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Consistent Pricing and Hedging of an FX Options Book

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In the foreign exchange (FX) options market away-from-the-money options are quite actively traded, and quotes for the same type of instruments are available everyday with very narrow spreads (at least for the main currencies). This makes it possible to devise a procedure for extrapolating the implied volatilities of non-quoted options, providing us with reliable data to which to calibrate our favorite model.

In this article, we test the goodness of the Brigo, Mercurio and Rapisarda (2004) model as far as some fundamental practical implications are concerned. This model, which is based on a geometric Brownian motion with time-dependent coefficients that are not known initially and whose value is randomly drawn at an infinitesimal future time, can accommodate very general volatility surfaces and, in case of the FX options market, can lead to a perfect fit to the main volatility quotes.

We first show the fitting capability of the model with an example from real market data. We then support the goodness of our calibration by providing a diagnostic on the forward volatilities implied by the model. We also compare the model prices of some exotic options with the corresponding ones given by a market practice. Finally, we show how to derive bucketed sensitivities to volatility and how to hedge accordingly a typical options book.

Keywords: foreign exchange options market, uncertain Black-Scholes parameters, calibration, forward volatilities, bucketed sensitivities, options book hedging.

JEL Classification Numbers: C14, C15, C61, F31, G13, G24

1. Introduction

In the foreign exchange (FX) options market away-from-the-money options are quite actively traded, and quotes for the same type of instruments are available everyday with very narrow spreads (at least for the main currencies). This makes it possible to devise a procedure for extrapolating the implied volatilities of non-quoted options, providing us with reliable data to which one can calibrate one's favorite alternative to the Black and Scholes (1973) (BS) model.

Brigo, Mercurio and Rapisarda (2004) have proposed an extension to the BS model where both the volatility and interest rates are stochastic in a very simple way. In this model, with uncertain volatility and uncertain interest rates (UVUR), the underlying asset evolves as a geometric Brownian motion with time-dependent coefficients, which are not known initially, and whose value is randomly drawn at an infinitesimal future time.

As stressed by the authors themselves, the UVUR model can accommodate very general volatility surfaces and, in case of the FX options market, one can achieve a perfect fitting to the main volatility quotes.

In this article, we test the goodness of this model as far as some fundamental practical implications are concerned. First of all, we ourselves show the fitting capability of the model with an example from real market data. We then support the goodness of our calibration by providing a diagnostic on the forward volatilities implied by the model. We also compare the model prices of some exotic options with the corresponding ones given by a market practice. Finally, we show how to derive bucketed sensitivities to volatility and how to hedge accordingly a typical options book.

The article is organized as follows. Section 2 provides a brief description of the FX options market and its volatility quotes. Section 3 introduces the UVUR model and describes its analytical tractability. Section 4 deals with an example of calibration to real market data. Section 5 illustrates a forward volatility surface and some forward volatility curves implied by the previously calibrated parameters. Section 6 deals with the issue of pricing exotic options. Section 7 considers an explicit example of volatility hedging applied to a given options book. Section 8 concludes the article.

2. A Brief Description of the FX Options Market

A stylized fact in the FX market is that options are quoted depending on their Delta, and not their strike as in other options market. This basically reflects the sticky Delta rule, according to which implied volatilities do not vary, from a day to the next, if the related moneyness remains the same. To state it differently, when the underlying exchange rate moves, and the Delta of an option changes accordingly, a different implied volatility has then to be plugged into the corresponding Black and

Scholes (1973) formula.

The FX options market is characterized by three volatility quotes up to relatively long expiries (at least for the EUR/USD exchange rate): i) the at-the-money (ATM), ii) the risk reversal (RR) for the 25Δ call and put, iii) the (vega-weighted) butterfly (VWB) with 25Δ wings[1)]. From these market quotes, one can easily infer the implied volatilities for the 25Δ call and put, and then build upon them an entire smile for the range going from a 5Δ put to a 5Δ call[2)].

2.1. The market quotes

We denote by $S(t)$ the value of a given exchange rate, say the EUR/USD, at time t. We set $S_0 := S(0) > 0$ and denote, respectively, by $P^d(0,t)$ and $P^f(0,t)$ the domestic and foreign discount factors for maturity t. We then consider a market maturity T. The Delta, at time 0, of a European call with strike K, maturity T and volatility σ is given by

$$P^f(0,T)\Phi\left(\frac{\ln\frac{S_0 P^f(0,T)}{KP^d(0,T)} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right),$$

where Φ denotes the standard normal distribution function[3)]. The market quotes for maturity T are defined as follows.

The ATM volatility is that of a 0Δ straddle, whose strike, for each given expiry, is chosen so that the related put and call have the same Δ but with different signs. Denoting by σ_{ATM} the ATM volatility for the expiry T, the ATM strike K_{ATM} can be immediately derived:

$$K_{ATM} = S_0 \frac{P^f(0,T)}{P^d(0,T)} e^{\frac{1}{2}\sigma_{ATM}^2 T} \tag{1}$$

The RR is a structure where one buys a call and sells a put with a symmetric Delta. The RR is quoted as the difference between the two implied volatilities, $\sigma_{25\Delta c}$ and $\sigma_{25\Delta p}$ to plug into the Black and Scholes formula for the call and the put respectively. Denoting such a price, in volatility terms, by σ_{RR}, we have:

$$\sigma_{RR} = \sigma_{25\Delta c} - \sigma_{25\Delta p} \tag{2}$$

The VWB is built up by selling a quantity of ATM straddle and buying a quantity of 25Δ strangle, in such a way the resulting structure has a zero Vega. The butterfly's price in volatility terms, σ_{VWB}, is then defined by:

$$\sigma_{VWB} = \frac{\sigma_{25\Delta c} + \sigma_{25\Delta p}}{2} - \sigma_{ATM} \tag{3}$$

[1)] In accordance with the market jargon, we drop the "%" sign after the level of the Delta, so that a 25Δ call is one whose Delta is 0.25. Analogously, a 25Δ put is one whose Delta is −0.25.
[2)] Notice that a $x\Delta$ call is equivalent to a $(P^f(0,T) - x)\Delta$ put, with P^f defined below.
[3)] Notice that this Delta can be interpreted as the discounted probability of ending in the money under the measure associated with the numeraire $S(t)/P^f(0,t)$.

