§ 5-4 Solution Methods for Systems of Algebraic Equation (Difference Equations)

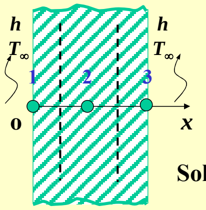
1. *Direct methods:* **(e.g. Gaussian Eliminiation)**
Solve in a systematic manner
Following a series of well-defined steps

- **Be more suitable for systems with relatively small numbers of equations.**

2. *Iterative methods:* **(e.g. Gauss-Seidel Iteration Method)**
Start with an initial guess for the solution, and iterate until solution converge

- **Be preferred for large systems.**

Example 5-1 Solution for difference equations of a plane wall's heat conduction



Difference equations of heat cond. are

$$T_1 - 3T_2 + T_3 = 10$$
$$-T_1 + T_2 - 2T_3 = -13$$
$$2T_1 + 5T_2 + T_3 = 4$$

Solution 1: Direct Method (Gaussian Eliminiation)

$T_1 = 2$, $T_2 = -1$, and $T_3 = 5$

Solution 2: Iterative methods (Gauss-Seidel Iteration Method)

$$T_1 = 10 + 3T_2 - T_3$$
$$T_2 = (4 - 2T_1 - T_3)/5$$
$$T_3 = (-13 + T_1 - T_2)/(-2)$$

Iteration 0 : **Guess** $T_1 = 0$, $T_2 = 0$ and $T_3 = 0$ as initial values.

Iteration 1: **absolute error**

$T_1 = 10 + 3T_2 - T_3 = 10 + 3 \times 0 - 0 = 10$ **! !** **10**
$T_2 = (4 - 2T_1 - T_3)/5 = [4 - 2 \times 10 - 0]/5 = -3.2$
$T_3 = (-13 + T_1 - T_2)/(-2) = [-13 + 10 - (-3.2)]/(-2) = -0.1$

Iteration 2:

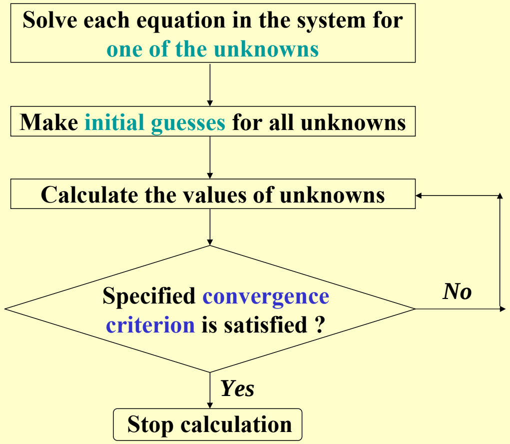
$T_1 = 10 + 3T_2 - T_3 = 10 + 3 \times (-3.2) - (-0.1) = 0.5$ **! !** **9.5**
$T_2 = (4 - 2T_1 - T_3)/5 = [4 - 2 \times 0.5 + 0.1]/5 = 0.62$
$T_3 = (-13 + T_1 - T_2)/(-2) = [-13 + 0.5 - 0.62]]/(-2) = 6.56$

⋮

		absolute error	relative error
Iteration 70:	$T_1 = 1.99$, $T_2 = -0.99$, $T_3 = 5.01$		
Iteration 80:	$T_1 = 1.99$, $T_2 = -1.00$, $T_3 = 5.01$	0.01	1 %
Iteration 90:	$T_1 = 2.00$, $T_2 = -1.00$, $T_3 = 5.00$	0.01	0.2 %

Converge: absolute error ≤ 0.01 or relative error ≤ 0.5%

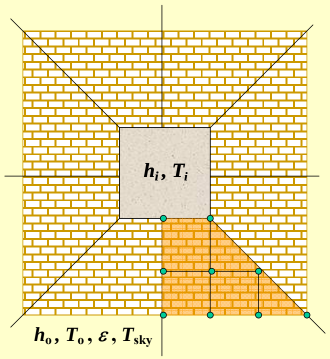
Iterative methods (Gauss-Seidel Iteration Method):




```
PROGRAM GAUSS
DATA T1,T2,T3/.0,.0,.0/
DO 20  K=K+1
  T1_former=T1
  T2_former=T2
  T3_former=T3
  T1 = 10 + 3*T2 - T3
  T2 = (4 - 2*T1 - T3)/5.
  T3 = (-13 + T1 - T2)/(-2.)
  err_1= abs((T1-T1_former)/T1_former)
  err_2= abs((T2-T2_former)/T2_former)
  err_3= abs((T3-T3_former)/T3_former)
  WRITE (*,10)K,T1,T2,T3
  if(err_1.le.0.005).and.(err_2.le.0.005).and.err_2.le.0.005))
  then goto 30
10 FORMAT (' ', 13, 3(F8.1))
20 CONTINUE
30 END
```

