The Kyoto Economic Review 74(1):47–64 (June 2005)

Real Options in an Oligopoly Market

Masaaki Kijima[1] and Takashi Shibata[2]

[1]Corresponding author, Faculty of Economics, Kyoto University. E-mail: kijima@econ.kyoto-u.ac.jp
[2]Faculty of Economics, Kyoto University. E-mail: shibata@econ.kyoto-u.ac.jp

This paper considers strategic entry decisions in an oligopoly market when the underlying state variable follows a geometric Brownian motion. It is shown that, even in the oligopoly case, three types of equilibria exist as in the duopoly case. Also, the firm value as well as the trigger point for investment decision increases as the market uncertainty increases. The presence of strategic interactions may push the firms to invest earlier than the trigger point. Nevertheless, the investment decision is always later than the traditional NPV criterion, even when the number of potential firms becomes infinity.

Keywords: investment decision, strategic substitution, NPV criterion, sequential investment, simultaneous investment

JEL Classification Numbers: D81, G31, D92, O31

1. Introduction

The aim of this paper is to analyze the strategies of firms in an oligopoly market when the firms have the opportunity to make an irreversible investment. The revenue of each firm is uncertain depending on the future condition of the market. By using the option pricing theory, we calculate the opportunity values of the firms and study their strategic entry decisions under the game-theoretic real options framework.

When a firm has the possibility to make an irreversible investment with future uncertainty, there is an option to delay. By analogy with a financial call option, it is optimal to delay exercising the option in the hope of gaining a higher payoff in the future[1)]. Using this insight, the real options approach improves the traditional net-present-value (NPV) method by incorporating the flexibility on decision making (see, e.g., McDonald and Siegel, 1986). An excellent overview of the real options approach is found in Dixit and Pindyck (1994) and Trigeorgis (1996).

[1)] See, e.g., Kijima (2002) for details of the option pricing theory.

Recently, the basic model has been extended in various ways[2]. Among them, Huisman (2001), Pawlina and Kort (2001) and Weeds (2002) considered the game-theoretic real options approach to incorporate strategic interactions between firms in a duopoly market. When related options are held by two firms with a first-mover advantage, each firm's ability to delay is undermined by the fear of pre-emption. In particular, under a winner-takes-all situation, the option will be exercised as soon as the payoff from doing so becomes marginally positive, and the traditional NPV rule resurfaces as the appropriate method of investment decision. On the other hand, if the market is not under such an extreme, the value of the option to delay should be evaluated under an environment with well-specified strategic interactions.

In the game-theoretic real options framework, assuming that the underlying market uncertainty follows a geometric Brownian motion, it is shown that there are three types of equilibria (i.e. pre-emptive leader–follower equilibria, joint-investment equilibria, and their mixture) in the case of strategic substitution[3]. The presence of strategic interactions may push a firm to invest earlier and, as for the basic model, the firm value as well as the optimal trigger point for the investment decision increases as the market uncertainty increases. These results have been extended to the case of diffusion processes with general volatility structure by Kijima and Shibata (2002).

In this paper, we extend the results in the duopoly case to an oligopoly market. That is, we consider strategic entry decisions in an oligopoly market with symmetric firms when the underlying state variable follows a geometric Brownian motion. Using the standard arguments, the value functions of each firm for various situations are obtained, i.e. when it is the leader or the follower. It is shown that the value functions are increasing and convex (concave, respectively) in the initial state variable for the follower (the leader). Based on these results and invoking classic results in game theory, we show that there always exists a critical point for each firm to pre-empt. In other words, even in the oligopoly case, there are three types of equilibria, pre-emptive leader–follower equilibria, joint-investment equilibria, and their mixture. The critical point is decreasing in the number of active firms in the market and always smaller than the optimal trigger point in the ordinary real options criterion. Nevertheless, the critical point for pre-emption can never be smaller than the traditional NPV criterion.

Even in a highly competitive market, the value of the option to delay exists and there is a possibility that many (but may not be all) firms start investing simulta-

---

[2] Other extensions include Alvarez and Stenbacka (2001) and Bernardo and Chowdhry (2002). Bernardo and Chowdhry (2002) considered an *incomplete* model where a firm cannot observe the state variable completely. Alvarez and Stenbacka (2001) considered a multi-stage technology project in which investment opportunities are available at each stage of the project. The multi-stage real option is similar to a *compound* option studied by Geske (1979) in the finance literature.

[3] This result has apparent economic implications in the context of industrial organization theory. That is, competition can introduce a strategic benefit to investment against the incumbent monopolist's claim that the introduction of competition will reduce future profits and delay investments. See, e.g., Nielsen (2002) for details.

neously (i.e. a mixture of joint and sequential investments) at the critical point for pre-emption. As a result, the investment decision may have been too early to produce a positive profit. This may explain why many Internet companies entered the market even though they have not yet been profitable[4)].

When the number of potential firms in a market becomes infinity, Grenadier (2002) claimed that the critical point for investment decision converges to the traditional NPV criterion (i.e. simultaneous investment). The claim is justified, because there is no need to wait in such a competitive market when the payoff from exercising the option becomes marginally positive. Although Grenadier (2002) considered a different model from ours, this paper also shows that there only exists a joint-investment equilibrium with probability one. However, the critical point can be different from the NPV criterion, and the investment decision may be too early to produce a positive profit.

This paper is organized as follows. In the next section, we describe the model and provide the information necessary for what follows. Section 3 calculates the firm values in the case of sequential investments. It is shown that optimal threshold for the k^{th}-to-invest is decreasing in k and increasing in the market uncertainty. Based on the results, we then consider equilibria in the oligopoly market in Section 4. We shall show that there exist not only joint-investment equilibria but also sequential-investment equilibria (pre-emptive leader–follower equilibria) in our oligopoly market. Section 5 considers a huge market including the case of infinitely many firms, and Section 6 concludes this paper. Proofs are given in Appendix.

2. The Setup

Consider $n \in \mathbb{N}$ firms in a market with the opportunity to make an irreversible investment. All firms can be active on the market to produce a single product and compete with each other to maximize their profits. The revenue of each firm is uncertain depending on the future condition of the market. We describe this uncertainty by state variable $X = (X_t)_{t\in\mathbb{R}_+}$ which evolves in time according to a geometric Brownian motion given by

$$\frac{\mathrm{d}X_t}{X_t} = \mu \mathrm{d}t + \sigma \mathrm{d}z_t, \qquad t \geq 0, \tag{1}$$

where $(z_t)_{t\in\mathbb{R}_+}$ denotes the standard Brownian motion, and where the mean growth rate μ as well as the volatility σ is a positive constant. As usual, the mean growth rate is assumed to be strictly less than the risk-neutral discount factor r, i.e. $\mu < r$. Also, it is assumed throughout that the firms are risk neutral.