For the given expiry T, the two implied volatilities $\sigma_{25\Delta c}$ and $\sigma_{25\Delta p}$ can be immediately identified by solving a linear system. We obtain:

$$\sigma_{25\Delta c} = \sigma_{ATM} + \sigma_{VWB} + \frac{1}{2}\sigma_{RR} \tag{4}$$

$$\sigma_{25\Delta p} = \sigma_{ATM} + \sigma_{VWB} - \frac{1}{2}\sigma_{RR} \tag{5}$$

The two strikes corresponding to the 25Δ put and 25Δ call can be derived, after straightforward algebra, from their definitions:

$$\begin{aligned} K_{25\Delta p} &= S_0 \frac{P^f(0,T)}{P^d(0,T)} e^{-\alpha\sigma_{25\Delta p}\sqrt{T}+\frac{1}{2}\sigma^2_{25\Delta p}T} \\ K_{25\Delta c} &= S_0 \frac{P^f(0,T)}{P^d(0,T)} e^{\alpha\sigma_{25\Delta c}\sqrt{T}+\frac{1}{2}\sigma^2_{25\Delta c}T} \end{aligned} \tag{6}$$

where $\alpha := -\Phi^{-1}(\frac{1}{4}/P^f(0,T))$ and Φ^{-1} is the inverse normal distribution function. We stress that, for typical market parameters and for maturities up to two years, $\alpha > 0$ and

$$K_{25\Delta p} < K_{ATM} < K_{25\Delta c}$$

Starting from the implied volatilities $\sigma_{25\Delta p}$, $\sigma_{25\Delta c}$ and σ_{ATM} and the related strikes, one can finally build the whole implied volatility smile for expiry T. A consistent construction procedure is given, for instance, in Castagna and Mercurio (2004).

An example of market volatility quotes is given in Table 1 and the associated implied volatility surface is shown in Figure 1.

3. The UVUR Model

We assume that the exchange rate dynamics evolves according to the uncertain volatility model with uncertain interest rates proposed by Brigo, Mercurio and Rapisarda (2004). In this model, the exchange rate under the domestic risk neutral measure follows

$$dS(t) = \begin{cases} S(t)[(r^d(t) - r^f(t))\,dt + \sigma_0\,dW(t)] & t \in [0,\varepsilon] \\ S(t)[(\rho^d(t) - \rho^f(t))\,dt + \sigma(t)\,dW(t)] & t > \varepsilon \end{cases} \tag{7}$$

where $r^d(t)$ and $r^f(t)$ are, respectively, the domestic and foreign instantaneous forward rates for maturity t, σ_0 and ε are positive constants, W is a standard Brownian motion, and (ρ^d, ρ^f, σ) is a random triplet that is independent of W and takes values in the set of N (given) triplets of deterministic functions:

$$(\rho^d(t), \rho^f(t), \sigma(t)) = \begin{cases} (r^d_1(t), r^f_1(t), \sigma_1(t)) & \text{with probability } \lambda_1 \\ (r^d_2(t), r^f_2(t), \sigma_2(t)) & \text{with probability } \lambda_2 \\ \quad\vdots & \quad\vdots \\ (r^d_N(t), r^f_N(t), \sigma_N(t)) & \text{with probability } \lambda_N \end{cases}$$

where the λ_i are strictly positive and add up to one. The random value of (ρ^d, ρ^f, σ) is drawn at time $t = \varepsilon$.

The intuition behind the UVUR model is as follows. The exchange rate process is nothing but a BS geometric Brownian motion where the asset volatility and the (domestic and foreign) risk free rates are unknown, and one assumes different (joint) scenarios for them.

The volatility uncertainty applies to an infinitesimal initial time interval with length ε, at the end of which the future values of volatility and rates are drawn. Therefore, S evolves, for an infinitesimal time, as a geometric Brownian motion with constant volatility σ_0, and then as a geometric Brownian motion with the deterministic drift rate $r_i^d(t) - r_i^f(t)$ and deterministic volatility $\sigma_i(t)$ drawn at time ε.

In this model, both interest rates and volatility are stochastic in the simplest possible manner. As already noted by Brigo, Mercurio and Rapisarda (2004), uncertainty in the volatility is sufficient by itself to accommodate implied volatility smiles (σ_{RR} close to zero), whereas uncertainty in interest rates must be introduced to capture skew effects (σ_{RR} far from zero).

Setting $\mu_i(t) := r_i^d(t) - r_i^f(t)$ for $t > \varepsilon$, $\mu_i(t) := r^d(t) - r^f(t)$ and $\sigma_i(t) = \sigma_0$ for $t \in [0, \varepsilon]$ and each i, and

$$M_i(t) := \int_0^t \mu_i(s)\,ds, \quad V_i(t) := \sqrt{\int_0^t \sigma_i^2(s)\,ds}$$

we have that the density of S at time $t > \varepsilon$ is the following mixture of lognormal densities:

$$p_t(y) = \sum_{i=1}^{N} \lambda_i \frac{1}{yV_i(t)\sqrt{2\pi}} \exp\left\{-\frac{1}{2V_i^2(t)}\left[\ln\frac{y}{S_0} - M_i(t) + \tfrac{1}{2}V_i^2(t)\right]^2\right\}. \tag{8}$$

Accordingly, European option prices are mixtures of BS prices. For instance the arbitrage-free price of a European call with strike K and maturity T is

$$\begin{aligned} P^d(0,T)\sum_{i=1}^{N}\lambda_i \Bigg[S_0 e^{M_i(T)} \Phi\left(\frac{\ln\frac{S_0}{K} + M_i(T) + \frac{1}{2}V_i^2(T)}{V_i(T)}\right) \\ -K\Phi\left(\frac{\ln\frac{S_0}{K} + M_i(T) - \frac{1}{2}V_i^2(T)}{V_i(T)}\right)\Bigg]. \end{aligned} \tag{9}$$

Further details can be found in Brigo, Mercurio and Rapisarda (2004).

