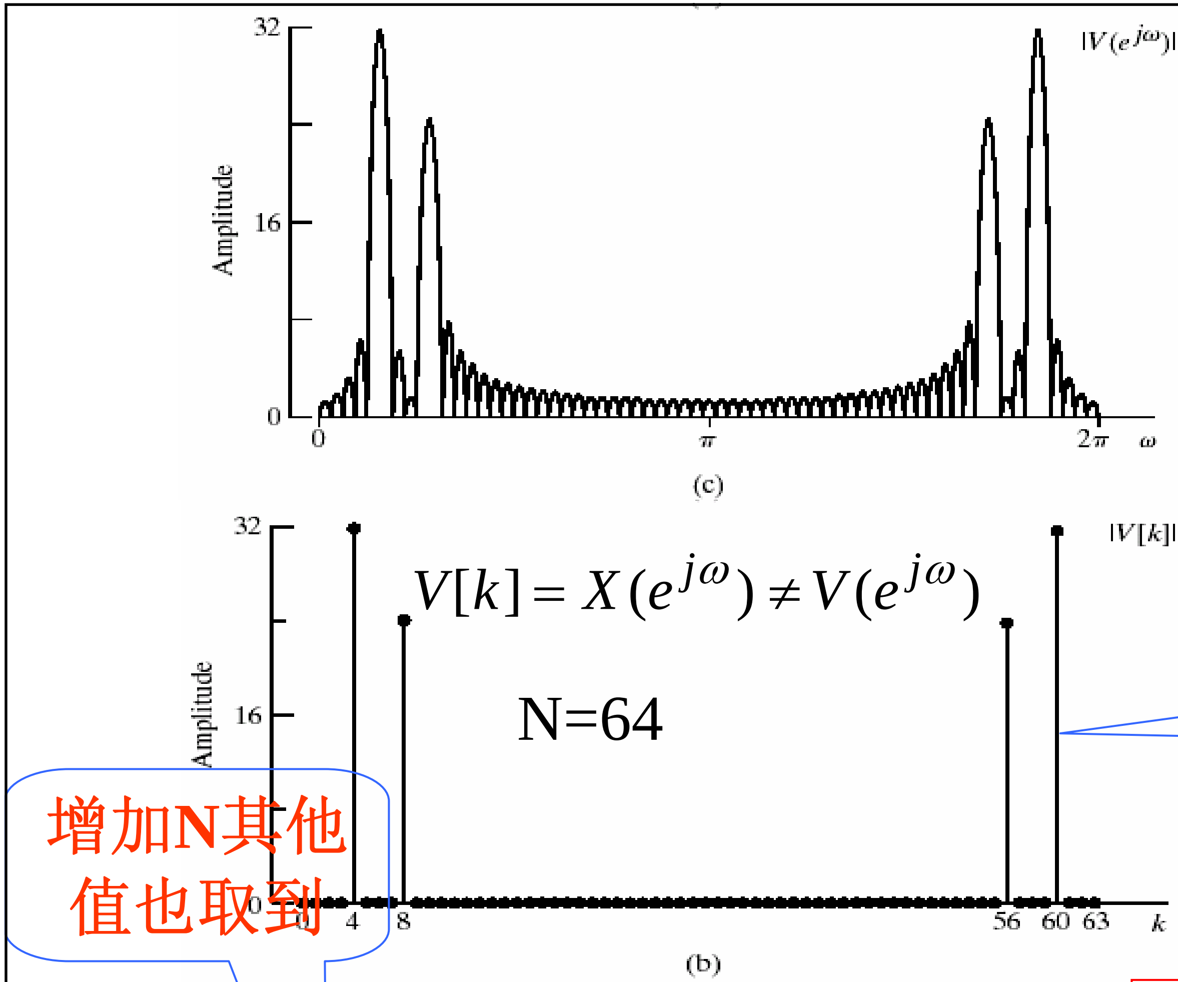


Figure 10.6

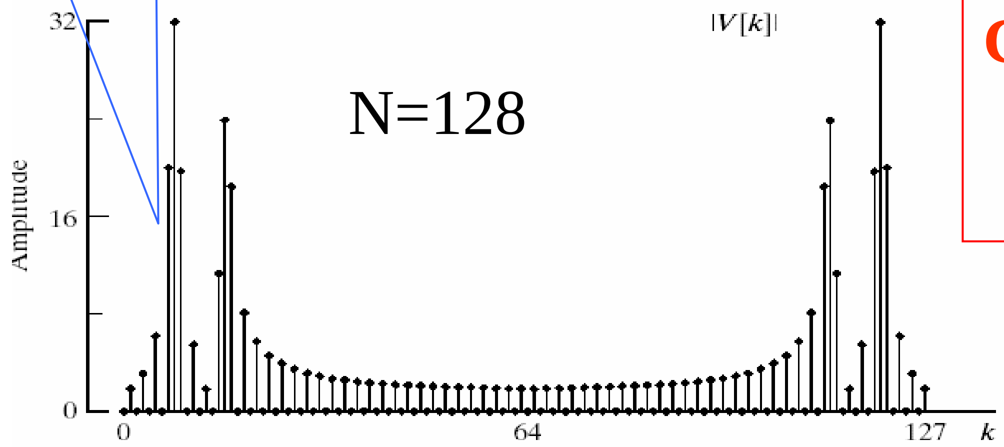


Figure 10.7

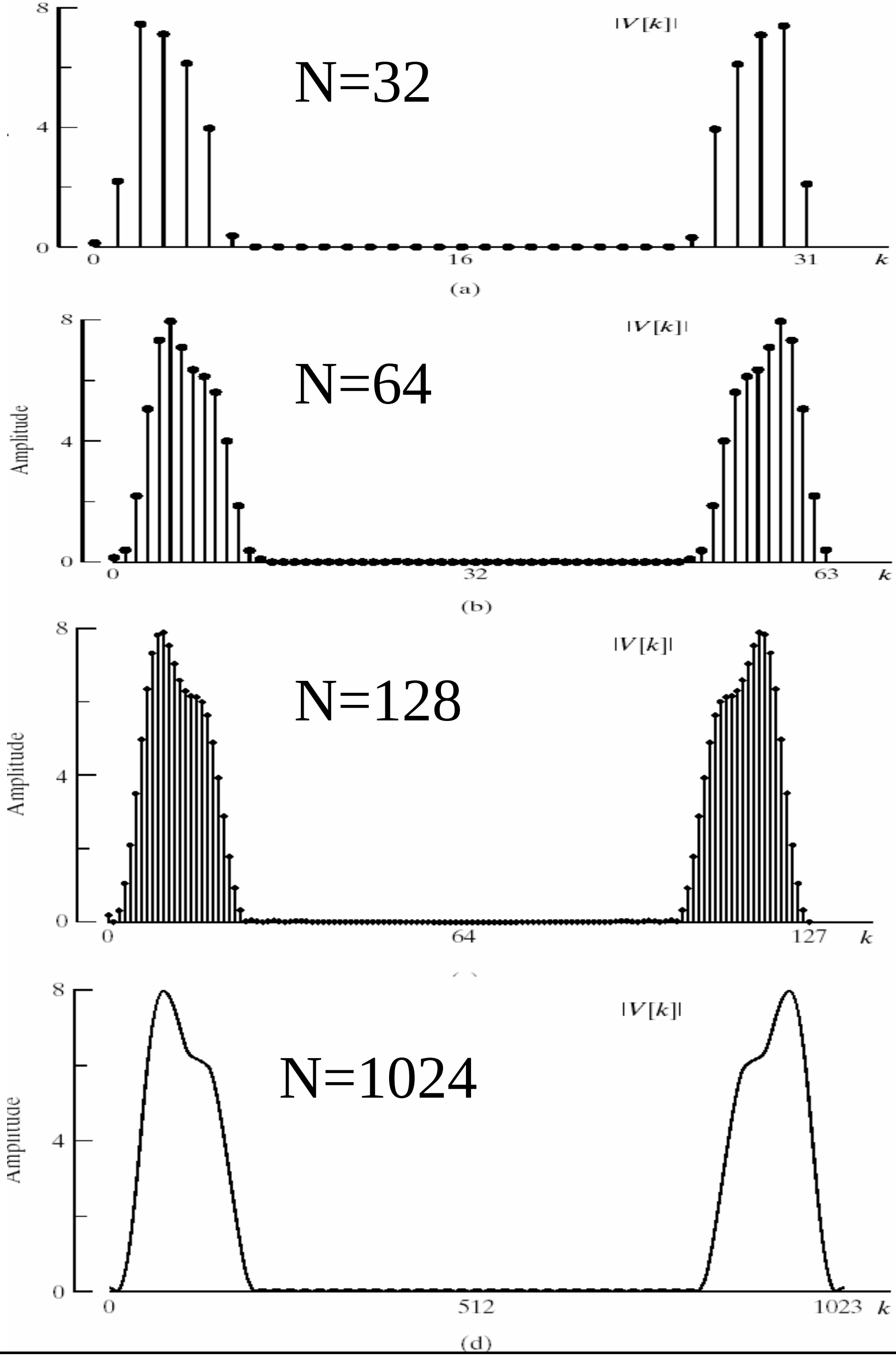
EXAMPLE

$$example\ 10.5: x[n] = (\cos(\frac{2\pi}{16} n$$

$$+ 0.75\cos(\frac{2\pi}{8} n))R_{64}[n]$$

$$w[n]'s \quad length \quad L = 64$$

Conclusion : increase N can fine the sampling of the spectrum.