Suppose that k, $k \leq n$, firms are active on the market. The revenue flow of each active firm is given by $\rho(x) = \pi_k x$, $\pi_k \in \mathbb{R}_+$, when $X_t = x$. The constants π_k are

[4)] Schwartz and Moon (2001) developed a different model to explain this phenomenon without considering strategic interactions.

non-increasing in k, $k = 1, 2, \dots, n$, meaning that an investment is less profitable when more firms have invested. This situation is called a *strategic substitution* and often observed in a market consisting of competing firms[5]. The sunk cost for each firm to adopt the investment opportunity is assumed to be a constant and equal to $I \in \mathbb{R}_+$. That is, all firms on the market have the same technology (symmetric firms).

Each firm seeks an optimal adoption strategy to the investment opportunity. However, since revenue is uncertain, the adoption time is a random variable. At each time t, the firm must decide whether it should adopt the investment opportunity or not based on the information $\mathcal{F}_t$ available at time t[6]. Hence, the adoption time is a stopping time.

The adoption strategy for each firm is not only a stopping time, but also it should take the non-cooperative competing firms' strategies into account. In other words, we seek an optimal adoption strategy to maximize the value function of each firm within the non-cooperative game-theoretic framework. It should be noted that the optimal adoption time may be infinity; in this case, the optimal strategy is never to invest.

3. Sequential Investments

Suppose that the constants π_k are strictly decreasing in k for the sake of simplicity. Then, there always exists a first-mover advantage and each firm's ability to delay is undermined by the fear of pre-emption. In this section, we calculate the value function of each firm when the firms make investments sequentially in time.

Let τ_j denote the adoption time of firm j, and let x_j be the associated trigger point (threshold), i.e. $\tau_j = \inf\{t \in \mathbb{R}_+ : X_t = x_j\}$. Associated with the collection of the adoption (stopping) times are the ordered sequence of random times $\tau_{(1)} \leq \tau_{(2)} \leq \cdots \leq \tau_{(n)}$. That is, by definition, $\tau_{(1)} = \min\{\tau_1, \tau_2, \dots, \tau_n\}$, and

$$\tau_{(k+1)} = \min\{\tau_j : j \in \{1, \dots, n\}, \tau_j > \tau_{(k)}\}, \qquad k = 1, 2, \dots, n-1.$$

In particular, $\tau_{(n)} = \max\{\tau_1, \tau_2, \dots, \tau_n\}$. Similarly, we denote the ordered sequence of the thresholds by $x_{(k)}$, $k = 1, 2, \dots, n$.

Dynamic games are usually solved backwards. Hence, we start with our analyses by assuming that $(n-1)$ firms have already invested.

3.1. The Value Function for the Last-to-Invest

Suppose that the current time is $t \in \mathbb{R}_+$, and assume that $\{\tau_{(n-1)} < t\}$, i.e. there remains only one firm that has not invested before time t. In this monopoly situa-

---

[5] The case of strategic compliment can be studied similarly. However, this case yields no strategic interaction and we omit the results.

[6] Throughout the paper, we fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, \mathbb{P})$, where the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ is generated by the state variable X. The filtration represents the resolution over time of the information commonly available to the firms. It is assumed that $\mathcal{F}_0 = \{\Omega, \emptyset\}$, i.e., no information is available at time 0.

tion, the value function of the firm is given by

$$V_{(n)}(x) = \operatorname*{ess\,sup}_{\tau_{(n)}\in\mathcal{T}_t} \mathbb{E}^x \left[\int_{\tau_{(n)}}^{\infty} \mathrm{e}^{-r(s-t)} \pi_n X_s \mathrm{d}s - \mathrm{e}^{-r(\tau_{(n)}-t)} I \,\middle|\, \mathcal{F}_t \right], \tag{2}$$

where $\mathcal{T}_t$ is the set of stopping times not less than t, and where $\mathbb{E}^x[\cdot|\mathcal{F}_t]$ denotes the time t conditional expectation operator given that $X_t = x$. For notational simplicity, we shall write $\mathbb{E}^x[\cdot]$ for the time 0 conditional expectation operator $\mathbb{E}^x[\cdot|\mathcal{F}_0]$. Also, the optimal adoption threshold and time are denoted by $x^*_{(n)}$ and $\tau^*_{(n)}$, respectively, i.e., $\tau^*_{(n)} = \inf\{t \in \mathbb{R}_+ : X_t = x^*_{(n)}\}$.

Using the strong Markov property and the time homogeneity of the geometric Brownian motion X, the value function (2) is rewritten as

$$V_{(n)}(x) = \mathbb{E}^x \left[\mathrm{e}^{-r\tau^*_{(n)}} \right] \mathbb{E}^{x^*_{(n)}} \left[\int_0^{\infty} \mathrm{e}^{-rt} \pi_n X_t \mathrm{d}t \right]. \tag{3}$$

The next proposition can be shown by solving the value-matching condition. The proof is given in the appendix.

Proposition 3.1. *The optimal threshold $x^*_{(n)}$ to adopt the investment opportunity is given by*

$$x^*_{(n)} = \frac{\beta}{\beta-1} \frac{r-\mu}{\pi_n} I. \tag{4}$$

The trigger point $x^*_{(n)}$ is strictly increasing in volatility σ. The Laplace transform of the optimal adoption time $\tau^*_{(n)}$ is given by

$$\mathbb{E}^x \left[\mathrm{e}^{-r\tau^*_{(n)}} \right] = \left(\frac{x}{x^*_{(n)}} \right)^{\beta}, \qquad x < x^*_{(n)}.$$

Here, β is the positive root of the quadratic equation $Q(y) = 0$, where

$$Q(y) := \frac{1}{2}\sigma^2 y^2 + \left(\mu - \frac{1}{2}\sigma^2 \right) y - r.$$

Accordingly, the optimal adoption time $\tau^*_{(n)}$ is finite with probability one. The value function (3) is obtained as

$$V_{(n)}(x) = \begin{cases} \left(\dfrac{x}{x^*_{(n)}} \right)^{\beta} \left\{ \dfrac{\pi_n}{r-\mu} x^*_{(n)} - I \right\}, & x^*_{(n)} > x, \\ \dfrac{\pi_n}{r-\mu} x - I, & x^*_{(n)} \le x. \end{cases} \tag{5}$$

Note that, from (4),

$$\frac{\pi_n}{r-\mu} x^*_{(n)} - I = \frac{I}{\beta-1}.$$

It follows that the value function is alternatively written as

$$V_{(n)}(x) = \left(\frac{x}{x^*_{(n)}} \right)^{\beta} \frac{I}{\beta-1}, \qquad x^*_{(n)} > x.$$