The analytical tractability at the initial time is extended to all those derivatives which can be explicitly priced under the BS paradigm. In fact, the expectations of functionals of the process (7) can be calculated by conditioning on the possible values of (ρ^d, ρ^f, σ), thus taking expectations of functionals of a geometric Brownian motion. Denoting by E the expectation under the risk-neutral measure, any smooth

Table 1 EUR/USD volatility quotes as of 12 February 2004

	σ_{ATM}	σ_{RR}	σ_{VWB}
1W	11.75%	0.50%	0.190%
2W	11.60%	0.50%	0.190%
1M	11.50%	0.60%	0.190%
2M	11.25%	0.60%	0.210%
3M	11.00%	0.60%	0.220%
6M	10.87%	0.65%	0.235%
9M	10.83%	0.69%	0.235%
1Y	10.80%	0.70%	0.240%
2Y	10.70%	0.65%	0.255%

payoff V_T at time T has a no-arbitrage price at time $t = 0$ given by

$$V_0 = P^d(0,T)\sum_{i=1}^{N}\lambda_i E\left\{V_T \middle| (\rho^d = r_i^d, \rho^f = r_i^f, \sigma = \sigma_i)\right\} = \sum_{i=1}^{N}\lambda_i V_0^{\mathrm{BS}}(r_i^d, r_i^f, \sigma_i) \quad (10)$$

where $V_0^{\mathrm{BS}}(r_i^d, r_i^f, \sigma_i)$ denotes the derivative's price under the BS model when the risk free rates are r_i^d and r_i^f and the asset (time-dependent) volatility is σ_i.

The advantages of model (7) can be summarized as follows: i) explicit dynamics; ii) explicit marginal density at every time (mixture of lognormals with different means and standard deviations); iii) explicit option prices (mixtures of BS prices) and, more generally, explicit formulas for European-style derivatives at the initial time; iv) explicit transitions densities, and hence future option prices; v) explicit (approximated) prices for barrier options and other exotics[4)]; vi) potentially perfect fitting to any (smile-shaped or skew-shaped) implied volatility curves or surfaces.

4. An Example of Calibration

We consider an example of calibration to EUR/USD market data as of 12 February 2004, when the spot exchange rate was 1.2832.

In Table 1 we report the market quotes of EUR/USD σ_{ATM}, σ_{RR} and σ_{VWB} for the relevant maturities from one week (1W) to two years (2Y), while in Table 2 we report the corresponding domestic and foreign discount factors.

The implied volatility surface that is constructed from the basic volatility quotes is shown in Table 3, for the major Deltas, and in Figure 1, where for clearness' sake we plot the implied volatility in terms of put Deltas ranging from 5% to 95% and for the same maturities as in Table 1.

[4)] As an example, the closed-form formula for the price of an up and out call under the UVUR model is reported in Appendix A.

Table 2 Domestic and foreign discount factors for the relevant maturities

	T (in years)	$P^d(0,T)$	$P^f(0,T)$
1W	0.0192	0.999804	0.999606
2W	0.0384	0.999595	0.999208
1M	0.0877	0.999044	0.998179
2M	0.1726	0.998083	0.996404
3M	0.2493	0.997187	0.994803
6M	0.5014	0.993959	0.989548
9M	0.7589	0.990101	0.984040
1Y	1.0110	0.985469	0.978479
2Y	2.0110	0.960102	0.951092

Table 3 EUR/USD volatility quotes as of 12 February 2004

	$10\Delta p$	$25\Delta p$	$35\Delta p$	ATM	$35\Delta c$	$25\Delta c$	$10\Delta c$
1W	11.96%	11.69%	11.67%	11.75%	11.94%	12.19%	12.93%
2W	11.81%	11.54%	11.52%	11.60%	11.79%	12.04%	12.78%
1M	11.60%	11.39%	11.39%	11.50%	11.72%	11.99%	12.77%
2M	11.43%	11.16%	11.15%	11.25%	11.48%	11.76%	12.60%
3M	11.22%	10.92%	10.90%	11.00%	11.23%	11.52%	12.39%
6M	11.12%	10.78%	10.76%	10.87%	11.12%	11.43%	12.39%
9M	11.04%	10.72%	10.71%	10.83%	11.09%	11.41%	12.39%
1Y	11.00%	10.69%	10.68%	10.80%	11.06%	11.39%	12.38%
2Y	11.02%	10.63%	10.60%	10.70%	10.94%	11.28%	12.34%

In order to exactly fit both the domestic and foreign zero coupon curves at the initial time, the following no-arbitrage constraints must be imposed for each t[5]:

$$\begin{aligned} \sum_{i=1}^{N} \lambda_i e^{-\int_0^t r_i^d(u)\,du} &= P^d(0,t) \\ \sum_{i=1}^{N} \lambda_i e^{-\int_0^t r_i^f(u)\,du} &= P^f(0,t) \end{aligned} \tag{11}$$

Our calibration is then performed by minimizing the sum of squared percentage differences between model and market volatilities of the 25Δ puts, ATM puts and 25Δ calls, while respecting the constraint (11). Given that ε is arbitrarily small, we considered the limit case $\varepsilon = 0$ in the calculation of option prices (9)[6].

