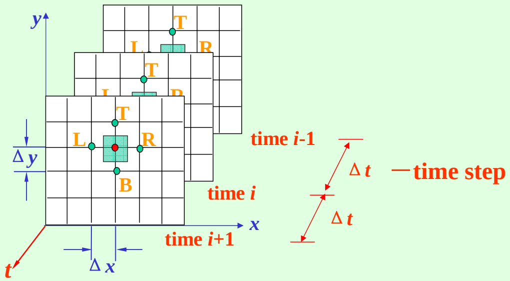
§ 5-5 Transient Heat Conduction

❑ Discretization



Δ x, Δ y — space step

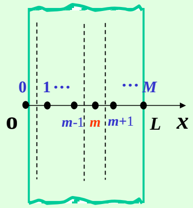
- Transient heat conduction is dependent with time and space
- Numerical method of transient problems requires discretization in time in addition to discretizaion in space

1) Finite difference method

e.g. Transient heat conduction in a plane wall

- Differential equation of heat conduction is

$$\frac{\partial^2 T}{\partial x^2} + \frac{\dot{g}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$



- Replace differentials by finite differences

$$\left(\frac{\partial T}{\partial t}\right)_m \approx \frac{T_m^{i+1} - T_m^i}{\Delta t} \qquad t_{i+1} = t_i + \Delta t$$

$$\left(\frac{\partial^2 T}{\partial x^2}\right)_m \approx \frac{T_{m-1}^{?} - 2T_m^{?} + T_{m+1}^{?}}{\Delta x^2} \qquad (m = 1, 2, 3, \ldots, M\text{-}1) \text{— interior nodes}$$

- Assume α is constant

$$\frac{T_{m-1}^{?} - 2T_m^{?} + T_{m+1}^{?}}{\Delta x^2} + \frac{\dot{g}_m^{?}}{k} = \frac{1}{\alpha}\frac{T_m^{i+1} - T_m^i}{\Delta t}$$

2 formats
- 1) previous time step i : Explicit method
- 2) new time step $i+1$: Implicit method

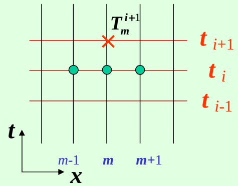
▪ Explicit method

$$\frac{T_{m-1}^{i} - 2T_m^{i} + T_{m+1}^{i}}{\Delta x^2} + \frac{\dot{g}_m^{i}}{k} = \frac{1}{\alpha}\frac{T_{m+1}^{i+1} - T_m^i}{\Delta t}$$

$$T_{m-1}^{i} - 2T_m^{i} + T_{m+1}^{i} + \frac{\dot{g}_m \Delta x^2}{k} = \frac{\Delta x^2}{\alpha \Delta t}(T_m^{i+1} - T_m^i)$$

Define **Mesh Fourier number** $\Delta\tau = \dfrac{\alpha\Delta t}{\Delta x^2}$

$$T_m^{i+1} = \Delta\tau T_{m-1}^{i} + \Delta\tau T_{m+1}^{i} + (1\text{-}2\Delta\tau)T_m^i + \Delta\tau\frac{\dot{g}_m^i \Delta x^2}{k}$$

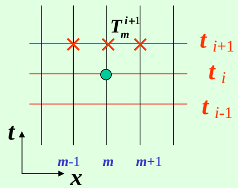


▪ Implicit method

$$\frac{T_{m-1}^{i+1} - 2T_m^{i+1} + T_{m+1}^{i+1}}{\Delta x^2} + \frac{\dot{g}_m^{i+1}}{k} = \frac{1}{\alpha}\frac{T_{m+1}^{i+1} - 2T_m^i}{\Delta t}$$

$$(1+2\Delta\tau)T_m^{i+1} = \Delta\tau T_{m-1}^{i+1} + \Delta\tau T_{m+1}^{i+1} + T_m^i + \Delta\tau\frac{\dot{g}_m^{i+1} \Delta x^2}{k}$$

All nodal temperatures have to be solved simultaneously for each time step.



2) Energy balance method

For each Control Volume

$$\sum_{\text{all sides}} \dot{Q} + \dot{g}V_{\text{element}} = \frac{\Delta E_{\text{element}}}{\Delta t} \begin{cases} = 0 & \textbf{Steady state} \\ = \rho V_{\text{element}} C_p \dfrac{\Delta T}{\Delta t} & \textbf{Transient state} \end{cases}$$

- **Transient state**

If expressed at i: Explicit method

$$\sum_{\text{all sides}} \dot{Q} + \dot{g}V_{\text{element}} = \rho V_{\text{element}} C_p \frac{T_m^{i+1} - T_m^i}{\Delta t}$$

If expressed at $i+1$: implicit method

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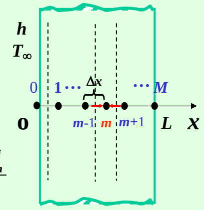
e.g. Transient heat conduction in a plane wall