3.2. The Value Function for the Second Last-to-Invest

Suppose that $\{\tau_{(n-2)} < t\}$, i.e. there remain two firms that have not invested before time t. In this duopoly situation, the value function of the leader among the two firms is given by

$$
\begin{aligned}
V_{(n-1)}(x) &= \mathbb{E}^x\left[\int_{\tau^*_{(n-1)}}^{\tau^*_{(n)}} \mathrm{e}^{-r(s-t)}\pi_{n-1}X_s\mathrm{d}s - \mathrm{e}^{-r(\tau^*_{(n-1)}-t)}I + \int_{\tau^*_{(n)}}^{\infty} \mathrm{e}^{-r(s-t)}\pi_n X_s\mathrm{d}s \,\middle|\, \mathcal{F}_t\right] \\
&= \mathbb{E}^x\left[\int_{\tau^*_{(n-1)}}^{\infty} \mathrm{e}^{-r(s-t)}\pi_{n-1}X_s\mathrm{d}s - \mathrm{e}^{-r(\tau^*_{(n-1)}-t)}I \,\middle|\, \mathcal{F}_t\right] \\
&\qquad - \mathbb{E}^x\left[\int_{\tau^*_{(n)}}^{\infty} \mathrm{e}^{-r(s-t)}(\pi_{n-1}-\pi_n)X_s\mathrm{d}s \,\middle|\, \mathcal{F}_t\right] \\
&= \mathbb{E}^x\left[\mathrm{e}^{-r\tau^*_{(n-1)}}\right]\mathbb{E}^{x^*_{(n-1)}}\left[\int_0^{\infty} \mathrm{e}^{-rs}\pi_{n-1}X_s\mathrm{d}x\right] \\
&\qquad -\mathbb{E}^x\left[\mathrm{e}^{-r\tau^*_{(n)}}\right]E^{x^*_{(n)}}\left[\int_0^{\infty} \mathrm{e}^{-rs}(\pi_{n-1}-\pi_n)X_s\mathrm{d}x\right],
\end{aligned}
$$

where $\tau^*_{(n-1)}$ denotes the optimal adoption time for the leader. The optimal adoption time $\tau^*_{(n)}$ for the follower has been obtained above. It is well known (see, e.g., Huisman, 2001) that, as in (4), the optimal threshold $x^*_{(n-1)}$ to adopt the investment opportunity is given by

$$
x^*_{(n-1)} = \frac{\beta}{\beta-1}\frac{r-\mu}{\pi_{n-1}}I > 0.
$$

Since $\pi_{n-1} > \pi_n$, we have $x^*_{(n-1)} < x^*_n$. The value function is obtained as

$$
V_{(n-1)}(x) = \begin{cases}
\left(\dfrac{x}{x^*_{(n-1)}}\right)^{\beta}\dfrac{I}{\beta-1} - \left(\dfrac{x}{x^*_{(n)}}\right)^{\beta}\dfrac{\pi_{n-1}-\pi_n}{r-\mu}x^*_{(n)}, & x < x^*_{(n-1)}, \\
\dfrac{\pi_{n-1}}{r-\mu}x - I - \left(\dfrac{x}{x^*_{(n)}}\right)^{\beta}\dfrac{\pi_{n-1}-\pi_n}{r-\mu}x^*_{(n)}, & x^*_{(n-1)} \le x < x^*_{(n)}, \\
\dfrac{\pi_n}{r-\mu}x - I, & x^*_{(n)} \le x.
\end{cases} \tag{6}
$$

Note here that, when $X_t \geq x^*_{(n)}$, there is no need to wait for the investment and, hence, a simultaneous investment occurs to the both firms.

3.3. The Value Function for the General Case

Finally, we consider the general case. Suppose that $\{\tau_{(k-1)} < t\}$, i.e. there are k firms that have not invested before time t. In this oligopoly situation, the value function of the leader among the k firms is given by

$$
V_{(k)}(x) = \mathbb{E}^x\left[\int_{\tau^*_{(k)}}^{\tau^*_{(k+1)}} \mathrm{e}^{-r(s-t)}\pi_k X_s\mathrm{d}s - \mathrm{e}^{-r(\tau^*_{(k)}-t)}I \,\middle|\, \mathcal{F}_t\right]
$$

$$
\begin{aligned}
&+\mathbb{E}^x\left[\sum_{j=k+1}^{n-1}\int_{\tau^*_{(j)}}^{\tau^*_{(j+1)}} \mathrm{e}^{-r(s-t)}\pi_j X_s \mathrm{d}s \middle| \mathcal{F}_t\right] + \mathbb{E}^x\left[\int_{\tau^*_{(n)}}^{\infty} \mathrm{e}^{-r(s-t)}\pi_n X_s \mathrm{d}s \middle| \mathcal{F}_t\right] \\
= \ & \mathbb{E}^x\left[\int_{\tau^*_{(k)}}^{\infty} \mathrm{e}^{-r(s-t)}\pi_k X_s \mathrm{d}s - \mathrm{e}^{-r(\tau^*_{(k)}-t)} I \middle| \mathcal{F}_t\right] \\
&\qquad - \mathbb{E}^x\left[\sum_{j=k+1}^{n}\int_{\tau^*_{(j)}}^{\infty} \mathrm{e}^{-r(s-t)}(\pi_{j-1}-\pi_j) X_s \mathrm{d}s \middle| \mathcal{F}_t\right],
\end{aligned}
$$

where $\tau^*_{(j)}$ denotes the optimal adoption time for the j-th investor with $\tau^*_{(n+1)} = \infty$. Using the strong Markov property and the time homogeneity of the geometric Brownian motion X, we obtain that

$$
\begin{aligned}
V_{(k)}(x) &= \mathbb{E}^x\left[\mathrm{e}^{-r(\tau^*_{(k)}-t)}\left(\int_0^{\infty} \mathrm{e}^{-rs}\pi_k X_{\tau^*_{(k)}+s}\mathrm{d}s - I\right) \middle| \mathcal{F}_t\right] \\
&\qquad - \sum_{j=k+1}^{n}\mathbb{E}^x\left[\mathrm{e}^{-r(\tau^*_{(j)}-t)}\int_0^{\infty} \mathrm{e}^{-rs}(\pi_{j-1}-\pi_j) X_{\tau^*_{(j)}+s}\mathrm{d}s \middle| \mathcal{F}_t\right] \\
&= \mathbb{E}^x\left[\mathrm{e}^{-r\tau^*_{(k)}}\right]\mathbb{E}^{x^*_{(k)}}\left[\int_0^{\infty} \mathrm{e}^{-rs}\pi_k X_s \mathrm{d}s - I\right] \\
&\qquad - \sum_{j=k+1}^{n}\mathbb{E}^x\left[\mathrm{e}^{-r\tau^*_{(j)}}\right]\mathbb{E}^{x^*_{(j)}}\left[\int_0^{\infty} \mathrm{e}^{-rs}(\pi_{j-1}-\pi_j) X_s \mathrm{d}s\right].
\end{aligned}
$$

The next result can be proved similarly to Proposition 3.1.