EXAMPLE

$$example\ 10.7:$$

$$x[n] = (\cos(\frac{2\pi}{14}n)$$

$$+ 0.75\cos(\frac{4\pi}{15}n))$$

$$w[n] = kaiser\ (L = 32, \beta = 5.48)$$

$$N = 32 \sim 1024$$

Conclusion: increase N can't increase frequency resolution.

Figure 10.9

EXAMPLE

MATLAB分析窗长对DFT的影响

$$f(t) = \cos(2\pi f_1 t) + \cos(2\pi f_2 t),\ f_1 = 2Hz,\ f_2 = 2.5Hz,\ f_s = 64Hz,$$

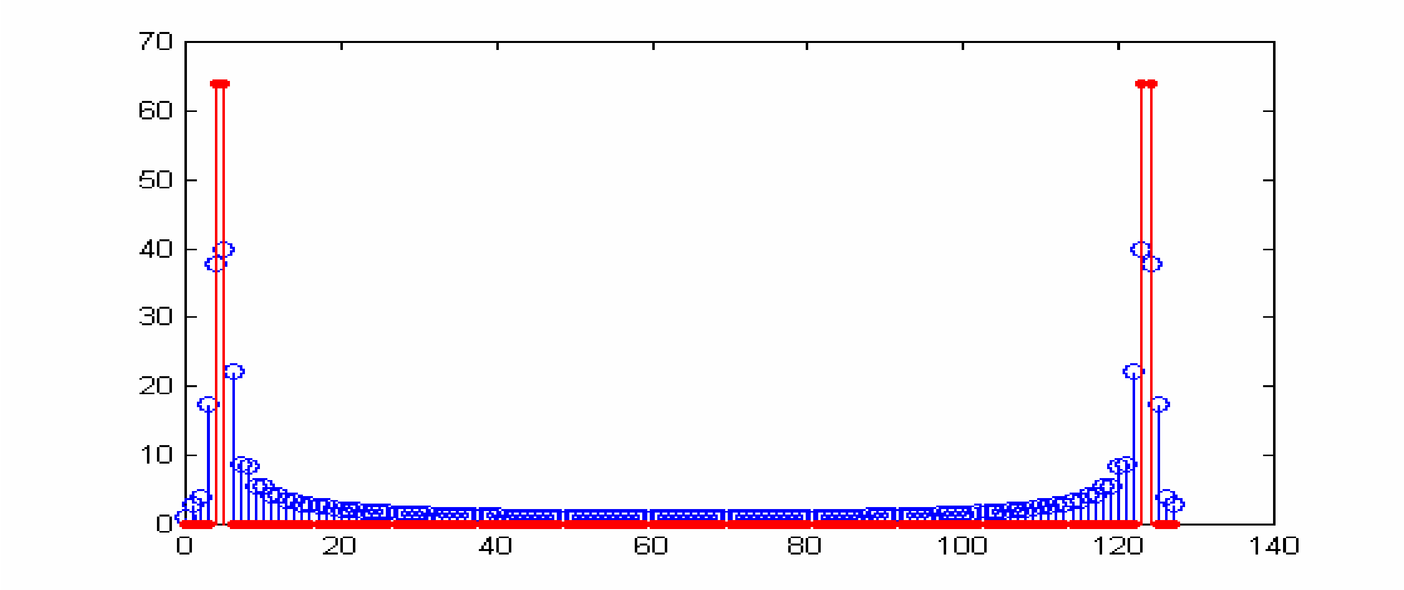
$$f_1[n] = \begin{cases} f(t)|_{t=nT} & 0 \le n \le 63 \\ 0 & 64 \le n < 128 \end{cases},T = 1/f_s$$

$$f_2[n] = f(t)|_{t=nT} \qquad 0 \le n < 128$$

分别作 128 点 DFT，比较二者的不同。

F1[n]:窗长64，矩形窗，DFT点数128
F2[n]:窗长128，矩形窗，DFT点数128

```
L1=64;            L2=128; N=128;  T=1/64
n1=0:L1-1;        x1=cos(2*pi*2*n1*T)+ cos(2*pi*2.5*n1*T)
n2=0:L2-1;        x2=cos(2*pi*2*n2*T)+ cos(2*pi*2.5*n2*T)
k=0:N-1;          X1=fft(x1,N);      X2=fft(x2,N)
stem(k,abs(X1));          hold on;            stem(k,abs(X2),'r.');
```