Interior nodes m :

$$\sum_{\text{all sides}} \dot{Q} + \dot{g}V_{\text{element}} = \rho V_{\text{element}} C_p \frac{T_m^{i+1} - T_m^i}{\Delta t}$$



- **Explicit method**

$$-kA\frac{T_m^i - T_{m-1}^i}{\Delta x} + kA\frac{T_{m+1}^i - T_m^i}{\Delta x} + \dot{g}_0^i A\Delta x = \rho A\Delta x C_p \frac{T_m^{i+1} - T_m^i}{\Delta t}$$

$$T_{m-1}^i - 2T_m^i + T_{m+1}^i + \frac{\dot{g}_m \Delta x^2}{k} = \frac{\Delta x^2}{\alpha \Delta t}(T_m^{i+1} - T_m^i)$$

$$T_m^{i+1} = \Delta\tau T_{m-1}^i + \Delta\tau T_{m+1}^i + (1-2\Delta\tau)T_m^i + \Delta\tau\frac{\dot{g}_m^i \Delta x^2}{k}$$

Mesh Fourier number: $\Delta\tau = \dfrac{\alpha\Delta t}{\Delta x^2}$

- **Implicit method**

$$-kA\frac{T_m^{i+1} - T_{m-1}^{i+1}}{\Delta x} + kA\frac{T_{m+1}^{i+1} - T_m^{i+1}}{\Delta x} + \dot{g}_0^{i+1} A\Delta x = \rho A\Delta x C_p \frac{T_m^{i+1} - T_m^i}{\Delta t}$$

$$(1+2\Delta\tau)T_m^{i+1} = \Delta\tau T_{m-1}^{i+1} + \Delta\tau T_{m+1}^{i+1} + T_m^i + \Delta\tau\frac{\dot{g}_m^{i+1}\Delta x^2}{k}$$

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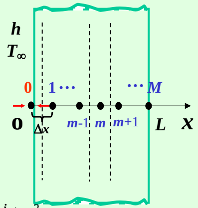
e.g. Transient heat conduction in a plane wall

Left boundary node 0 with Convection B.C.

$$\sum_{\text{all sides}} \dot{Q} + \dot{g}V_{\text{element}} = \rho V_{\text{element}} C_p \frac{T_m^{i+1} - T_m^i}{\Delta t}$$



- **Explicit method**

$$hA(T_\infty - T_0^i) + kA\frac{T_1^i - T_0^i}{\Delta x} + \dot{g}_0^i A\frac{\Delta x}{2} = \rho A\frac{\Delta x}{2}C_p\frac{T_0^{i+1} - T_0^i}{\Delta t}$$

$$T_0^{i+1} = (1 - 2\Delta\tau - 2\Delta\tau\frac{h\Delta x}{k})T_0^i + 2\Delta\tau T_1^i + 2\Delta\tau\frac{h\Delta x}{k}T_\infty + \Delta\tau\frac{\dot{g}_0^i\Delta x^2}{k}$$

Define **Mesh Biot number** $\Delta\text{Bi} = \dfrac{h\Delta x}{k}$

$$T_0^{i+1} = (1 - 2\Delta\tau - 2\Delta\tau\cdot\Delta\text{Bi})T_0^i + 2\Delta\tau T_1^i + 2\Delta\tau\cdot\Delta\text{Bi}T_\infty + \Delta\tau\frac{\dot{g}_0^i\Delta x^2}{k}$$

- **Implicit method**

$$hA(T_\infty - T_0^{i+1}) + kA\frac{T_1^{i+1} - T_0^{i+1}}{\Delta x} + \dot{g}_0^{i+1} A\frac{\Delta x}{2} = \rho A\frac{\Delta x}{2}C_p\frac{T_0^{i+1} - T_0^i}{\Delta t}$$

❑ Stability Criterion

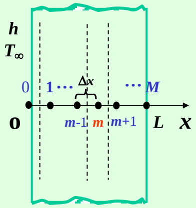
Limitation on time step Δt

The stability criterion is satisfied if the coefficients of all in the temperature expressions (called the primary coefficients) are greater than or equal to zero for all nodes.

All the terms in an expression involving for a particular node must be grouped together before this criterion is applied.

e.g. Transient heat conduction in a plane wall

- **Explicit method**



Interioir nodes:

$$T_m^{i+1} = \Delta\tau T_{m-1}^i + \Delta\tau T_{m+1}^i + (1-2\Delta\tau)T_m^i + \Delta\tau \frac{\dot{g}_m^i \Delta x^2}{k}$$

$$1 - 2\Delta\tau \geq 0 \Longrightarrow \Delta t \leq \frac{1}{2}\frac{\Delta x^2}{\alpha}$$

