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The Discontinuous Trend Unit Root Test with a Break Interval

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Dickey and Fuller proposed tests for the unit root hypotheses in a uni-variate time series. Perron (1989) extended the t-ratio type unit-root tests so that they allow for a break in the deterministic trend and/or in the intercept term. In practice, it seems not easy to specify the break point correctly. Zivot and Andrews (1992) proposed a test in which the break point is estimated by repeated calculations. Morimune and Nakagawa (1999) studied the effect of a misspecified break point on the Perron tests, and the accuracy of the asymptotic expression is examined under various specifications of the error. This paper proposes to set an interval that possibly covers a break point in the Perron tests. The χ^2 type test statistic which is termed Ψ and defined by the equation (9) is calculated for all possible sub-intervals, and the mean of all Ψ values is used as a test statistic. The critical values of the mean-Ψ test are calculated by simulation.

Keywords: unit-root test, discontinuous-trend, break-interval
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1. Introduction

Dickey and Fuller proposed the tests for unit root hypotheses in a uni-variate time series. Perron (1989) extended the t-ratio type unit-root tests so that they allow for a break in the deterministic trend and/or in the intercept term. In practice, it seems difficult to specify the break point correctly. Zivot and Andrews (1992) proposed a test in which the break point is statistically determined but their test does not necessarily lead to an empirically satisfactory break point. Morimune and Nakagawa (1999) studied the effect of a misspecified break point on the Perron tests, and the accuracy of the asymptotic expression is examined under various specifications of the error. This paper proposes to set a break interval that possibly covers a break point in the Perron tests. It is difficult to specify a break point but easier to set a break interval in empirical studies. Tests lose power by setting break intervals, but it helps to avoid misspecifying the break point. The unit root test is less susceptible to the choice of a particular break point.

Test is applied to the US macro series. In this application, the break intervals used in our study is varied from the shortest 1930–1930 to the longest 1930–1941 interval in most cases. The χ^2 type test statistic Ψ defined by the equation (9) is calculated for all possible sub-intervals, and the mean of all Ψ values is chosen as a test statistic. Simulation under the null hypothesis is used to find the critical values of the mean-Ψ test.

2. Model

The alternative regression of the test by Perron (1989) for the unit root which allows for a break in the deterministic trend as well as in the intercept term is

$$y_t = \Sigma_{i=1,2}(\alpha_i^+ + \beta_i^+ t)DU_{it} + \gamma D_{B+1,t} + u_t,\ u_t = (1+\phi)u_{t-1} + \varepsilon_t \tag{1}$$

where ε_t is the white noise with variance σ^2. This regression equation is called the C model by Perron. The A and B tests by Perron include the shifting intercept but the common trend terms or the shifting intercept and trend terms where the latter is continuous at the break point, respectively. They are formulated in Appendix A. Extensions of our analyses to A and B tests are straightforward[1]. The null hypothesis of the test is H_0: $\phi = 0$. The sub-interval dummy variables DU_{1t} and DU_{2t} are 1 for $1 \le t \le B$ and $B+2 \le t \le T$, and 0 otherwise, respectively, $D_{B+1,t}$ is a shock dummy variable which is 1 when t is at the break point B + 1, and 0 otherwise. This shock dummy variable has an effect of jumping the observation at the break point in the estimation. If the Cochrane-Orcutt transformation is applied to the equation (1), it is recast as

$$\Delta y_t = \phi y_{t-1} + \Sigma_{i=1,2}(\alpha_i + \beta_i^+ \phi t)DU_{it} + \gamma D_{B+1,t} + \varepsilon_t. \tag{2}$$

In this equation, α_i is newly defined. The Dickey-Fuller type t-test is derived as the t-ratio of the ϕ coefficient on y_{t-1} in this regression equation neglecting the nonlinear constraints on the trend coefficients. Lack of the shock dummy variable causes inconsistency of the test. See Morimune and Nakagawa (1999, 2001). Using the shock dummy variables to jump observations in the break interval (B, B + m) that is m consecutive points after B, the regression equation under the alter-

[1] Estimating (1) by OLS and calculating residuals $\hat{u}_t$, the test statistic is the t-ratio of the ϕ coefficient in the regression $\Delta\hat{u}_t = \phi\hat{u}_{t-1} + \gamma D_{B+1,t}$ + error. This approach to the unit root test is found in Schmidt and Phillips (1992), Oya and Toda (1995), and Morimune and Nakagawa (2001). The F-ratio type test denoted Ψ does not follow from this formulation. (Once ϕ is estimated, α^+ and β^+ coefficients can be re-estimated using $\hat{\phi}$. This leads to the nonlinear estimation of ϕ and the resulting test statistics may have more complicated properties than that of the t-ratio of the ϕ coefficient in (2)).

native hypothesis of the test is specified as

$$\Delta y_t = \phi y_{t-1} + \Sigma_{i=1,2}(\alpha_i + \beta_i t)DU_{it} + \Sigma_{i=1,m}\gamma_i D_{B+i,t} + \varepsilon_t. \quad t = 2, ..., T \tag{3}$$

In this equation, β_i is newly defined. Each of the shock dummy variable $D_{B+i,t}$ is 1 when t is B + i and 0 otherwise, i = 1, 2, ..., m. These m observations are not used in estimating coefficients such as ϕ, α, and β. This alternative modeling may reduce the risk of specifying an erroneous break point when the break point is not known.