Proposition 3.2. *The optimal threshold $x^*_{(k)}$ to adopt the investment opportunity is given by*

$$
x^*_{(k)} = \frac{\beta}{\beta-1}\frac{r-\mu}{\pi_k} I. \tag{7}
$$

The trigger point $x^*_{(k)}$ is strictly increasing in k and in volatility σ. The Laplace transform of the optimal adoption time $\tau^*_{(k)}$ is given by

$$
\mathbb{E}^x\left[\mathrm{e}^{-r\tau^*_{(k)}}\right] = \left(\frac{x}{x^*_{(k)}}\right)^{\beta}, \qquad x < x^*_{(k)}.
$$

In the standard real options analyses, i.e. if the market is monopoly, the threshold for making investment decision increases as the volatility increases. Proposition 3.2 shows that this result also holds even in our oligopoly market.

The key issue in Proposition 3.2 is that the optimal threshold $x^*_{(k)}$ depends only on π_k, not on the other revenue structures. Also, from (7), we have

$$
\frac{\pi_k}{r-\mu}x^*_{(k)} - I = \frac{I}{\beta-1}, \qquad k = 1, 2, \ldots, n. \tag{8}
$$

It follows from Proposition 3.2 that the value function is obtained as

$$V_{(k)}(x) = \begin{cases} \left(\dfrac{x}{x^*_{(k)}}\right)^{\beta} \dfrac{I}{\beta-1} - \displaystyle\sum_{j=k+1}^{n} \left(\dfrac{x}{x^*_{(j)}}\right)^{\beta} \dfrac{(\pi_{j-1}-\pi_j)}{r-\mu} x^*_{(j)}, & x < x^*_{(k)}, \\ \dfrac{\pi_k}{r-\mu} x - I - \displaystyle\sum_{j=k+1}^{n} \left(\dfrac{x}{x^*_{(j)}}\right)^{\beta} \dfrac{(\pi_{j-1}-\pi_j)}{r-\mu} x^*_{(j)}, & x^*_{(k)} \le x < x^*_{(k+1)}. \end{cases} \tag{9}$$

If $X_t \ge x^*_{(k+1)}$, the leader should invest immediately to leave the situation with $(k-1)$ firms being not invested before time t.

Remark 3.1. In the traditional NPV approach, the trigger point to invest is given by

$$x^{\mathrm{N}}_{(j)} = \frac{r-\mu}{\pi_j} I, \qquad j = 1, 2, \ldots, n. \tag{10}$$

Since $\beta > 1$ in our setting, it follows from (7) that $x^{\mathrm{N}}_{(j)} < x^*_{(j)}$ for all $j = 1, 2, \ldots, n$. Hence, the investment timing in the real options approach is always delayed compared to the NPV criterion.

4. Equilibria

In this section, we focus on Markov perfect equilibria so that the outcome of the non-cooperative game takes one of two forms, a pre-emptive leader–follower outcome or a joint-investment outcome, depending on parameter values[7]. To this end, we first discuss the properties of the value functions under the strategic interactions.

Let $V^o_{(k)}(x)$ ($V^m_{(k)}(x)$, respectively) be the value function for a firm when the firm has not (has) invested for the k^{th}-to-invest opportunity.

4.1. The Duopoly Case

As before, we start with our analyses by assuming that $(n-2)$ firms have already invested. Suppose that two firms i and j are under strategic interaction, and assume that firm i decides to invest immediately. Then, from (6), the value function for firm i (the leader) is given by

$$V^m_{(n-1)}(x) = \frac{\pi_{n-1}}{r-\mu} x - I - \left(\frac{x}{x^*_{(n)}}\right)^{\beta} \frac{\pi_{n-1}-\pi_n}{r-\mu} x^*_{(n)}, \qquad x < x^*_{(n)}. \tag{11}$$

Note that the value function $V^m_{(n-1)}(x)$ satisfies the value-matching condition $V^m_{(n-1)}(x^*_{(n)}) = V^m_{(n)}(x^*_{(n)})$, i.e.

$$V^m_{(n-1)}(x^*_{(n)}) = \frac{\pi_n}{r-\mu} x^*_{(n)} - I. \tag{12}$$

---

[7] See, e.g., Fudenberg and Tirole (1985) for details.

On the other hand, from (5), the value function for firm j (the follower) is given by

$$V^o_{(n)}(x) = \left(\frac{x}{x^*_{(n)}}\right)^\beta \left\{\frac{\pi_n}{r-\mu}x^*_{(n)} - I\right\}, \qquad x < x^*_{(n)}. \tag{13}$$

Note that the value function $V^o_{(n)}(x)$ satisfies not only the value-matching condition but also the smooth-pasting condition $V^{o\prime}_{(n)}(x^*_{(n)}) = V^{m\prime}_{(n)}(x^*_{(n)})$, because it is determined so as to optimize the trigger point $x^*_{(n)}$. Here, the prime denotes the derivative. It follows that the value function $V^o_{(n)}(x)$ satisfies

$$V^o_{(n)}(x^*_{(n)}) = \frac{\pi_n}{r-\mu}x^*_{(n)} - I, \qquad V^{o\prime}_{(n)}(x^*_{(n)}) = \frac{\pi_n}{r-\mu}. \tag{14}$$

It is readily seen that the value function $V^m_{(n-1)}(x)$ is strictly concave in $x \in (0, x^*_{(n)})$, while the value function $V^o_{(n)}(x)$ is strictly convex in $x \in (0, x^*_{(n)})$ since $\beta > 1$. These value functions are depicted in Figure 1. Here, the straight line represents the value function $V^m_{(n)}(x)$, $x \geq 0$.

Since $V^m_{(n-1)}(x)$ is strictly concave and $V^o_{(n)}(x)$ is strictly convex in $x \in (0, x^*_{(n)})$, there exists a unique root $x^{\mathrm{P}}_{(n-1)} \in (0, x^*_{(n)})$ for the equation $V^m_{(n-1)}(x) = V^o_{(n)}(x)$. The existence follows from the conditions (12) and (14). Also, $x^{\mathrm{P}}_{(n-1)} < x^*_{(n-1)}$ (to be proved later). Therefore, we conclude that, in the duopoly market, firm i has an incentive to invest immediately when $x^{\mathrm{P}}_{(n-1)} < x < x^*_{(n-1)}$, since we have $V^m_{(n-1)}(x) > V^o_{(n)}(x)$ for $x^{\mathrm{P}}_{(n-1)} < x < x^*_{(n)}$. These results have been obtained by Huisman (2001), Pawlina and Kort (2001) and Weeds (2002).