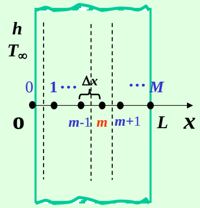
Boundary nodes 0 (Conv B.C.):

$$T_0^{i+1} = (1 - 2\Delta\tau - 2\Delta\tau \cdot \Delta\mathrm{Bi})T_0^i + 2\Delta\tau T_1^i + 2\Delta\tau \cdot \Delta\mathrm{Bi} T_\infty + \Delta\tau \frac{\dot{g}_0^i \Delta x^2}{k}$$

$$1 - 2\Delta\tau - 2\Delta\tau \cdot \Delta\mathrm{Bi} \geq 0 \Longrightarrow \Delta t \leq \frac{\Delta x^2}{2\alpha\left(1 + \frac{h\Delta x}{k}\right)} = \frac{1}{2\alpha\left(\frac{1}{\Delta x^2} + \frac{h}{k\Delta x}\right)}$$

So, $\Delta t \leq \dfrac{1}{2\alpha\left(\frac{1}{\Delta x^2} + \frac{h}{k\Delta x}\right)}$ $\quad \Delta t_{max} = \dfrac{1}{2\alpha\left(\frac{1}{\Delta x^2} + \frac{h}{k\Delta x}\right)}$ $\quad \Delta x \downarrow,\ \Delta t_{max} \downarrow$

- **Implicit method**



Interioir nodes:

$$(1+2\Delta\tau)T_m^{i+1} = \Delta\tau T_{m-1}^{i+1} + \Delta\tau T_{m+1}^{i+1} + T_m^i + \Delta\tau \frac{\dot{g}_m^{i+1} \Delta x^2}{k}$$

- **Unconditionally stable (any time step is usable)**
- **But, the smaller the time step, the better the accuracy of the solution.**

❖ Compare two formats

Formats	Advantages	Disadvantages
Explicit	Easy to implement	Limitation on Δt
Implicit	Unconditionally stable (large time step)	Simultaneous solution

§ 5-6 Controlling the Numerical Error

1. Discretization / Truncation error

2. Round-off error

1. Discretization / Truncation error

Be due to replacing the derivatives by differences in the numerical equation for each steps.

e.g. $\left(\frac{\partial^2 T}{\partial x^2}\right)_m^i = \left(\frac{1}{\alpha}\frac{\partial T}{\partial t}\right)_m^i$

- $\left(\frac{\partial T}{\partial t}\right)_m^i \approx \frac{T_m^{i+1} - 2T_m^i}{\Delta t}$

error $\propto (\Delta t)^2$
$\Delta t \downarrow$, *error* $\downarrow$

$$T(x_m, t_i + \Delta t) = T(x_m, t_i) + \frac{\partial T(x_m, t_i)}{\partial x}\Delta t + \frac{\partial^2 T(x_m, t_i)}{\partial x^2}\frac{\Delta t^2}{2!} + \frac{\partial^3 T(x_m, t_i)}{\partial x^3}\frac{\Delta t^3}{3!} + \cdots$$

$$\left.\frac{\partial T(x_m, t_i)}{\partial x}\right|_m \approx \frac{T(x_m, t_i + \Delta t) - T(x_m, t_i)}{\Delta x} = \frac{T_m^{i+1} - T_m^i}{\Delta x}$$

- $\left(\frac{\partial^2 T}{\partial x^2}\right)_m^i \approx \frac{T_{m-1}^i - 2T_m^i + T_{m+1}^i}{\Delta x^2}$

error $\propto (\Delta x)^4$
$\Delta x \downarrow$, *error* $\downarrow$

2. Round-off error

$\propto$ **(The number of computations)**

Be caused by the (computer's) use of a limited number of significant digits and continuously rounding (or chopping) off the digits it cannot retain.

Single precision: **keep 7 significant digits**

√ **Double precision:** **keep 15 significant digits**

44444.666666
- **Be rounded to** **44444.67**
- **Be Chopped off** **44444.66**

$\downarrow$ *error* , $\downarrow$**The number of computations** $\Rightarrow$ $\uparrow \Delta t$ **or/and** $\uparrow \Delta x$

❖ **Total error** **Mesh size Δx ?** **Time step Δt ?**

❖ Review of Heat Conduction (Chapter 2 - 5)

1. Fourier's law of heat conduction (+ energy balance)

Infinitely small element | Finite control volume

2. • Differential equation **• Difference equation**

3. Problems	Analytical solution (1-D)		Numerical solution (multidimensional)
	Plane wall cylinder sphere		
Steady	Specified temp. B.C. (§ 2-5) Convection B.C. (§ 3-1, § 3-4) With heat generation (§ 2-6)	Mulitlayer (§ 3-3) : Thermal resistance network	0
	Fin (§ 3-6)		$\sum_{\text{all sides}} \dot{Q} + \dot{g}V_{\text{element}} =$
Transient	$Bi_V \geq 0.1$: τ>0.2, Heisler chart (§ 4-2) $Bi_V \leq 0.1$: Lumped system (§ 4-1)		
	Semi-infinite solid: (§ 4-3)		$\rho V_{\text{element}} C_p \frac{T_m^{i+1} - T_m^i}{\Delta t}$
	Mulitdimensional system (§ 4-4) : Production solution		