The null hypothesis $\phi = 0$ automatically implies $\beta = 0$ as can be seen by the equation (2). The null hypothesis leads to the regression equation with a break in the intercept term

$$\Delta y_t = \Sigma_{i=1,2}\alpha_i DU_{it} + \Sigma_{i=1,m'} \quad \gamma_i D_{B+i,t} + \varepsilon_t.$$

However, the null regression used in Section 5 is

$$\Delta y_t = \alpha + \Sigma_{i=1,m'} \quad \gamma_i D_{B+i,t} + \varepsilon_t. \quad t = 2, ..., T \tag{4}$$

by the same reason as that found by Zivot and Andrews (1992). The equation (4) is nested by the alternative regression (3). The break interval (B, B + m’) under the null hypothesis needs not be the same as that under the alternative hypothesis. It can be shorter under the alternative hypothesis. However, m’ is set equal to m in all the tests below, for simplicity.

3. Outlier Models

The regression equation (3) is augmented as

$$\Delta y_t = \phi y_{t-1} + \Sigma_{i=1,2}(\alpha_i + \beta_i t)DU_{it} + \Sigma_{i=1,m}\gamma_i D_{B+i,t} + \Sigma_{k=1,s}\theta_k \Delta y_{t-k} + \varepsilon_t, \tag{5}$$

t = S + 2, ..., T. This augmented equation can be interpreted as outlier models by which the sudden break point is transformed into gradual changes. Assume A(L) to be a finite order polynomial equation of the lag operator so that A(0) = 0, and the zeroes of the equation 1 – A(L) = 0 lie outside the unit circle. The same condition apply to other polynomial equations such as B(L) and C(L) below. Perron (1989) defines an additive outlier model as

$$A(L)\{y_t - \alpha_1 - \beta_1 t - (\alpha_2 + \beta_2 t)DU^*_{2t}\} = B(L)\varepsilon_t, \tag{6}$$

where the regression is written in terms of the common intercept, trend terms, and the outlier terms, and DU^*_{2t} is 1 for $B + 1 \le t \le T$ since shock dummy is not included in the equation. Equation (5) follows from dividing both sides of (6) by B(L), and expanding A(L)/B(L) as $\{\Delta - \phi L - \Sigma_{i=1,s}\theta_i L^i\Delta\}$. $((\Delta - \Sigma_{i=1,s}\theta_i L^i\Delta)(\alpha_2$

$+ \beta_2 t)DU^*_{2t} = \sum_{i=0,s} \theta_i D_{B+1,t-i} + \alpha DU_{2t}$ by using the same notations for newly defined coefficients freely, and the properties of shock dummy variable such that $D_{B+1,t-S} = D_{B+1+S,t}$). Similarly, since $u_t = \varepsilon_t/(1 - \phi L)$ by equation (1), an innovative outlier model with autoregressive error is

$$(1 - \phi L)A(L)(y_t - \alpha_1 - \beta_1 t) = (1 - \phi L)B(L)(\alpha_2 + \beta_2 t)DU^*_{2t} + C(L)\varepsilon_t. \qquad (7)$$

The outlier part can be combined together with the error term if $(1-\phi L)B(L)$ and $C(L)$ are the same as in Perron (1989). This is in line with Fox (1972). Equation (5) follows from dividing both sides of the equation by C(L), and expanding A(L)/C(L) and B(L)/C(L) as before. The left hand side leads to an augmented autoregressive equation and a step function, and the right hand side leads to a step function and shock dummy variables.

4. Test Statistics

The number of observations in the first and the second intervals are B and T − B − m, respectively, since the break interval includes m observations. The t-ratio of the ϕ coefficient in the equation (3) is a simple extension of the $\hat{\tau}_\tau$ test statistic by Dickey and Fuller (1981). Denote this t-ratio $\hat{\tau}_\tau$ again, the weak convergence under the null hypothesis is given by

$$\hat{\tau} \Rightarrow \frac{\sum_{i=1,2} \lambda_i \int_0^1 \tilde{B}_i(r) d\tilde{B}_i(r)}{\sqrt{\sum_{i=1,2} \lambda_i^2 \int_0^1 \tilde{B}_i(r)^2 dr}}, \qquad (8)$$

where λ_i, i = 1, 2, are the break ratios of the two intervals B/(T − m) and (T − B − m)/(T − m), respectively, the sum of which is one. The demeaned and detrended Brownian motions are $\tilde{B}_i(r) = B_i(r) - \int_0^1 B_i(s)ds - 12(r - \frac{1}{2}) \int_0^1 B_i(s)(s - \frac{1}{2})ds$, i = 1, 2 defining the mutually independent standard Brownian motions $B_i(r)$. The number of observations m in the break interval can increase in this asymptotic analysis. B and (T − B − m) must increase so that B/(T − m) and (T − B − m)/(T − m) converge to fixed values. The $\hat{\tau}_\tau$ test is consistent.