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[5] We can safely use the same λ's both for the domestic and foreign risk-neutral measures, since such probabilities do not change when changing measure due to the independence between W and (ρ^d, ρ^f, σ).
[6] We notice that, setting $\varepsilon = 0$, σ_0 is no longer an optimization parameter.

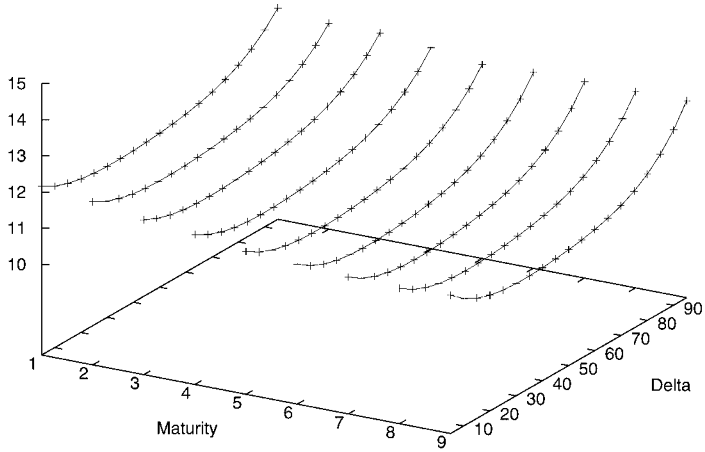

Figure 1 EUR/USD implied volatilities (in percentage points) as of 12 February 2004.

Given the high degrees of freedom at hand, we set $N = 2$ and assumed that the domestic rate ρ^d is deterministic and equal to r^d, so that the first constraint in (11) is automatically satisfied. In fact, sticking to only two scenarios and assuming uncertainty only in the asset volatility σ and foreign rate ρ^f is sufficient, in the considered case and many others as well, to achieve a perfect calibration to the three main volatility quotes for all maturities simultaneously.

To speed up the calibration procedure, we resorted to a non parametric estimate of functions ρ^f and σ, assuming r_i^f and σ_i, $i = 1, 2$, to be constant over each interval defined by consecutive market maturities. In such a way, we could apply an iterative procedure and calibrate one implied volatility curve at a time, starting from the first maturity and up to the last. Precisely, we set $t_0 := 0$, $t_1 := 1W$, $t_2 := 2W$, $t_3 := 1M$, $t_4 := 2M$, $t_5 := 3M$, $t_6 := 6M$, $t_7 := 9M$, $t_8 := 1Y$, $t_9 := 2Y$, and denoted by $r_{i,j}^f$ and $\sigma_{i,j}$ the constant values assumed, respectively, by r_i^f and σ_i, $i = 1, 2$, on the intervals $[t_{j-1}, t_j)$ $j = 1, \ldots, 9$. At each maturity t_j, we then optimized over $r_{1,j}^f$, $\sigma_{1,j}$ and $\sigma_{2,j}$, which are the only free parameters at the j-th step appearing in formula (9), given that we expressed $r_{2,j}^f$ as a function of $r_{1,j}^f$ by the second constraint in (11), and also given the previously obtained values $r_{1,1}^f, \ldots, r_{1,j-1}^f$, $\sigma_{1,1}, \ldots, r_{1,j-1}^f$ and $\sigma_{2,1}, \ldots, r_{1,j-1}^f$.

The perfect fit to three main volatilities for each maturity holds true for many different specifications of the probability parameter λ_1. We then chose an optimal λ_1 by calibrating the whole implied volatility matrix in Table 3, under the constraint that the three main quotes are reproduced exactly. We obtained $\lambda_1 = 0.625$. The values of the other model parameters are shown in Table 4.

In Table 5 we show our calibration's errors in absolute terms: the model perfectly fits the three main volatilities for each maturity and performs quite well for almost every level of Delta. The performance slightly degenerates for extreme wings. However, the largest error is quite acceptable, given also that market bid-ask spreads are typically higher.

The perfect calibration to the basic volatility quotes is essential for a Vega breakdown along the strike and maturity dimensions. This is extremely helpful to traders, since it allows them to understand where their volatility risk is concentrated. The possibility of such a Vega break down is a clear advantage of the UVUR model. In general, the calculation of bucketed sensitivities is neither straightforward nor even possible when we depart from the BS world. In fact, classical and widely used stochastic-volatility models, like those of Hull and White (1987) or Heston (1993), can not produce bucketed sensitivities. A trader is then typically compelled to resort to a dangerous and unnatural parameter hedging or to an overall Vega hedge based on a parallel shift of the implied volatility surface.

In Section 7 we will show how to calculate a Vega break down and, accordingly, how to hedge a book of exotic options in terms of plain-vanilla instruments.

5. The Forward Volatility Surfaces

The quality of calibration to implied volatility data is usually an insufficient criterion for judging the goodness of an alternative to the BS model. In fact, a trader is also interested in the evolution of future volatility surfaces, which are likely to have a strong impact both in the pricing and especially in the hedging of exotic options.

Once the deterministic (time-dependent) volatility σ and interest rates ρ^d and ρ^f are drawn at time ε, we know that the model (7) behaves as a BS geometric Brownian motion, thus leading to flat implied volatility curves for each given maturity. This is certainly a drawback of the model. However, the situation improves sensibly if we consider forward implied volatility curves.

A forward implied volatility is defined as the volatility parameter to plug into the BS formula for forward starting option to match the model price.