思考：如何用加窗**DFT**求周期为**N**的周期序列的**DFS**？

即使峰值正好被取样到，

且谱线间隔=主瓣宽的一半。

条件：采用矩形窗；**DFT**点数=窗长。

EXAMPLE 同前例：手工求解

$L1=64, L2=128, N1=N2=128$

$f(t)=\cos(2\pi f_1 t)+\cos(2\pi f_2 t),\ f_1=2Hz,\ f_1=2.5Hz,\ f_s=64Hz,$

$$f_1[n]=\begin{cases} f(t)|_{t=nT} & 0\le n\le 63 \\ 0 & 64\le n<128 \end{cases}, T=1/f_s$$

$$f_2[n]=f(t)|_{t=nT} \qquad 0\le n<128$$

$F_1[k]$和$F_2[k]$分别是 $f_1[n]$和$f_2[n]$的128点DFT

比较 $F_1[k]$和$F_2[k]$的谱线， k为何值时谱线较高 ？

解 :（1）谱线间隔一样： $\Delta f=\dfrac{f_s}{N}=0.5Hz$

（2）两个分量的位置一样 ： $f_1=k_1\Delta f, k_1=4$

$f_2=k_2\Delta f, k_2=5$，

根据对称性还有 $k_3=128-k_1=124, k_4=128-k_2=123$

（3）主瓣宽：

情况 1： $\Delta_{ml1}=4\pi/L1=4\Delta f, k=3,6,128-3,128-6$等谱线幅度也较高

情况 2： $\Delta_{ml2}=4\pi/L2=2\Delta f$,其他谱线为 0

结论：

（1）关于加窗：增大窗长或改变窗形状（例如选矩形窗）可提高频率分辨率；

改变窗形状可改变旁瓣相对幅度。