The χ^2 type test Ψ is also used for testing the unit root. The sum of the squared residuals denoted RSS hereafter is calculated under the null as well as the alternative regression equation (4) and (3), respectively. Ψ is defined as

$$\Psi = \frac{RSS(4) - RSS(3)}{RSS(3)/((T-1) - m - 5)}. \qquad (9)$$

The distribution of Ψ under the null hypothesis is calculated by Perron (1989) and Perron and Vogelsang (1993). It is well known that the asymptotic distributions of $\hat{\tau}_\tau$ and Ψ tests under the null hypothesis are not affected by augmenting the regression equations to higher order terms.

5. Hansen Test

Since the length of break interval m is unknown, we calculate the test statistic under the assumption that the break interval is a sub-interval (B, B + k), k = 1, 2, ..., m, of the maximum break interval (B, B + m). In the case of Ψ test, Ψ_k in which a break interval is taken as the sub-intervals (B, B + k) is calculated for each k, k = 1, 2, ..., m, and denoted $\{\Psi_1, \Psi_2, ..., \Psi_m\}$. The minimum, mean, median, and maximum of m values $\{\Psi_1, \Psi_2, ..., \Psi_m\}$ with *a priori* given maximum value of m are used as test statistics. Similarly, $\hat{\tau}_\tau$ test statistic is calculated for various subintervals, and the minimum, mean, median, and maximum of m squared values of $\hat{\tau}_\tau$ are used as the test statistics. Since minimum, mean, median, and maximum lead to similar results, the mean test statistic is used in the next section (Hansen (1996) and Zivot and Andrews (1992)).

Even if the break point is not in the interval (B, B + k) for some values of k, the statistic Ψ_k does not diverge under the null model since only the shock dummy variables are allowed under the null model. Misspecification of the break point does not affect the asymptotic distribution (Montanes (1997)). This ensures that the mean Ψ_k converges in distribution as the number of observations diverges. Simulations were used to calculate the critical values of these tests. Since results followed from various test statistics were similar, the mean statistic is used in the empirical studies.

The null model satisfies the following conditions. (1) The break point lies in the interval (B, B + m). (2) Only shock dummy variables reflect the break interval. (3) Deviations from the mean function follow a unit root process. If either of these conditions is violated, the null model is rejected, and the alternative model is accepted.

6. US Macro Series

The tests are applied to the US macro series. The lag orders of the augmented regressions are chosen by the same rule as Perron. The highest order term is kept in the regression when the t-ratio is larger than 1.6, but removed when it is less than 1.6 in most series. Choosing twelve for m *a priori*, the Ψ test statistic is calculated for the twelve break intervals starting from the shortest 1930–1930 break point to the longest 1930–1941 break interval. All results are summarized in Table 1.

6.1. Nominal GNP (1909–1988)

The A-model is used for this series as Perron did where the trend term is common but intercept terms are different in the sub-intervals. Figures 1.1 shows the natural log of the original series and the first difference. Ψ and the squared $\hat{\tau}_\tau$ curves are plotted in Figure 1.2. These two curves move closely. Ψ is almost significant by the 1% test in the 1930–1930 break point test, but is insignificant by the 10% test when the break interval is longer than two years. Since the nominal GNP returned to the 1930 level in 1938, it may be fair to say that the unit root

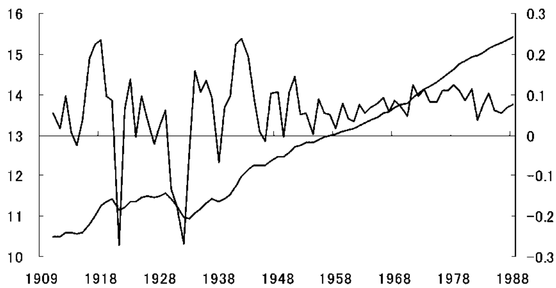

Figure 1.1 Natural log of nominal GNP and the first difference.

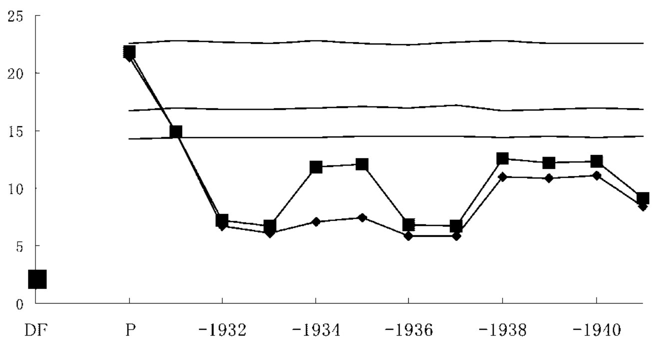

Figure 1.2 Interval Ψ and squared τ ratios (A model). From the top, 1%, 5%, 10% critical values of Ψ test statistic, Ψ, τ^2 and DF stats are plotted. DF is the squared Dickey-Fuller value, and P is the Perron value.

hypothesis cannot be rejected by the Ψ test once the break interval is taken into account. (Mean-Ψ is 11.2, and the 10% value is 12.6.) This diagnosis coincides with the standard DF test without a break. (The t-ratio of ϕ is −1.5.) The A-model and the DF regression model are different only in the break in the intercept term. The 1930–1930 break point test is solely giving the trend stationary result.