Table 4 Calibrated parameters for each maturity

	$r^f_{1,j}$	$\sigma_{1,j}$	$\sigma_{2,j}$
1W	9.82%	9.23%	15.72%
2W	5.14%	8.96%	15.36%
1M	5.47%	8.90%	15.21%
2W	3.44%	8.26%	15.21%
3W	2.84%	7.79%	14.72%
6M	3.09%	7.92%	15.05%
9M	3.11%	7.96%	14.90%
1Y	2.79%	7.81%	15.13%
2Y	3.02%	7.51%	15.44%

Table 5 Absolute differences (in percentage points) between model and market implied volatilities

	$10\Delta p$	$25\Delta p$	$35\Delta p$	ATM	$35\Delta c$	$25\Delta c$	$10\Delta c$
1W	0.00%	0.00%	0.00%	0.00%	0.01%	0.00%	0.01%
2W	0.00%	0.00%	0.00%	0.00%	0.01%	0.00%	0.01%
1M	0.01%	0.00%	0.00%	0.00%	0.01%	0.00%	0.01%
2M	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.01%
3M	0.00%	0.00%	0.00%	0.00%	0.01%	0.00%	0.01%
6M	−0.02%	0.00%	0.01%	0.00%	0.00%	0.00%	−0.01%
9M	−0.02%	0.00%	0.00%	0.00%	0.00%	0.00%	−0.01%
1Y	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
2Y	0.02%	0.00%	0.00%	0.00%	0.00%	0.00%	0.01%

Table 6 Comparison between ATM implied volatilities as of 12^{th} February 2004 and three month forward ATM implied volatilities

	$12^{th} February\, 2004$	$three - month\, fwd$
1W	11.75%	10.63%
2W	11.60%	10.63%
1M	11.50%	10.63%
2M	11.25%	10.64%
3M	11.00%	10.65%
6M	10.87%	10.66%
9M	10.83%	10.65%
1Y	10.80%	10.63%
2Y	10.70%	10.62%

A forward starting option with forward start date T_1 and maturity T_2 is an option where the strike price is set as a proportion α of the spot price at time T_1. In case of a call, the payoff at time T_2 is

$$[S(T_2) - \alpha S(T_1)]^+$$

whose BS price at time 0 is

$$\begin{aligned} S_0\Bigg[& P^f(0,T_2)\Phi\left(\frac{\ln \frac{P^d(0,T_1)P^f(0,T_2)}{\alpha P^d(0,T_2)P^f(0,T_1)} + \frac{1}{2}\sigma(T_1,T_2,\alpha)^2(T_2-T_1)}{\sigma(T_1,T_2,\alpha)\sqrt{T_2-T_1}}\right) \\ & - \alpha\frac{P^d(0,T_2)}{P^d(0,T_1)}P^f(0,T_1)\Phi\left(\frac{\ln \frac{P^d(0,T_1)P^f(0,T_2)}{\alpha P^d(0,T_2)P^f(0,T_1)} - \frac{1}{2}\sigma(T_1,T_2,\alpha)^2(T_2-T_1)}{\sigma(T_1,T_2,\alpha)\sqrt{T_2-T_1}}\right)\Bigg], \end{aligned} \tag{12}$$

where $\sigma(T_1, T_2, \alpha)$ denotes the forward volatility for the interval $[T_1, T_2]$ and "moneyness" α.

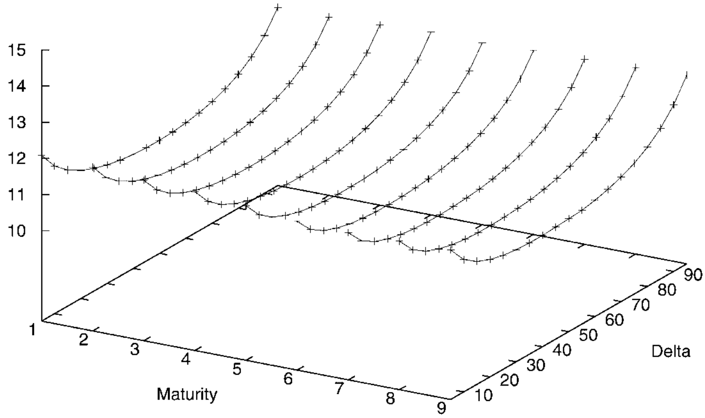

Figure 2 The three-month forward volatility surface.

In Figure 2 we show the three-month forward volatility surface that is implied by the previous calibration. Such a surface is the graph of function $\sigma(T_1, T_2, \alpha)$ for different values of T_2 and α, with T_1 set to 0.25 (three months). For a more consistent plot and a better homogeneity of values, we replaced α with Δ, thus using different α's for different maturities. The α for given maturity T_2 and Δ was calculated as the moneyness of the plain vanilla option with the same Δ and same time to maturity $T_2 - T_1$. In Table 6 we compare the ATM volatilities as of 12^{th} February 2004 and the three-month forward ATM implied volatility. The level of the surface, as is clear from the ATM volatilities, keeps a regular term structure. The shape of the surface also looks consistent with the initial one.

Similar plots can be obtained by considering different forward start dates T_1. This provides a strong empirical support to model (7), since its forward volatility surfaces are regular and realistic in that they do not differ too much from the initial one.

As a further example, in Figure 3 we show the forward evolution of the three-month implied volatility smile. To this end, we set $T_2 = T_1 + 0.25$ and considered the forward implied volatility curves for $T_1 \in \{1W, 2W, 1M, 2M, 3M, 6M, 9M, 1Y, 2Y\}$. The evolution is sensible and realistic also in this case: the shape of the smile keeps the features usually observed in the market.

6. Pricing Exotic Options

In this section, we will briefly describe the empirical procedure used by many practitioners in the market to account for implied volatility smiles in the pricing of non-quoted instruments. We will also compare the prices of some exotic options obtained with the market practice with those coming from the UVUR model with $N = 2$.