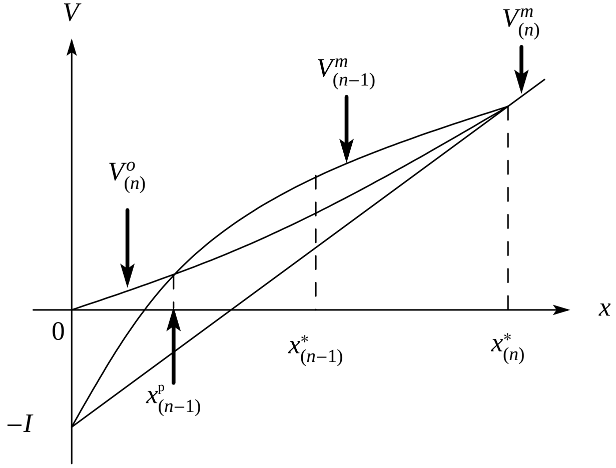

Figure 1 Value functions for the duopoly market.

4.2. The Oligopoly Case

Suppose that $(k-1)$ firms have already invested, and assume that firm i, among the remaining $(n-k+1)$ firms, decides to invest immediately. Then, from (9), the value function for firm i (the leader) is given by

$$V^m_{(k)}(x) = \frac{\pi_k}{r-\mu}x - I - \sum_{j=k+1}^{n}\left(\frac{x}{x^*_{(j)}}\right)^{\beta}\frac{\pi_{j-1}-\pi_j}{r-\mu}x^*_{(j)}, \qquad x < x^*_{(k+1)}. \tag{15}$$

The value function $V^m_{(k)}(x)$ represents the present value of the profit obtained after carring out the investment, and satisfies the value-matching condition $V^m_{(k)}(x^*_{(k+1)}) = V^m_{(k+1)}(x^*_{(k+1)})$.

On the other hand, suppose that firm i decides to be a follower and take the $(k+1)^{\text{th}}$ investment opportunity. The value function for firm i in this case is given by

$$V^o_{(k+1)}(x) = \left(\frac{x}{x^*_{(k+1)}}\right)^{\beta}\frac{I}{\beta-1} - \sum_{j=k+2}^{n}\left(\frac{x}{x^*_{(j)}}\right)^{\beta}\frac{\pi_{j-1}-\pi_j}{r-\mu}x^*_{(j)}, \qquad x < x^*_{(k+1)}. \tag{16}$$

The value function $V^o_{(k+1)}(x)$ represents the option value to delay the investment opportunity, and satisfies not only the value-matching condition $V^o_{(k+1)}(x^*_{(k+1)}) = V^m_{(k+1)}(x^*_{(k+1)})$, but also the smooth-pasting condition $V^{o\prime}_{(k+1)}(x^*_{(k+1)}) = V^{m\prime}_{(k+1)}(x^*_{(k+1)})$.

Lemma 4.1. *For all* $k = 1, 2, \ldots, n-1$, *we have the following.*

(1) *The value function* $V^m_{(k)}(x)$ *is strictly concave in* $x \in (0, x^*_{(k+1)})$, *while the value function* $V^o_{(k+1)}(x)$ *is strictly convex in* $x \in (0, x^*_{(k+1)})$.

(2) *We have* $V^m_{(k)}(x) > V^m_{(k+1)}(x)$ *and* $V^o_{(k)}(x) > V^o_{(k+1)}(x)$ *for* $x \le x^*_{(k)}$.

Collecting the information obtained above, we can depict the value functions $V^m_{(k)}(x)$ and $V^o_{(k+1)}(x)$ as in Figure 2. Also, since $V^m_{(k)}(x)$ is strictly concave and $V^o_{(k+1)}(x)$ is strictly convex in $x \in (0, x^*_{(k+1)})$, there exists a unique root $x^{\mathrm{P}}_{(k)} \in (0, x^*_{(k+1)})$ for the equation $V^m_{(k)}(x) = V^o_{(k+1)}(x)$.

As to the critical points $x^{\mathrm{P}}_{(k)}$, we have the following.

Proposition 4.1. *The critical value* $x^{\mathrm{P}}_{(k)}$ *is strictly decreasing in k, and* $x^{\mathrm{P}}_{(k)} < x^*_{(k)}$ *for all* $k = 1, 2, \ldots, n-1$.

The next result is a direct consequence of the properties of the value functions stated above.

Proposition 4.2. *In the oligopoly market, we have* $V^m_{(k)}(x) > V^o_{(k+1)}(x)$ *for* $x^{\mathrm{P}}_{(k)} < x < x^*_{(k+1)}$. *Hence, firm i has an incentive to invest immediately when* $x^{\mathrm{P}}_{(k)} < x < x^*_{(k)}$.

We are now in a position to state our main result. Recall from Proposition 4.2 that each firm has an incentive to pre-empt.

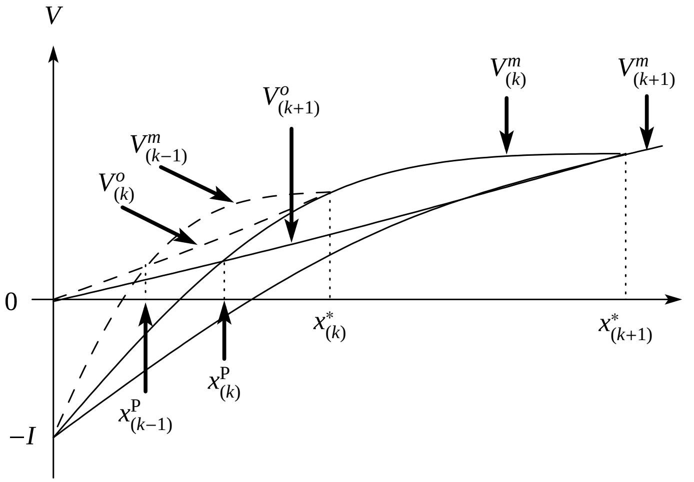

Figure 2 Value functions for the oligopoly market.

Theorem 4.1. *Suppose that there are n firms that have not invested in the market. Then, we have the following outcomes for the initial state x.*

(0) *If $x < x^{\mathrm{P}}_{(1)}$, then either*
- (0-1) *each firm invests at time $\tau^{\mathrm{P}}_{(1)}, \tau^{\mathrm{P}}_{(2)}, \cdots, \tau^{\mathrm{P}}_{(n-1)}, \tau^*_{(n)}$ in sequence,*
- (0-2) *some m firms invest at time $\tau^{\mathrm{P}}_{(1)}$, and the other $(n-m)$ firms invest simultaneously or sequentially after time $\tau^{\mathrm{P}}_{(m+1)}$, or*
- (0-3) *all firms invest simultaneously at time $\tau^{\mathrm{P}}_{(1)}$.*

(1) *If $x^{\mathrm{P}}_{(1)} \le x < x^{\mathrm{P}}_{(2)}$, then either*
- (1-1) *one firm invests now, the other firms invest at $\tau^{\mathrm{P}}_{(2)}, \cdots, \tau^{\mathrm{P}}_{(n-1)}, \tau^*_{(n)}$ in sequence,*
- (1-2) *some m, $m > 1$, firms invest simultaneously now, the other $(n-m)$ firms invest simultaneously or sequentially after time $\tau^{\mathrm{P}}_{(m+1)}$, or*
- (1-3) *all firms invest simultaneously now.*