6.2. Real Wage (1900–1988)

The C-model defined by equation (3) is used in this series. The lag order for this series is selected by the same rule as the nominal GDP, but the selection started from sixth term instead of twelfth term since longer lag orders resulted in positive ϕ values. Time series is explosive if ϕ is positive. Figures 2.1 shows the results on the natural log and the first difference. Ψ and the squared $\hat{\tau}_\tau$ are plotted in Figure 2.2. The squared $\hat{\tau}_\tau$ values show that the null hypothesis of unit root cannot be rejected by the break interval as well as the 1930–1930 break point test. The Ψ values are insignificant for most break intervals including the 1930–1930 break point test. Then, the null hypothesis of the unit root cannot be rejected by this test either. (Mean-Ψ is 10.9, and the 10% value is 14.3.) This result is the same by the standard DF test without a break. (The t-ratio of ϕ is −1.7.)

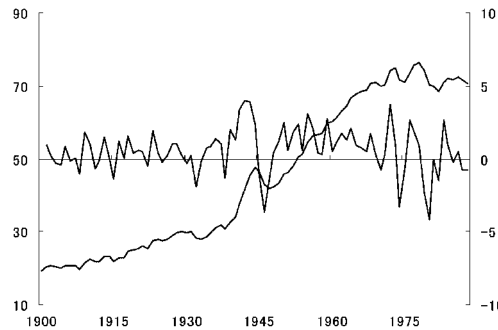

Figure 2.1 Natural log of real wage and the first difference.

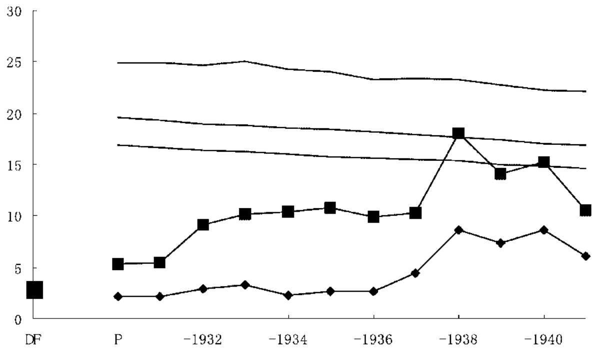

Figure 2.2 Interval Ψ and τ^2 ratios (C model).

6.3. Real GNP (1909–1988)

The A-model is used where the trend term is common but intercept terms are different in the sub-intervals. The differences between Ψ and $\hat{\tau}_\tau$ values are small. The null hypothesis is rejected by the 5% test up to 1930–1937 interval, and even by the 1% test for the longer intervals. The $\hat{\tau}_\tau$ test gives the similar results as the Ψ test. The 1930–1930 break point test is also significant. This series is trend stationary. (Mean-Ψ is 21.9, and the 1% value is 18.8.) Similarly, the standard DF test without a break is marginally significant by the 5% test. (The t-ratio of ϕ is –3.5.)

6.4. GNP Deflator (1989–1988)

The A-model is used where the trend term is common but intercept terms are different in the sub-intervals. Ψ curve is flat. Results of the break point and break interval tests in this series are all insignificant. (Mean-Ψ is 9.7, and the 10% value is 12.6.) This result is the same as the DF test without a break. (The t-ratio of ϕ is –1.6.) There remains the inconsistency among the nominal GNP, real GNP and the GNP deflator series. The real GNP is marginally significant but others are insignificant. The DF test without a break also brings about the same inconsistency.

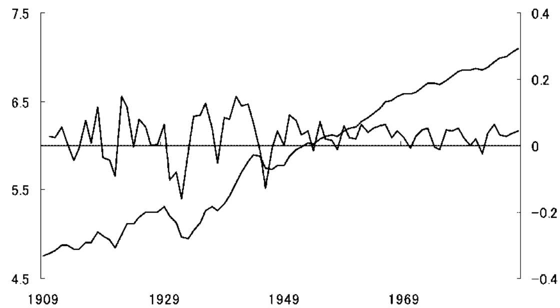

Figure 3.1 Natural log of real GNP and the first difference.

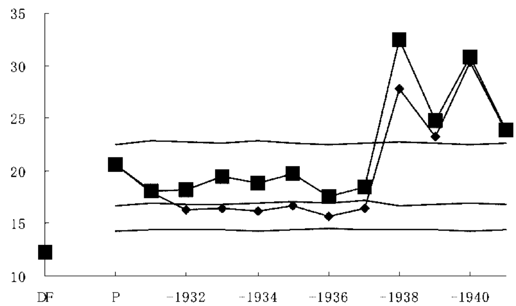

Figure 3.2 Interval Ψ and τ^2 ratios (A model).