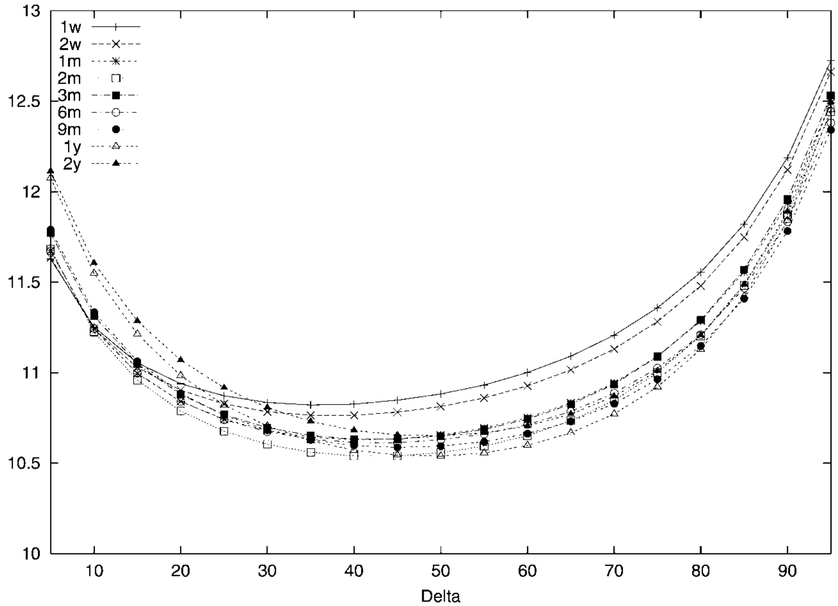

Figure 3 The three-month implied volatility smiles starting at different forward times.

Market practitioners tend to stick to a BS constant-volatility model to price exotic options, but they also adopt some rules of thumbs, based on hedging arguments, to include the volatility surface into the pricing. To cope with a smile-shaped volatility surface, traders hedge their positions by keeping low exposures not only in classical Greeks like Delta, Gamma and Vega, but also in some higher order Greeks like the DVegaDvol (a.k.a. Volga) and the DVegaDSpot (a.k.a. Vanna). The Volga measures the sensitivity of the Vega of the option with respect to a change in the implied volatility, whereas the Vanna measures the sensitivity of the Vega with respect to the a change in the underlying spot price. The Volga can be thought of as a sensitivity with respect to the volatility of the implied volatility, whereas the Vanna as a sensitivity with respect to the correlation between the underlying asset and the implied volatility. By setting the Vega, the Vanna and the Volga of the hedged portfolio equal to zero, traders try to minimize the model risk deriving from using BS, which is manifestly inconsistent with the reality.

The trader's procedure for pricing an exotic option can be summarized as follows. First, he/she prices the option with the BS formula by plugging into it the ATM volatility. He/she then calculates the option's Vega, Vanna and Volga. The related exposures can be hedged by buying and selling appropriate numbers of out-of-the-money and at-the-money options. Since the most liquid options for each expiry are the ATM calls (or puts) and the 25Δ calls and puts, the three exposures are eventually hedged by means of combinations of such options. Once the hedging

Table 7 Market data for EUR/USD as of 31st March 2004

	σ_{ATM}	σ_{RR}	σ_{VWB}	$P^d(0,T)$	$P^f(0,T)$
1W	13.50%	0.00%	0.19%	0.9997974	0.9996036
2W	11.80%	0.00%	0.19%	0.9995851	0.9992202
1M	11.95%	0.05%	0.19%	0.9991322	0.9983883
2M	11.55%	0.15%	0.21%	0.9981532	0.9966665
3M	11.50%	0.15%	0.21%	0.9972208	0.9951018
6M	11.30%	0.20%	0.23%	0.9941807	0.9902598
9M	11.23%	0.23%	0.23%	0.9906808	0.9855211
1Y	11.20%	0.25%	0.24%	0.9866905	0.9807808
2Y	11.10%	0.20%	0.25%	0.9626877	0.9550092

Table 8 UVUR Model prices compared with BS and BS plus market adjustments

	BS Value	BS+Adj	UVUR
Up&Out call	0.0041	0.0047	0.0049
Down&Out put	0.0169	0.0148	0.0150

portfolio is built, it is priced with the proper market implied volatilities, yielding its true market value, and then with a constant at-the-money volatility. The difference between the two values is added to the BS price of the exotic option, thus incorporating, via the above hedging procedure, the market smile into the pricing. This add-on is usually weighted by the survival probability when a barrier option is involved.

This is as far as market practice is concerned. We now provide two examples showing that exotic options prices implied by (7) do not significantly depart from those given by the above procedure. This can be viewed as a further argument supporting the UVUR model.

The exotic options we consider are two barrier options: an Up&Out call and a Down&Out put. Valuations are based on the EUR/USD market data as of 31 March 2004 (shown in Table 7), with the EUR/USD spot rate set at 1.2183. We first price the two options with the BS model, we then calculate the related adjustments by the market's rule of thumb explained above, and finally compare the adjusted prices with those implied by the UVUR model. In the UVUR model barrier options prices are consistently calculated according to formula (10), that is we simply use a combination of BS barrier option formulae by plugging, for each scenario, the integrated volatility corresponding to the claim's expiry[7)].

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7) This formula is not exact since actual BS barrier option prices depend on the whole term structure of the instantaneous volatility and not on its mean value only. However, such prices can not be expressed in closed form and our approximation turns out to be extremely accurate in most FX market conditions. A complete catalogue of alternative approximations for BS barrier option prices, in presence of a term structure of volatility, can be found in Rapisarda (2003).