(k) *If $x^{\mathrm{P}}_{(k)} \le x < x^{\mathrm{P}}_{(k+1)}$, then either*
- (k-1) *some k firms invest now, the other $(n-k)$ firms invest at $\tau^{\mathrm{P}}_{(k+1)}, \cdots, \tau^{\mathrm{P}}_{(n-1)}, \tau^*_{(n)}$ in sequence,*
- (k-2) *some m, $m > k$, firms invest simultaneously now, the other $(n-m)$ firms invest simultaneously or sequentially after time $\tau^{\mathrm{P}}_{(m+1)}$, or*
- (k-3) *all firms invest simultaneously now.*

(n) *If $x \ge x^*_{(n)}$, then all firms invest simultaneously now.*

From Theorem 4.1, we note that there is the possibility of sequential equlibria in our oligopoly market. Note also that, in the interval $\{x^{\mathrm{P}}_{(k)} \le x < x^{\mathrm{P}}_{(k+1)}\}$, although the outcome that only one of the firms invests is a Pareto optimum, there is the possibility that more than two firms invest simultaneously. As a result, option values to delay the investment are substantially undermined and the value functions for some firms may become negative.

Nevertheless, the investment decision is always delayed compared to the NPV counterpart, as the next result reveals.

Proposition 4.3. *For each $k = 1, 2, \ldots, n-1$, we have $x^{\mathrm{P}}_{(k)} > x^{\mathrm{N}}_{(k)}$, where the NPV criterion $x^{\mathrm{N}}_{(k)}$ is defined in* (10).

More precisely, consider the situation that no firms are active in the market. If firm i adopts the NPV criterion to enter the market, the firm invests as soon as the underlying state variable reaches the trigger point $x^{\mathrm{N}}_{(1)}$. As a result, the value function of firm i becomes

$$V_i(x) = \frac{\pi_1}{r-\mu}x - I - \sum_{j=2}^{n}\left(\frac{x}{x^*_{(j)}}\right)^{\beta}\frac{\pi_{j-1}-\pi_j}{r-\mu}x^*_{(j)}, \qquad x^{\mathrm{N}}_{(1)} \le x < x^{\mathrm{P}}_{(1)}.$$

Of course, this value function can be negative until the state variable x becomes sufficiently large.

On the other hand, even though firm j adopts the real options criterion to enter the market, the fear of pre-emption may push some firms to invest as soon as the state variable reaches the critical point $x^{\mathrm{P}}_{(1)}$ under strategic interactions. As a result, if k firms invest simultaneously, the value function of firm j becomes, from (15),

$$V_j(x) = \frac{\pi_k}{r-\mu}x - I - \sum_{j=k+1}^{n}\left(\frac{x}{x^*_{(j)}}\right)^{\beta}\frac{\pi_{j-1}-\pi_j}{r-\mu}x^*_{(j)}, \qquad x^{\mathrm{P}}_{(1)} \le x < x^*_{(1)}.$$

But, again, this value can be negative before the underlying state variable x becomes sufficiently large.

Finally, we consider the comparative statics of the critical point $x^{\mathrm{P}}_{(k)}$ for pre-emption with respect to volatility σ.

Proposition 4.4. *The critical point $x^{\mathrm{P}}_{(k)}$ for pre-emption is increasing in volatility σ if and only if*

$$\log\frac{x^{\mathrm{P}}_{(k)}}{x^*_{(k+1)}} + \frac{y_k - 1}{\beta y_k - \beta + 1} < 0, \tag{17}$$

where $y_k = \pi_k/\pi_{k+1} > 1$.

Proposition 4.4 states that, when the critical point $x^{\mathrm{P}}_{(k)}$ for pre-emption is close to the optimal trigger point $x^*_{(k+1)}$, the critical point is decreasing in volatility. However, when the critical point $x^{\mathrm{P}}_{(k)}$ is far from $x^*_{(k+1)}$, the critical point is increasing in volatility. This is interesting, because the optimal trigger point $x^*_{(k)}$ is monotone increasing with respect to the volatility.

5. Real Options in a Huge Market

In this section, we consider the case of infinitely many firms. That is, suppose that there are infinitely many potential firms that can enter a market. As before, we assume that the revenue flow of each active firm is given by $\rho(x) = \pi_k x$ when there are k active firms in the market. It is assumed that the constant π_k is decreasing in k and converges to a constant π as $k \to \infty$[8].

To this end, suppose first that there exists some n such that $\pi_k = \pi$ for all $k \geq n$. That is, there is no strategic competition after $(n-1)$ firms have invested. Then, no firms feel the fear of pre-emption and they behave as firms in a monopoly market. Hence, the value function for each firm in this case is exactly the same as that for the last-to-invest firm discussed in Section 3. More precisely, the optimal trigger point x^* for each firm is given by (4) with π_n being replaced by π. The optimal adoption time τ^* is finite with probability one, and the value function is obtained as (5) with π_n and $x^*_{(n)}$ being replaced by π and x^*, respectively.

Consider next the case that $(n-2)$ firms have invested before time t. If $\pi_{n-1} > \pi_n = \pi$, then there exists a first-mover advantage and a firm may invest earlier than its optimal trigger point. In fact, we are in the same situation as Proposition 4.1 and a firm has an incentive to invest earlier as far as $V^m_{(n-1)}(x) > V^o_{(n)}(x)$ holds.

The general case is also the same as that in Section 4. That is, a firm has an incentive to invest earlier as far as $V^m_{(k-1)}(x) > V^o_{(k)}(x)$, $x^{\mathrm{P}}_{(k)} \leq x < x^*_{(k)}$, when there are $(k-1)$ firms that have not invested yet. Note that these value functions depend only on n, not on the number of remaining firms. Accordingly, the critical point $x^{\mathrm{P}}_{(k)}$ for pre-emption remains the same. The next proposition summarizes.

Proposition 5.1. *Consider a market with infinitely many potential firms. If there exists some n such that $\pi_k = \pi$ for all $k \geq n$, then we have the same equilibria as Theorem 4.1.*

Recall that, in Proposition 5.1, n represents the number of firms that the first-mover advantage disappears completely. It is readily seen that the value functions $V^o_{(k)}$ and $V^m_{(k)}$ are decreasing in n. Hence, in such a highly competitive market, firms become less profitable as n increases. Also, while the optimal trigger point $x^*_{(k)}$ as well as the NPV critical point $x^{\mathrm{N}}_{(k)}$ is independent of n, the critical point $x^{\mathrm{P}}_{(k)}$ for pre-emption is affected by the increase of n.