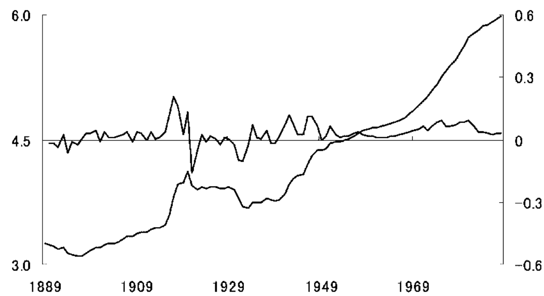

Figure 4.1 Natural log of GNP deflator and the first difference.

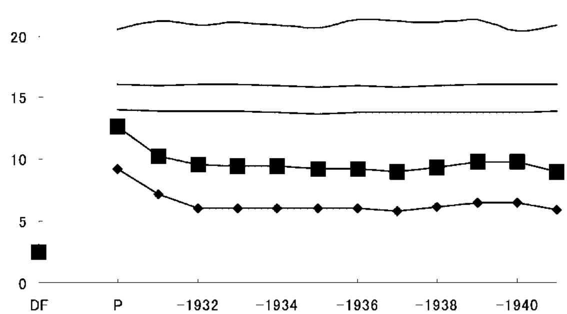

Figure 4.2 Interval Ψ and τ^2 ratios (A model).

6.5. Real GNP per capita (1909–1988)

The A-model is used where the trend term is common but intercept terms are different in the sub-intervals. This series is trend stationary once a break interval longer than two is taken into account. (Mean-Ψ is 23.3, and the 1% value is 18.8.) This is opposite from the 1930–1930 break point test which is insignificant by the 10% test. Ψ and $\hat{\tau}_\tau$ values are very close. Traces of the test statistics are similar to those in the Figure 3 of the real GNP. Since the t-ratio of ϕ in the DF test is −3.5 which is marginally significant by the 5% test, only the 1930–1930 break point and the 1930–1931 break interval tests lead to the insignificant result. This series is trend stationary.

6.6. Nominal Wage (1900–1988)

This series is trend stationary by the 5% test if the 1930–1930 break point test is used. It is non-stationary if the break interval is between two to four years. It is trend stationary by the 5% test if the break interval is longer than or equal to five years. (Mean-Ψ is 19.3, and the 1% value is 18.6.) This series is trend stationary. The Ψ ratio is mostly accounted by the $\hat{\tau}_\tau$ value, and both test statistics give the same results. It seems adequate to take the break interval longer than four years by checking the Figure 6.1. The DF test without a break is insignificant. (The t-ratio of ϕ is −1.1.)

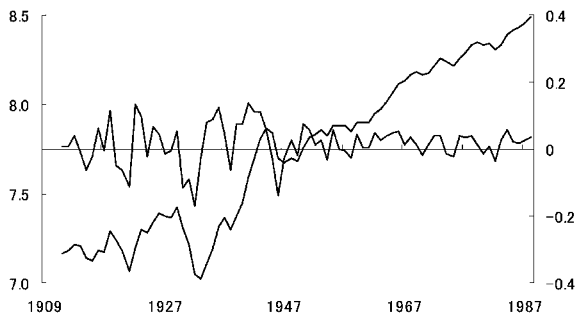

Figure 5.1 Natural log of real GNP per capita, and the first difference.

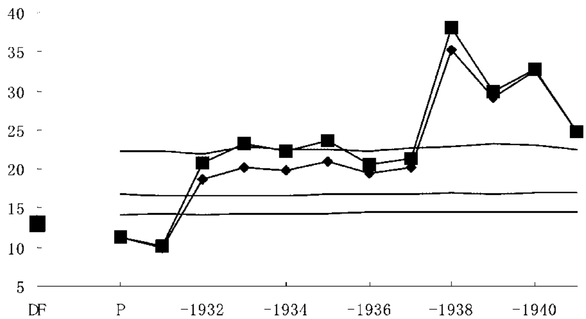

Figure 5.2 Interval Ψ and τ^2 ratios (A model).

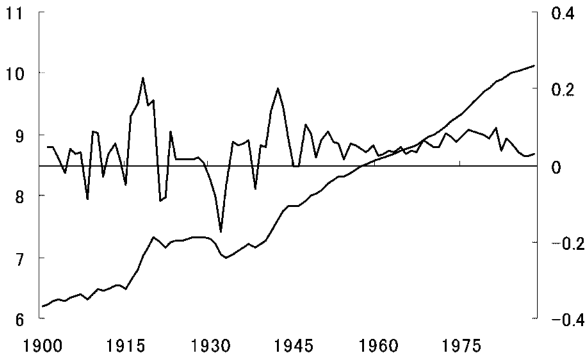

Figure 6.1 Natural log of nominal wage and the first difference.