Results are displayed in Table 8. The first option is a EUR call/USD put struck at 1.2250 with knock out at 1.3100, expiring in 6 months. The BS price is 0.0041 US$ and the adjustment to this theoretical value is positive and equal to 0.0006 US$. The UVUR model evaluates this option 0.0049 US$. The second option is a EUR put/USD call struck at 1.2000 and knocked out at 1.0700, expiring in 3 months. The BS price is 0.0169 US$ and, in this case, the market adjustment is negative and equal to -0.0021 US$. The UVUR price is again very close to that implied by the market's practice. The model, therefore, seems to be consistent with market's adjustments and prices, at least in the EUR/USD market case.

In presence of steep skews as in the USD/JPY market, however, the accordance between the market procedure and the UVUR model may worsen considerably. There are in fact particular combinations of strikes and barrier levels such that the corrections implied by the two approaches have opposite signs. One may wonder whether this is an indication that the UVUR model misprices certain derivatives. The answer, however, seems to be negative in general. In fact, using the Heston (1993) model as a reference, whenever the UVUR price is significantly different than that implied by the market approach, the Heston price is definitely more in accordance with the former than the latter. This is another argument in favor of the UVUR model.

In the next section we show how to use the UVUR model also in the management of an options book.

7. Hedging a Book of Exotic Options

As pointed out by Brigo, Mercurio and Rapisarda (2004), model (7) can be efficiently used for the valuation of a whole options book. This is essentially due to the possibility of pricing analytically most derivatives in the FX market. Our practical experience is that it takes a few seconds to value a book with 10000 options, half of which exotics, including the time devoted to calibration. This is an impossible task to achieve with any known stochastic volatility model.

The consistent valuation of his/her book is not, however, the only concern of an options trader. Hedging is usually an even more important issue to address. In this section, we will show how to hedge, by means of model (7), changes of a portfolio's value due to changes in market volatilities.

From a theoretical point of view, the UVUR model is characterized by market incompleteness, due to the randomness of the asset's volatility. In principle, therefore, a contingent claim can be hedged by means of the underlying asset and a given option. In practise, however, there are several sources of randomness that are not properly accounted for in the theory. This is why traders prefer to implement alternative hedging strategies, like those based on Vega bucketing, as we illustrate in the following.

We already noticed that, under (7), a Vega break down is possible thanks to the model capability of exactly reproducing the fundamental volatility quotes. The sensitivity of a given exotic to a given implied volatility is readily obtained by applying the following procedure. One shifts such a volatility by a fixed amount $\Delta\sigma$, say ten basis points. One then fits the model to the tilted surface and calculate the price of the exotic, π_{NEW}, corresponding to the newly calibrated parameters. Denoting by π_{INI} the initial price of the exotic, its sensitivity to the given implied volatility is thus calculated as:

$$\frac{\pi_{NEW} - \pi_{INI}}{\Delta\sigma}$$

For a better sensitivity we can also calculate the exotic price under a shift of $-\Delta\sigma$. However, if $\Delta\sigma$ is small enough (even though not too small), the improvement tends to be negligible.

In practice, it can be more meaningful to hedge the typical movements of the market implied volatility curves. To this end, we start from the three basic data for each maturity (the ATM and the two 25Δ call and put volatilities), and calculate the exotic's sensitivities to: i) a parallel shift of the three volatilities; ii) a change in the difference between the two 25Δ wings; iii) an increase of the two wings with fixed ATM volatility[8)]. In this way we should be able to capture the effect of a parallel, a twist and a convexity movements of the implied volatility surface. Once these sensitivities are calculated, it is straightforward to hedge the related exposure via plain vanilla options, namely the ATM calls or puts, 25Δ calls and 25Δ puts for each expiry.

A further approach that can be used for hedging is the classical *parameter hedging*. In this case, one calculates the variations of the exotic derivative price with respect to the parameters of the model, namely the forward volatilities and the foreign forward rates. We assume that the parameter λ is constant[9)].

If we have a number n of hedging instruments equal to the number of parameters, we can solve a linear system $Ax = b$, where b is a $(n \times 1)$ vector with the exotic's sensitivities obtained by an infinitesimal perturbation of the n parameters, and A is the $(n \times n)$ matrix whose i-th row contains the variations of the n hedging instruments with respect the i-th parameter. The instruments we use are, as before, the ATM puts, 25Δ calls and 25Δ puts for each expiry. Since the model is able to perfectly fit the price of these hedging instruments, we have a one to one relation between the sensitivities of the exotic with respect to the model parameters, and its variations with respect to the hedging instruments. More formally, denoting by π the exotic option's price, by p the model parameters' vector and by R the market's data vector, we have:

$$\frac{d\pi}{dR} = \frac{\partial\pi}{\partial R} + \frac{\partial\pi}{\partial p}\frac{\partial p}{\partial R}$$

Exact calibration allows therefore an exact calculation of the matrix $\frac{\partial p}{\partial R}$.

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8) This is actually equivalent to calculating the sensitivities with respect to the basic market quotes.

9) This can be justified by the fact that λ turns out to mainly accommodate the convexity of the volatility surface, which, as measured by the butterfly, is typically very stable. Besides, the effect of a change in convexity is well captured also by the difference between the volatilities in the two scenarios (when $N = 2$).

Table 9 Quantities of plain vanilla options to hedge the barrier options according to the BS model

	25Δ put	25Δ call	ATM put
Up&Out call	79,008,643	54,195,790	−127,556,533
Down&Out put	−400,852,806	−197,348,566	496,163,095

Table 10 Quantities of plain vanilla options to hedge the barrier options according to the UVUR model with the scenario approach

	25Δ put	25Δ call	ATM put
Up&Out call	76,409,972	42,089,000	−117,796,515
Down&Out put	−338,476,135	−137,078,427	413,195,436

We now show how the barrier options of the previous section can be hedged in terms of plain vanillas under both the scenarios and parameter hedging procedures, presenting also a BS based hedging portfolio for both options. Using again the market data as of 31 March 2004, we assume that both exotics have a nominal of 100,000,000 US$ and calculate the nominal values of the ATM puts, 25Δ calls and 25Δ puts that hedge them.