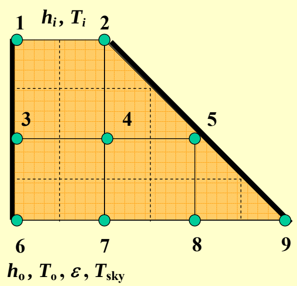
$T_1 = 10+3T_2-T_3$
$T_2 = (4-2T_1- T_3)/5$
$T_3 = (-13+T_1-T_2)/(-2)$

Example 5-2 Heat loss through Chimneys



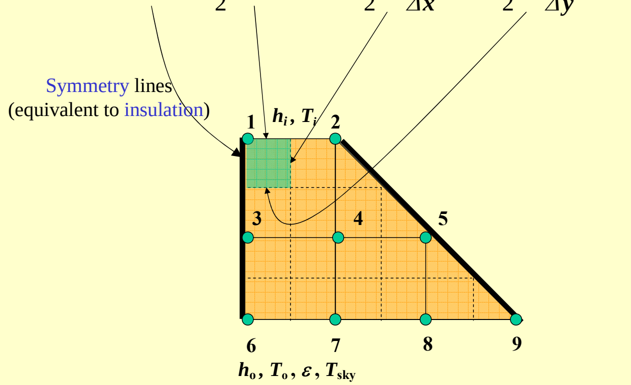
Square mesh: Δx, Δy

Thermal symmetry—mirror



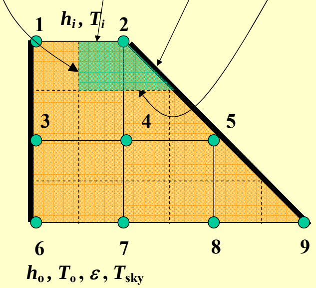
Solution: Energy Balance: $\sum_{\text{all side}} \dot{Q} + \dot{g}V_{\text{element}} = 0$

Node 1: $0 + h_i \dfrac{\Delta x}{2}(T_i - T_1) - k\dfrac{\Delta y}{2}\dfrac{T_1 - T_2}{\Delta x} - k\dfrac{\Delta x}{2}\dfrac{T_1 - T_3}{\Delta y} = 0$



Solution: **Energy Balance:** $\sum_{\text{all side}} \dot{Q} + \dot{g}V_{\text{element}} = 0$

Node 2: $-k\frac{\Delta y}{2}\frac{T_2-T_1}{\Delta x}+h_i\frac{\Delta x}{2}(T_i-T_2)+0-k\Delta x\frac{T_2-T_4}{\Delta y}=0$

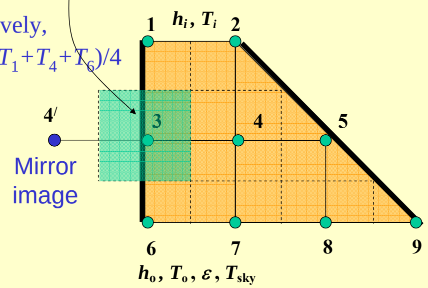


Solution: **Energy Balance:** $\sum_{\text{all side}} \dot{Q} + \dot{g}V_{\text{element}} = 0$

Node 3: $0+k\frac{\Delta x}{2}\frac{T_1-T_3}{\Delta y}+k\Delta y\frac{T_4-T_3}{\Delta x}-k\frac{\Delta x}{2}\frac{T_3-T_6}{\Delta y}=0$

$\Delta x = \Delta y$ $\quad\therefore\ T_3=(2T_4+T_1+T_6)/4$

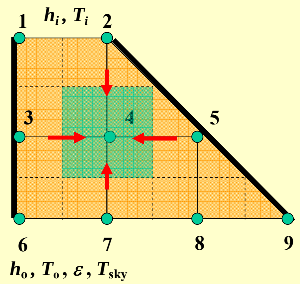
Alternatively,
$T_3=(T_4+T_1+T_4+T_6)/4$



Solution: **Energy Balance:** $\sum_{\text{all side}} \dot{Q} + \dot{g}V_{\text{element}} = 0$

Node 3: $T_3=(T_4+T_1+T_4+T_6)/4$

Node 4: $T_4=(T_3+T_2+T_5+T_7)/4$

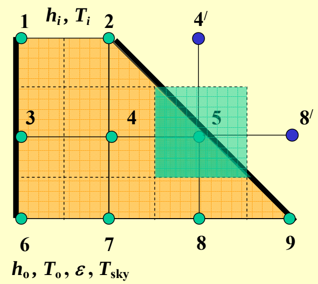


Solution: **Energy Balance:** $\sum_{\text{all side}} \dot{Q} + \dot{g}V_{\text{element}} = 0$