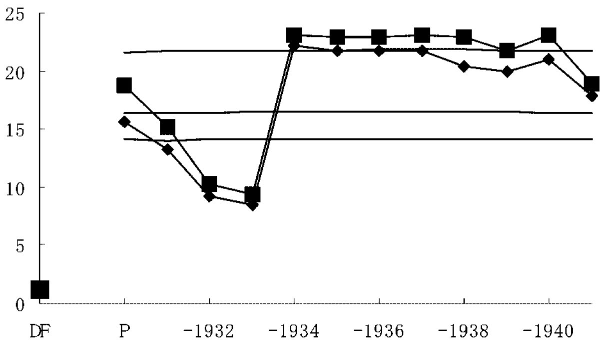

Figure 6.2 Interval Ψ and τ^2 ratios (A model).

6.7. Summary Table

Diagnoses of all tests are summarized in Table 1. The DF test without a break and the 1930–1930 break point test give different results only in five series. Among these five series, the break interval test supports the DF test in the nominal GNP and the GNP per capita. The break interval test supports the break point test in the nominal wage, S & P 500 and the velocity of money.

7. Conclusion

It is reasonable to specify a break interval instead of a break point since the test has a higher probability of including the true break point in the interval. This is also natural since, for example, the exact break point of the Japanese economy caused by the oil shock in 1973 is difficult to specify but an interval of a few years is easy to set. Setting a break interval does not guarantee avoiding a misspecification but it is more likely to avoid one. If the true break point is in a break interval, bias of the test is avoided even though the power of the test is less than that of a correct break point test. The break point test with an erroneously determined break point is inconsistent.

Table 1 Summary of tests

	DF	Interval	Perron	Mean-Ψ	CV	Model
Nominal GNP	DS	DS	TS (1%)*	11.2	14.1 (10%)	A
Real Wages	DS	DS	DS	10.9	17.6 (10%)	C
Real GNP	TS (5%)	TS (1%)	TS (5%)	21.9	21.2 (1%)	A
GNP Deflator	DS	DS	DS	9.7	13.5 (10%)	A
GNP per capita	TS (5%)	TS (1%)	DS*	23.3	21.2 (1%)	A
Nominal Wages	DS	TS (5%)*	TS (5%)*	19.3	20.4 (1%)	A
S & P500	DS	TS (5%)*	TS (1%)*	24.8	21.0 (5%)	C
Velocity	DS	TS (1%)*	TS (1%)*	26.4	20.1 (1%)	A
Real Rate of Interest	DS	DS	DS	14.2	13.7 (10%)	A
CPI	DS	DS	DS	11.2	13.5 (10%)	A
Employment	TS (5%)	TS (5%)	TS (1%)	19.0	15.6 (5%)	A
Money Stock	DS	DS	DS	8.5	13.5 (10%)	A
Unemployment Rate	TS (1%)	TS (1%)	TS (5%)	19.6	16.7 (1%)	No Trend
Industrial Product	DS	DS	DS	10.0	13.5 (10%)	A

DS and TS impliy difference stationarity and trend stationarity, respectively.
* marks series in which results of the Perron and DF tests are inconsistent, and CV implies critical values.

A formal procedure to select a correct interval is not proposed by this paper. We have proposed to calculate Ψ for plausible sub-intervals, and to use the mean of Ψ values as the test statistic. The same procedure applies to $\hat{\tau}_\tau$. A typical example is found in the nominal GNP series where the break interval and the break point tests show opposite results. It is concluded that the discontinuous trend unit root test is susceptible to the choice of a break interval.

The Zivot and Andrews (1992) test avoids specifying the break point prior to the test, and the break point is statistically determined by the first round test. However, there is no guarantee that the chosen break point is correctly specified. It seems appropriate to set a break interval so that it covers a break point selected by the Zivot and Andrew test. It is also shown in the Appendix C that the Ψ test statistic is a sum of the $(\hat{\tau}_\tau)^2$ statistic and the nuisance term.

Appendix A: Model A, B and C

The Model A which has an intercept break but no trend break can be formulated by

$$y_t = \Sigma_{i=1,2}\alpha_i^+ DU_{it} + \beta^+ t + \gamma D_{B+1,t} + u_t,\ u_t = (1+\phi)u_{t-1} + \varepsilon_t. \tag{A1}$$

The Cochrane-Orcutt transformation of (A1) yields

$$\Delta y_t = \phi y_{t-1} + \beta t + \Sigma_{i=1,2}\alpha_i DU_{it} + \gamma D_{B+1,t} + \varepsilon_t. \tag{A2}$$

The Model B has only a kinked trend break

$$y_t = \alpha + \beta_1^+ t + (\beta_2^+ - \beta_1^+)(t - B)DU_{2t} + \gamma D_{B+1,t} + u_t,\ u_t = (1 + \phi)u_{t-1} + \varepsilon_t \tag{A3}$$

and two trends are continuous at B. Model B can be transformed as

$$\Delta y_t = \phi y_{t-1} + \alpha_1 + \beta_1 t + \alpha_2 DU_{2t} + \beta_2(t - B)DU_{2t} + \gamma D_{B+1,t} + \varepsilon_t. \tag{A4}$$

If the model is with a break interval (B, B + m), it is necessary to include the m shock dummy variables as the equation 3. The Model C has both a trend and an intercept break which is formulated by (1).