Table 9 shows the hedging portfolio suggested by the BS model: the hedging plain vanilla options have the same expiry as the related barrier option and their quantities are chosen so as to zero the overall Vega, Vanna and Volga.

In Table 10 we show the hedging quantities calculated according to the UVUR model with the scenario approach. The expiry of the hedging plain vanilla options is once again the same as that of the corresponding barrier options. It is noteworthy that both the sign and order of magnitude of the hedging options are similar to those of the BS model.

In the last two Tables 11 and 12 we show the results for the parametric approach. In this case, the hedging portfolio is made of all the options expiring before or at the exotic's maturity, though the amounts are all negligible but the ones corresponding to the maturity of the barrier option. Also in this case, signs and order of magnitude of the hedging amounts seem to agree with those obtained under the BS model and the UVUR model with a scenario approach. This should be considered as a further advantage of the UVUR model, both in terms of market practice and ease of implementation.

8. Conclusions

Asset price models where the instantaneous volatility is randomly drawn at (an infinitesimal instant after) the initial time are getting some popularity due to their

Table 11 Quantities of plain vanilla options to hedge the six-month Up&Out call according to the UVUR model with the parametric approach

	25Δ put	25Δ call	ATM put
1W	5	49	−34
2W	5	−4	5
1M	21	14	−30
2M	−27	−28	43
3M	15	39	−37
6M	77,737,033	44,319,561	-116,151,192

Table 12 Quantities of plain vanilla options to hedge the three-month Down&Out put according to the UVUR model with the parametric approach

	25Δ put	25Δ call	ATM put
1W	−11	244	−169
2W	−150	−226	288
1M	−34	78	−49
2M	24	−6	−19
3M	−334,326,734	−145,863,908	397,433,268

simplicity and tractability. We mention, for instance, the recent works of Brigo, Mercurio and Rapisarda (2004) and Gatarek (2003), who considered an application to the LIBOR market model. Alternatives where subsequent draws are introduced have been proposed by Alexander, Brintalos and Nogueira (2003) and Mercurio (2002).

At the same time, these models encounter some natural criticism because of their very formulation, which seems to make little sense from the historical viewpoint. In this article, however, we try to demonstrate the validity of the above uncertain volatility models, focusing in particular on that proposed by Brigo, Mercurio and Rapisarda (2004). We verify that such a model well behaves when applied to FX market data. Precisely, we show that it leads to a very good fitting of market volatilities, implies realistic forward volatilities, and allows for a fast and consistent valuation and hedge of a typical options book.

Our tests on the model are indeed encouraging and may help in addressing the above natural criticism. We in fact believe that a model should be judged not only in terms of its assumptions but also in terms of its practical implications.

Appendix A: the price of an up-and-out call

The price at time $t = 0$ of an up-and-out call (UOC) with barrier level $H > S_0$, strike K and maturity T under model (7) is (approximately) given by

$$\begin{aligned}\mathrm{UOC}_0 =& 1_{\{K<H\}} \sum_{i=1}^{N} \lambda_i \Bigg\{ S_0 e^{c_1+c_2+c_3} \left[\Phi\left(\frac{\ln \frac{S_0}{K} + c_1 + 2c_2}{\sqrt{2c_2}} \right) - \Phi\left(\frac{\ln \frac{S_0}{H} + c_1 + 2c_2}{\sqrt{2c_2}} \right) \right] \\ & - K e^{c_3} \left[\Phi\left(\frac{\ln \frac{S_0}{K} + c_1}{\sqrt{2c_2}} \right) - \Phi\left(\frac{\ln \frac{S_0}{H} + c_1}{\sqrt{2c_2}} \right) \right] - H e^{c_3 + (\beta-1)(\ln \frac{S_0}{H} + c_1) + (\beta-1)^2 c_2} \\ & \cdot \left[\Phi\left(\frac{\ln \frac{S_0}{H} + c_1 + 2(\beta-1)c_2}{\sqrt{2c_2}} \right) - \Phi\left(\frac{\ln \frac{S_0 K}{H^2} + c_1 + 2(\beta-1)c_2}{\sqrt{2c_2}} \right) \right] \\ & + K e^{c_3 + \beta(\ln \frac{S_0}{H} + c_1) + \beta^2 c_2} \left[\Phi\left(\frac{\ln \frac{S_0}{H} + c_1 + 2\beta c_2}{\sqrt{2c_2}} \right) - \Phi\left(\frac{\ln \frac{S_0 K}{H^2} + c_1 + 2\beta c_2}{\sqrt{2c_2}} \right) \right] \Bigg\},\end{aligned} \tag{13}$$

where $1_{\{A\}}$ denotes the indicator function of the set A, and

$$\begin{aligned}
c_1 &= c_1^i := R_i^d(0,T) - R_i^f(0,T) - \tfrac{1}{2} V_i^2(0,T) \\
c_2 &= c_2^i := \tfrac{1}{2} V_i^2(0,T) \\
c_2 &= c_3^i := -R_i^d(0,T) \\
\beta &= \beta^i := -2 \frac{\int_0^T [R_i^d(t,T) - R_i^f(t,T) - \frac{1}{2} V_i^2(t,T)] V_i^2(t,T)\, dt}{\int_0^T V_i^4(t,T)\, dt} \\
R_i^x(t,T) &:= \int_t^T r_i^x(s)\, ds, \quad x \in \{d, f\}, \\
V_i^2(t,T) &:= \int_t^T \sigma_i^2(s)\, ds
\end{aligned}$$

For a thorough list of formulas we refer to Rapisarda (2003)[10].

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[10] These formulas, including the above (13), are only approximations, since no closed-form formula is available for barrier option prices under the BS model with time-dependent coefficients.

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