Node 3: $T_3=(T_4+T_1+T_4+T_6)/4$

Node 4: $T_4=(T_3+T_2+T_5+T_7)/4$

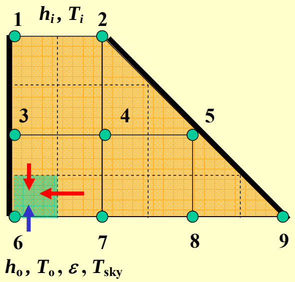
Node 5: $T_5=(T_4+T_8+T_4+T_8)/4$



Solution: **Energy Balance:** $\sum_{\text{all side}} \dot{Q} + \dot{g}V_{\text{element}} = 0$

Node 6:

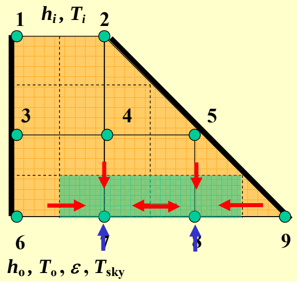
$$0 + k\frac{\Delta x}{2}\frac{T_3 - T_6}{\Delta y} + k\frac{\Delta y}{2}\frac{T_7 - T_6}{\Delta x} + h_o\frac{\Delta x}{2}(T_o - T_6) + \varepsilon\sigma\frac{\Delta x}{2}(T_{sky}^4 - T_6^4) = 0$$



Solution: **Energy Balance:** $\sum_{\text{all side}} \dot{Q} + \dot{g}V_{\text{element}} = 0$

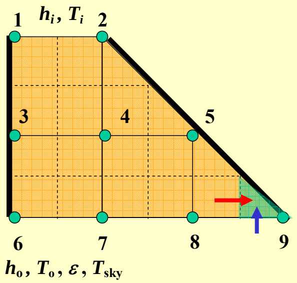
Node 7: $$k\frac{\Delta y}{2}\frac{T_6 - T_7}{\Delta x} + k\Delta x\frac{T_4 - T_7}{\Delta y} + k\frac{\Delta y}{2}\frac{T_8 - T_7}{\Delta x} + h_o\Delta x(T_o - T_7) + \varepsilon\sigma\Delta x(T_{sky}^4 - T_7^4) = 0$$

Node 8: $$k\frac{\Delta y}{2}\frac{T_7 - T_8}{\Delta x} + k\Delta x\frac{T_5 - T_8}{\Delta y} + k\frac{\Delta y}{2}\frac{T_9 - T_8}{\Delta x} + h_o\Delta x(T_o - T_8) + \varepsilon\sigma\Delta x(T_{sky}^4 - T_8^4) = 0$$



Solution: **Energy Balance:** $\sum_{\text{all side}} \dot{Q} + \dot{g}V_{\text{element}} = 0$

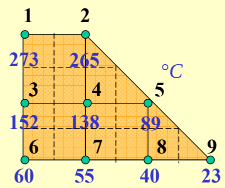
Node 9: $$k\frac{\Delta y}{2}\frac{T_8 - T_9}{\Delta x} + 0 + h_o\Delta x(T_o - T_9) + \varepsilon\sigma\frac{\Delta x}{2}(T_{sky}^4 - T_9^4) = 0$$



The system of difference equations in nodal temperature:

$T_1=(T_2+T_3+2865)/ 7$
$T_2=(T_1+T_4+2865)/ 8$
$T_3=(T_1+2T_4+T_6)/ 4$
$T_4=(T_2+T_3+T_5+T_7)/ 4$
$T_5=(2T_4+2T_8)/4$
$T_6=(T_2+T_3+456.2-0.3645\times 10^{-8}T_6^4)/ 3.5$
$T_7=(2T_4+T_6+T_8+912.4-0.729\times 10^{-8}T_7^4)/ 7$
$T_8=(2T_5+T_7+T_9+912.4-0.729\times 10^{-8}T_8^4)/ 7$
$T_9=(T_8+456.2-0.3645\times 10^{-8} T_9^4)/ 2.5$

Apply Gauss-seidel method



Note: The Gaussian elimination method is not applicable since it is limited to linear equation.

- The average temperature at the outer/inner surface of the chimney are

$$T_{\text{wall,out}} = \frac{(0.5T_6 + T_7 + T_8 + 0.5T_9)l}{(0.5+1+1+0.5)l} = 318.6\ K = 45.6\,°C$$

$$T_{\text{wall,in}} = \frac{(0.5T_1 + 0.5T_2)l}{(0.5+0.5)l} = 537.4\ K = 264.4\,°C$$

- Heat loss through 1-m-long section of the chimney

$$\dot{Q}_{\text{chimney}} = h_o A_o (T_{\text{wall,out}} - T_o) + \varepsilon\sigma A_o (T_{\text{wall,out}}^4 - T_{\text{sky}}^4) = 1291 + 702 = 1993\ \text{W}$$

$$\dot{Q}_{\text{chimney}} = h_i A_i (T_i - T_{\text{wall,in}}) = 1994\ \text{W}$$