Appendix B: The weak convergence of the t statistic

The regression model for the Dickey-Fuller type test is the equation 3. Applying the orthogonal transformation, (3) is

$$\Delta y_t = \phi y_{t-1}^+ + \Sigma_{i=1,2}\{\alpha_i^{**} + \beta_i^{**}(t - \bar{t}_i^+)\}DU_{it} + \Sigma_{i=1,m}\gamma_i^+ D_{B+i,t} + \varepsilon_t \tag{B1}$$

where the mean of y_{t-1} and trend in the first interval are $\bar{y}_1^+ \equiv \Sigma_{i=2,B}y_{t-1}/(B - 1)$ and $\bar{t}_1^+ \equiv (B + 2)/2$, respectively; the regression coefficient of y_{t-1} on the trend in the first interval is $\hat{\beta}_1^* \equiv \Sigma_{t=2,B}(t - \bar{t}_1^+)y_{t-1}/\Sigma_{t=2,B}(t - \bar{t}_1^+)^2$; similarly, $\bar{y}_2^+$, $\bar{t}_2^+$, and $\hat{\beta}_2^*$ are defined in the second interval; the residual series from the regression of y_{t-1} on all other explanatory variables in (3) is

$$y_t^+ \equiv \{y_{t-1} - \bar{y}_1^+ - \hat{\beta}_1^*(t - \bar{t}_1^+)\}DU_{1t} + \{y_{t-1} - \bar{y}_2^+ - \hat{\beta}_2^*(t - \bar{t}_2^+)\}(DU_{2t} - \Sigma_{i=1,m}D_{B+i,t}), \tag{B2}$$

and, finally, α_i^{**}, β_i^{**}, γ_i^+ are defined so that the equality holds. Since the explanatory variables are orthogonal, the estimate of ϕ is

$$\hat{\phi} = \frac{\Sigma_{t=2,T}\Delta y_t y_{t-1}^+}{\Sigma_{t=2,T}(y_{t-1}^+)^2} = \frac{\Sigma_{t=2,B}\Delta y_t y_{t-1}^+ + \Sigma_{t=B+m+1,T}\Delta y_t y_{t-1}^+}{\Sigma_{t=2,B}(y_{t-1}^+)^2 + \Sigma_{t=B+m+1,T}(y_{t-1}^+)^2}. \tag{B3}$$

The t statistic of ϕ is

$$\tau_\tau = \frac{\Sigma_{t=2,B}\Delta y_t y_{t-1}^+ + \Sigma_{t=B+m+1,T}\Delta y_t y_{t-1}^+}{\hat{\sigma}\sqrt{\Sigma_{t=2,B}(y_{t-1}^+)^2 + \Sigma_{t=B+m+1,T}(y_{t-1}^+)^2}}. \tag{B4}$$

The t-ratio of the ϕ coefficient is a simple extension of the $\hat{\tau}_\tau$ test statistic by

Dickey and Fuller (1981). Jumping observations over the break interval (B, B + m), the t-ratio is defined as

$$\hat{\tau}_\tau \Rightarrow \tau_\tau \equiv \frac{\sum_{i=1,2} \lambda_i \int_0^1 \tilde{B}_i(r)d\tilde{B}_i(r)}{\sqrt{\sum_{i=1,2} \lambda_i^2 \int_0^1 \tilde{B}_i(r)^2 dr}} \tag{B5}$$

The arrow implies the weak convergence under the null hypothesis, λs are the break ratio of the two intervals the sum of which is one, $\hat{\sigma}^2$ is the mean of the squared residuals calculated under the alternative regression, and $\tilde{B}_i(r)$, $i = 1, 2$, are the demeaned and detrended Brownian motions. It is easy to prove that $\hat{\tau}_\tau$ test is consistent. This formulation of the equation (B5) suggests other tests. For example, the t-ratio of β_i coefficient in (3) is an extension of the $\hat{\tau}_{\beta\tau}$ test statistic by Dickey and Fuller.