The trigger point for pre-emption can never be smaller than the traditional NPV criterion. However, even in a highly competetive market, there is a possibility that many firms start investing simultaneously (i.e. joint-investment equilibria). As a result, the investment decision may have been too early to produce a positive profit. This may explain why many Internet companies entered the market even though they have not yet been profitable.

---

[8] We assume that the constant π is equal to the cost expenditure I, i.e., the profit of each active firm converges to zero when the number of such firms increases to infinity.

Suppose next that π_k is strictly decreasing in $k = 1, 2, \ldots$. That is, there always exists a first-mover advantage in the market. Then, firms always feel the fear of pre-emption and there is no need to wait. Hence, as soon as the state variable reaches some critical point x^{P}, some firms decide to enter the market. The probability that only finitely many firms enter the market is zero. Therefore, there is some critical level that infinitely many firms enter the market simultaneously. The critical level is larger than $x^{\mathrm{N}}_{(1)}$, but can be smaller than the NPV criterion $x^{\mathrm{N}}_{(\infty)}$. The next theorem summarizes.

Theorem 5.1. *Suppose that π_k is strictly decreasing in $k = 1, 2, \ldots$. Then, there are only joint-investment equilibria. The critical point for investment decision is larger than $x^{\mathrm{N}}_{(1)}$, but can be smaller than the traditional NPV criterion $x^{\mathrm{N}}_{(\infty)}$.*

6. Concluding Remarks

In this paper, we analyze the strategies of firms in an oligopoly market when the firms have the opportunity to make an irreversible investment. By using the option pricing theory, we calculate the opportunity values of the firms and study their strategic entry decisions under the game-theoretic real options framework.

Assuming that the underlying market uncertainty follows a geometric Brownian motion, we show that there are three types of equilibria (i.e. pre-emptive leader–follower equilibria, joint-investment equilibria, and their mixture). The presence of strategic interactions may push a firm to invest earlier and, as for the ordinary real options model, the firm value as well as the trigger point increases as the market uncertainty increases. Moreover, it is shown that the critical point for pre-emption is always smaller than the trigger point.

In a highly competitive market, there is a possibility that many firms start investing simultaneously (i.e. a mixture of sequential and joint investments). As a result, the investment decision may have been too early to produce a positive profit. This may explain why many Internet companies entered the market even though they have not yet been profitable.

Finally, when the number of firms in the market becomes infinity, this paper shows that there only exist joint-investment equilibria. The critical point for investment decision is larger than $x^{\mathrm{N}}_{(1)}$, but can be smaller than the traditional NPV criterion $x^{\mathrm{N}}_{(\infty)}$.

However, the assumption of geometric Brownian motions for the underlying state variable is problematic, because Davis (1998) reported through an empirical work that the volatility of state variable varies over the life of the real options. Hence, as in Kijima and Shibata (2002), it is of great interest to extend our model to the case of diffusion processes with general volatility structure. Also, in an actual project, investment opportunities are usually available at each stage of the project. Hence, as in Alvarez and Stenbacka (2001), it is also of importance to extend our model to the compound option framework.

Appendix

Proof of Proposition 3.1. It is well known that the Laplace transform of the optimal adoption time $\tau^*_{(n)}$ satisfies the ordinary differential equation

$$\frac{x^2\sigma^2}{2}W''(x) + \mu x W'(x) = rW(x).$$

It follows that $W(x) = (x/x^*_{(n)})^\beta$ for some $x^*_{(n)}$. The optimal threshold $x^*_{(n)}$ is obtained by the smooth-pasting condition. That is,

$$x^*_{(n)} = \operatorname*{argmax}_y \left(\frac{x}{y}\right)^\beta \left\{\frac{\pi_n}{r-\mu}y - I\right\}.$$

Finally, differentiation of the threshold with respect to σ yields

$$\frac{\mathrm{d}x^*_{(n)}}{\mathrm{d}\sigma} = \frac{\partial x^*_{(n)}}{\partial\beta}\frac{\mathrm{d}\beta}{\mathrm{d}\sigma} = \left(\frac{-1}{(\beta-1)^2}\frac{r-\mu}{\pi_n}I\right)\frac{\mathrm{d}\beta}{\mathrm{d}\sigma} > 0$$

where we have used the fact that $\dfrac{\mathrm{d}\beta}{\mathrm{d}\sigma} < 0$. □

Proof of Lemma 4.1. The strict concavity of $V^m_{(k)}(x)$ in $x \in (0, x^*_{(k+1)})$ is obvious, since $\beta > 1$. To prove the strict convexity of $V^o_{(k+1)}(x)$, define the function $o_k(x) := V^o_{(k)}(x) - V^o_{(k+1)}(x)$. Then, we obtain $o_k(x) = x^\beta L_k$ with

$$L_k = \frac{\pi_k}{r-\mu}\left\{\frac{1}{(x^*_{(k)})^{\beta-1}} - \frac{1}{(x^*_{(k+1)})^{\beta-1}}\right\} - I\left\{\frac{1}{(x^*_{(k)})^\beta} - \frac{1}{(x^*_{(k+1)})^\beta}\right\},$$

where we have used Equation (8). Note that, from Proposition 3.1, we have $x^*_{(k)} > x^*_{(k+1)}$ so that

$$L_k \geq \left(\frac{\pi_k}{r-\mu}x^*_{(k)} - I\right)\left\{\frac{1}{(x^*_{(k)})^\beta} - \frac{1}{(x^*_{(k+1)})^\beta}\right\},$$

which is positive. It follows that

$$V^o_{(k)}(x) = \sum_{j=k}^n o_j(x) = x^\beta \sum_{j=k}^n L_j,$$

with $L_{n+1} = 0$, which is strictly convex since $\beta > 1$, proving the first part of the lemma.

To prove the second part, define the function $\gamma_k(x) := V^m_{(k)}(x) - V^m_{(k+1)}(x)$. Then,

$$\gamma_k(x) = \frac{\pi_k - \pi_{k+1}}{r-\mu}\left[x - x^\beta (x^*_{(k+1)})^{-\beta+1}\right].$$

It suffices to prove that $\gamma_k(0) = 0$, $\gamma_k(x^*_{(k+1)}) = 0$, and $\dfrac{\partial^2 \gamma_k(x)}{\partial x^2} < 0$. The first two equations are obvious. To prove the third assertion, we obtain

$$\frac{\partial^2 \gamma_k(x)}{\partial x^2} = \frac{\pi_k - \pi_{k+1}}{r-\mu} \left[-\beta(\beta-1) x^{\beta-2} (x^*_{(k+1)})^{-\beta+1} \right],$$

which is negative, since $\beta > 1$ and $r > \mu$. The other part is obvious from the proof of the first part. □