通常用增加窗长的方式增大分辨率，而不用改变窗形状，因为会增大旁瓣幅度。

但L太大，时间分辨率降低。

（2）关于频谱取样：点数>=窗长（满足频域取样定理）才能重构时域；

增大点数不能提高分辨率，但可取到每个细节，以至于线性内插就可得到较精细的连续频谱。但点数大运算量也大。

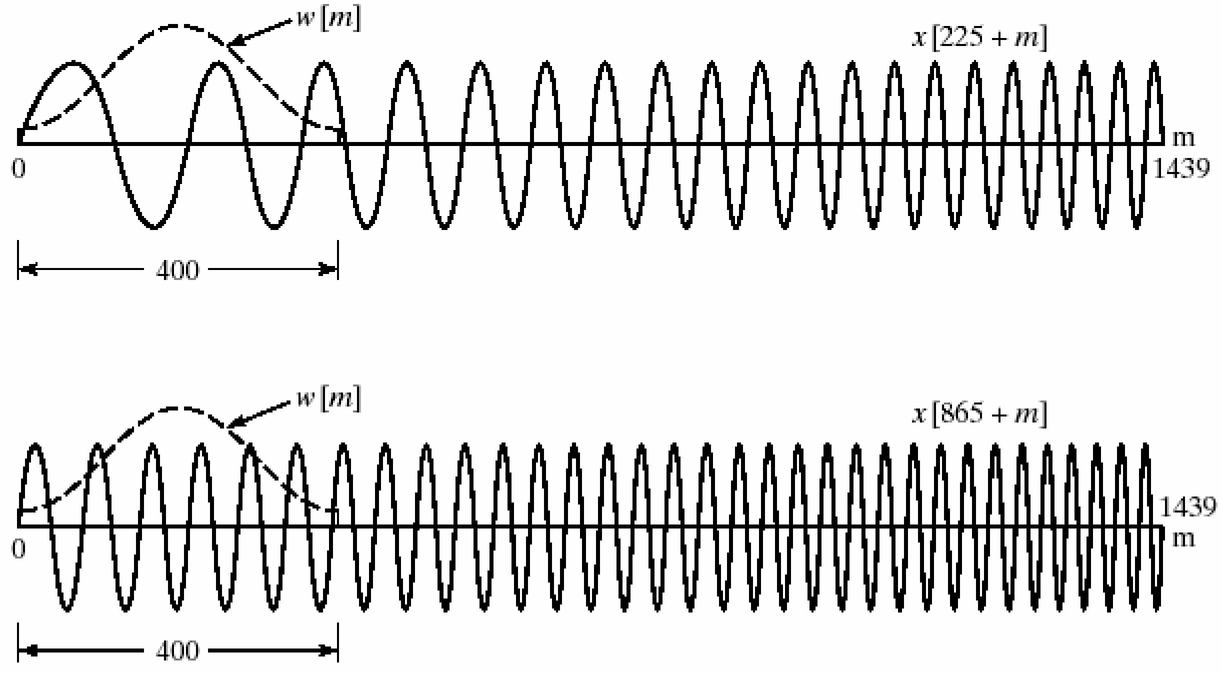
考虑到时间分辨率和运算量的要求，N和L的取值需折中。

10.3 the time-dependent Fourier transform

1.definition

$$X[n,\lambda) = \sum_{m=-\infty}^{\infty} x[n+m]w[m]e^{-j\lambda m}$$

$$x[n+m]w[n] = \frac{1}{2\pi}\int_0^{2\pi} X[n,\lambda)e^{j\lambda m}d\lambda$$



$$x[n] = \cos(\omega_0 n^2)$$

Figure 10.11

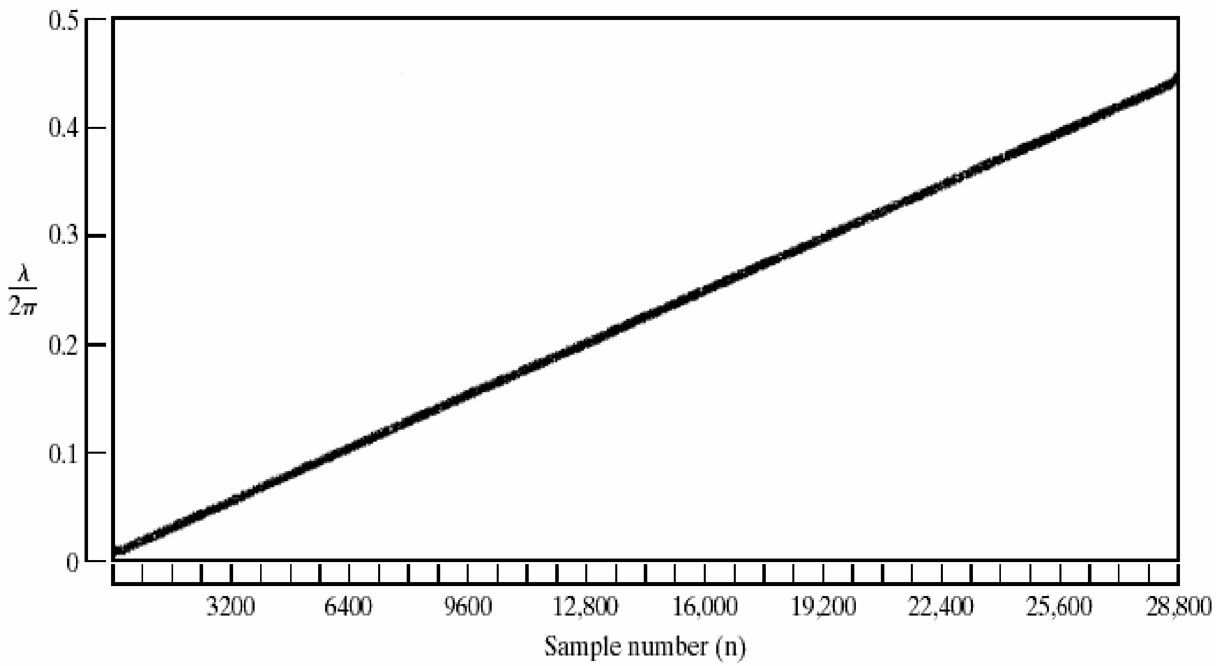


Figure 10.12

小结

10.1 用**DFT**分析信号的系统

10.2 用**DFT**分析正弦信号

分析正弦序列（稳定信号）得出结论：

加窗和频谱采样对频谱分析的影响

10.3 依时**DFT**

将**10.2**的结论应用于分析一般信号

要求：

加窗和取样分别对**DFT**谱线的影响

频率分辨率的概念，与窗长、窗形状和**DFT**点数的关系

会分析正弦信号**DFT**的谱线大致情况

作业和实验

10.21　10.31(a)-(c)

实验　**31**

34和**35**（画在一起）

36

37（**B**）

39（**A**）