Appendix C: Orthogonal Decomposition of Ψ

The F-ratio type test Ψ is also used for testing the unit root in the augmented regression model. The regression equation includes lagged differenced terms so that

$$\Delta y_t = \phi y_{t-1} + \Sigma_{i=1,2}[\alpha_i + \beta_i(t - \bar{t}_i)]DU_{it} + \Sigma_{i=1,m}\gamma_i D_{it} + \Sigma_{k=1,L}\theta_k \Delta y_{t-k} + \varepsilon_t,\ \ t = L + 2,...,T \tag{C1}$$

and the null regression is

$$\Delta y_t = \Sigma_{i=1,2}\alpha_i DU_{it} + \Sigma_{i=1,m}\gamma_i D_{it} + \Sigma_{k=1,L}\theta_k \Delta y_{t-k} + \varepsilon_t, \quad t = L + 2,...,T \tag{C2}$$

The regression equation (C2) is transformed as

$$\Delta y_t = \phi y^*_{t-1} + \Sigma_{i=1,2}\beta_i^{**} t_i^* + \Sigma_{i=1,2}\alpha_i^{**}DU_{it} + \Sigma_{i=1,m}\gamma_i D_{it} + \Sigma_{k=1,L}\theta^*_k \Delta y_{t-k} + \varepsilon_t, \tag{C3}$$

$t = L + 2, ..., T$, where y^*_{t-1} is the residual from regressing y_{t-1} on constant dummy variables, dummy trend variables, and also on all the lagged differenced variables; t_1^* and t_2^* are the residuals from regressing $(t - \bar{t}_1)DU_{1t}$ and $(t - \bar{t}_2)DU_{2t}$ on all lagged differenced variables, respectively. Defining Q as the matrix of the whole observations on the lagged variables, $(T - 1 - m - L) \times 2$ matrix T^* is defined as

$$T^* = (t_1^*, t_2^*) = \{I - Q(Q'Q)^{-1}Q'\}\begin{pmatrix} t_1 - \bar{t}_1 & 0 \\ 0 & t_2 - \bar{t}_2 \end{pmatrix}. \tag{C4}$$

The regression coefficients in (C3) are adjusted according to these transformations of variables, but the ϕ coefficient is untouched. Observations in the break interval are not used in these calculations of residuals. The t-ratio on ϕ is denoted $\hat{\tau}_\tau$. The F-ratio type statistic is decomposed as

$$\Psi = \frac{\mathrm{RSS(C2)} - \mathrm{RSS(C1)}}{\mathrm{RSS(C1)}/(\,(T-1-L)-5-m-L)} = (\hat{\tau}_t)^2 + \tau_t \Rightarrow (\tau_\tau)^2 + \chi^2(2) \quad \text{(C5)}$$

where

$$\tau_t = \frac{1}{\hat{\sigma}^2}\Delta y' T^*(T^{*\prime}T^*)^{-1}T^{*\prime}\Delta y. \quad \text{(C6)}$$

τ_t is the F-ratio statistic associated with β_1^{**} and β_2^{**} coefficients. The arrow implies the weak convergence again under the null hypothesis. τ_t is the χ^2 random variable with two degrees of freedom.

The difference between the two RSS is decomposed into the sum of two orthogonal terms. Implication of each term in this decomposition is of interest. The first term is the $(\hat{\tau}_\tau)^2$ test statistic. The second term does not affect the power of the test. This is proven by replacing Δy for a vector of ε_t under the null, and for $\Delta\varepsilon_t$ under the alternative hypotheses since a constant term is orthogonal to T^*. Since y_t is a stationary process with trend, Δy_t is a constant plus a process of $\Delta\varepsilon_t$ under the alternative hypothesis. Under the null hypothesis, Δy_t is a constant plus ε_t. This decomposition implies that τ_t is a nuisance for testing the unit root.

In the case of the equations (4) and (3), τ_t is decomposed as the sum $\tau_{t1} + \tau_{t2}$ where

$$\tau_{t1} = \frac{\{\sum_{t=2,B} \Delta y_t(t-\bar{t}_1)^2\}}{\hat{\sigma}^2 \sum_{t=2,B}(t-\bar{t}_1)^2}, \quad \tau_{t2} = \frac{\{\sum_{t=B+m+1,T} \Delta y_t(t-\bar{t}_2)\}^2}{\hat{\sigma}^2 \sum_{t=B+m+1,T}(t-\bar{t}_2)^2}, \quad \text{(C7)}$$

respectively, which are the squared t-ratios of β_1^{**} and β_2^{**} coefficients. Two extra terms are the correlation between Δy and the trend in each interval, in short. These terms do not reflect the violations from the null hypothesis. The asymptotic distribution is derived by replacing Δy_t for ε_t under the null hypothesis. Under the alternative hypothesis, Δy_t is replaced by $\Delta\varepsilon_t$, in short. Then a large Ψ value does not necessarily imply a large $\hat{\tau}_\tau$ value. It can be resulted from a high correlation between Δy and the trend in either or both of the two sub-intervals.

If a model is misspecified, and Δy_t is a trend plus ε_t under the null hypothesis, then τ_t term diverges to infinity even though $(\hat{\tau}_\tau)^2$ converges to zero as the sample size increases to the infinity ($\hat{\sigma}^2 = O(n^2)$, $\sum_{t=2,B}\Delta y_t(t-\bar{t}_1) = O(n^3)$, $\sum_{t=2,B}(t-\bar{t}_1)^2 = O(n^3)$, $\sum_{t=2,B}\Delta y_t y_{t-1}^+ = O(n^4)$, $\sum_{t=2,B}(y_{t-1}^+)^2 = O(n^5)$).

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