Proof of Proposition 4.1. The first assertion is a direct consequence of Lemma 4.1(2). To prove the second assertion, define the function $\phi_k(x) := V^m_{(k)}(x) - V^o_{(k+1)}(x)$, $k = 1, 2, \dots, n-1$. It suffices to prove that $\phi_k(x^*_{(k)}) > 0$. To this end, we note that

$$\phi_k(x) = \frac{\pi_k}{r-\mu} x - I - \left(\frac{x}{x^*_{(k+1)}} \right)^\beta \left\{ \frac{\beta}{\beta-1} \frac{\pi_k}{\pi_{k+1}} - 1 \right\} I, \qquad \text{(A.1)}$$

where we have used (7). Substituting $x^*_{(k)}$ into (A.1), we obtain

$$\phi_k(x^*_{(k)}) = I \left[\frac{1}{\beta-1} - \left(\frac{\pi_{k+1}}{\pi_k} \right)^\beta \left\{ \frac{\beta}{\beta-1} \frac{\pi_k}{\pi_{k+1}} - 1 \right\} \right].$$

Let us define the function $h_k(a) := (\beta-1)\phi_k(x^*_{(k)}) I^{-1}$ with $a = \pi_{k+1}/\pi_k \in (0, 1)$. Then,

$$h_k(a) = 1 - \beta a^{\beta-1} + (\beta-1) a^\beta.$$

Note that $h_k(a) > 0$ for all $a \in (0, 1)$ implies $\phi_k(x^*_{(k)}) > 0$. This holds true since $h_k(1) = 0$ and

$$h_k'(a) = \beta(\beta-1) a^{\beta-1} \left[1 - a^{-1} \right] < 0, \qquad 0 < a < 1.$$

Since $\phi_k(x^{\mathrm{P}}_{(k)}) = 0$, we conclude that $x^{\mathrm{P}}_{(k)} < x^*_{(k)}$. □

Proof of Proposition 4.3. Let $\phi_k(x)$ be defined as in the proof of Proposition 4.1. Then, $\phi_k(x^*_{(k+1)}) = \phi_k(x^{\mathrm{P}}_{(k)}) = 0$ and $\phi_k''(x) < 0$ for all x. Hence, $\phi_k(x) < 0$ and $x < x^*_{(k)}$ implies that $x < x^{\mathrm{P}}_{(k)}$. It follows from (A.1) that $\phi_k(x^{\mathrm{N}}_{(k)}) < 0$, since

$$\frac{\pi_k}{r-\mu} x^{\mathrm{N}}_{(k)} = I, \qquad \frac{\beta}{\beta-1} > 1, \qquad \frac{\pi_k}{\pi_{k+1}} > 1.$$

Hence, $x^{\mathrm{N}}_{(k)} < x^{\mathrm{P}}_{(k)}$, proving the proposition. □

Proof of Proposition 4.4. Let $\phi_k(x)$ be defined as in (A.1). Then, by definition, $x^*_{(k)}$ is the unique solution to the equation

$$\frac{\pi_k}{r-\mu}x - I = \left(\frac{x}{x^*_{(k+1)}}\right)^{\beta}\left\{\frac{\beta}{\beta-1}y_k - 1\right\}I, \qquad x < x^*_{(k)}, \tag{A.2}$$

where $y_k = \pi_k/\pi_{k+1} > 1$. Let us define

$$f(\beta) = \left(\frac{x}{x^*_{(k+1)}}\right)^{\beta}\left\{\frac{\beta}{\beta-1}y_k - 1\right\}.$$

Then, after some algebra, we obtain

$$f'(\beta) = \left(\frac{x}{x^*_{(k+1)}}\right)^{\beta}\left[\left\{\frac{\beta}{\beta-1}y_k - 1\right\}\log\frac{x}{x^*_{(k+1)}} + \frac{y_k-1}{\beta-1}\right].$$

Hence, $f'(\beta) < 0$ (> 0, respectively) at $x = x^{\mathrm{P}}_{(k)}$ if

$$\log\frac{x^{\mathrm{P}}_{(k)}}{x^*_{(k+1)}} + \frac{y_k - 1}{\beta y_k - \beta + 1} < 0 \ \ (> 0), \tag{A.3}$$

which is (17). That is, the right-hand side in (A.2) decreases (increases) around $x = x^{\mathrm{P}}_{(k)}$ if $f'(\beta) < 0$ (> 0). On the other hand, the left-hand side in (A.2) is independent of β. Therefore, the solution to (A.2) is decreasing (increasing) with respect to β if (A.3) holds. Since $\dfrac{\mathrm{d}\beta}{\mathrm{d}\sigma} < 0$, the result follows. □

Acknowledgements

This research is supported in part by a Grant-in-Aid for the 21st Century COE research. Authors also acknowledge the financial support of Daiwa Securities Group Inc.

References

Alvarez, L. H. R. and R. Stenbacka (2001) “Adoption of uncertain multi-stage technology projects: A real options approach”, *Journal of Mathematical Economics* 35: 71–97.

Bernando, A. E. and B. Chowdhry (2002) “Resources, real options, and corporate strategy”, *Journal of Financial Economics* 63: 211–234.

Davis, G. A. (1998) “Estimating volatility and dividend yield when valuing real options to invest or abandon”, *Quarterly Review of Economics and Finance* 38: 725–754.

Dixit, A. K. and R. S. Pindyck (1994) *Investment under uncertainty*. Princeton University Press. Princeton.

Fudenberg, D. and J. Tirole (1985) “Preemption and rent equalization in the adoption of new technology”, *Review of Economic Studies* 52: 383–401.

Geske, R. (1979) “The valuation of compound options”, *Journal of Financial Economics* 7: 63–81.

Grenadier, S. (2002) “Option exercise game: An application to the equilibrium investment strategies of firms”, *Review of Financial Studies*, 15: 691–721.

Huisman, K. J. M. (2001) “Technology investment: A game theoretic real options approach”, Ph.D Dissertation, Tilburg University.

Kijima, M. (2002), *Stochastic processes with applications to finance*. Chapman & Hall. London.

Kijima, M. and T. Shibata (2002) “Real options in a duopoly market with general volatility structure”, Working Paper, Kyoto University.

McDonald, R. and D. R. Siegel (1986) “The value of waiting to invest”, *Quarterly Journal of Economics* 101: 707–727.

Nielsen, M. J. (2002) “Competition and irreversible investment”, *International Journal of Industrial Organization* 20: 731–743.

Pawlina, G. and P. M. Kort (2001) “Real options in an asymmetric duopoly: Who benefits from your competitive disadvantage?” *Journal of Economics and Management Strategy*, forthcoming.

Schwartz, E. S. and M. Moon (2001) “Rational pricing of internet companies revisited”, *Financial Review* 36: 7–26.

Trigeorgis, L. (1996) *Real options: managerial flexibility and strategy in resource allocation*. MIT Press, Boston.

Weeds, H. (2002) “Strategic delay in a real options model of R&D competition”, *Review of Economic Studies* 69: 729–747.