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Homotopy Continuation Method of Arbitrary Order of Convergence for Solving the Hyperbolic Form of Kepler's Equation

M. A. Sharaf[1,*], M. A. Banajh[2] & A. A. Alshaary[2]
[1]*Department of Astronomy, Faculty of Science, King Abdul Aziz University, Jeddah, Saudi Arabia.*
[2]*Department of Mathematics, Girls College of Education, Jeddah, Saudi Arabia.*
[*]*e-mail: sharaf_adel@hotmail.com*

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Abstract. In this paper, an efficient iterative method of arbitrary integer order of convergence ≥ 2 has been established for solving the hyperbolic form of Kepler's equation. The method is of a dynamic nature in the sense that, moving from one iterative scheme to the subsequent one, only additional instruction is needed. Most importantly, the method does not need any prior knowledge of the initial guess. A property which avoids the critical situations between divergent and very slow convergent solutions that may exist in other numerical methods which depend on initial guess. Computational Package for digital implementation of the method is given and is applied to many case studies.

Key words. Homotopy method—hyperbolic Kepler's equation—initial value problem—orbit determination.

1. Introduction

Many instances of hyperbolic orbits occur in the solar system and recently, among the artificial satellites, lunar and solar probes. Moreover, in some cases of orbit determination for an elliptic orbit, it may very well happen (Escobal 1975) that during the solution process (usually iteration), the eccentricity e becomes greater than unity and the orbit becomes hyperbolic. Also, in the interplanetary transfer, the escape from the departure planet and the capture by the target planet involve hyperbolic orbits (Gurzadyan 1996). On the other hand, in orbit determination of visual binaries provisional hyperbolic orbits are used to represent the periastron section of high-eccentricity orbits of long and indeterminate period (Knudsen 1953). In fact, we should handle hyperbolic orbits frequently when integrating a perturbed motion with the initial condition of nearly parabolic orbits (Fukushima 1997).

From the above, it is then clear that the hyperbolic orbits not only exist naturally, but can also be used to solve some critical orbital situations.

The position–time relation in hyperbolic orbits is known as Kepler's equation for the hyperbolic case and is given as

$$M = e \sinh G - G; \quad 1 \leq e < \infty; \quad 0 \leq M < \infty, \tag{1.1}$$

where G is the eccentric anomaly for a hyperbolic orbit, and M is the mean anomaly (Danby 1988).

Equation (1.1) is transcendental and is usually solved by iterative methods, which in turn need: (a) initial guess and (b) an iterative scheme. In fact, these two points are not separated from each other but there is a full agreement that even accurate iterative schemes are extremely sensitive to initial guess. Moreover, in many cases the initial guess may lead to a drastic situation between divergent and very slow convergent solutions.

In the field of numerical analysis, very powerful techniques have been devoted (Allgower & George 1993) to solve transcendental equations without any prior knowledge of the initial guess. These techniques are known as homotopy continuation methods.

In the present paper, an efficient iterative method of arbitrary positive integer order of convergence ≥ 2 has been established for solving Kepler's equation (1.1) for hyperbolic orbits using homotopy continuation technique. The method does not need any prior knowledge of the initial guess, a property which avoids the critical situations between divergent and very slow convergent solutions, that may exist in other numerical methods depending on initial guess.

Computational package for digital implementation of the method is given and applied for many cases.

2. Basic formulations

2.1 *Homotopy continuation method for solving $Y(x) = 0$*

Suppose one wishes to obtain a solution of a single non-linear equation in one variable x (say)

$$Y(x) = 0, \tag{2.1}$$

where $Y: \mathbf{R} \rightarrow \mathbf{R}$ is a mapping which, for our application is assumed to be smooth, i.e., a map has as many continuous derivatives as it requires. Let us consider the situation in which no prior knowledge concerning the zero point of Y is available. Since we assume that such a prior knowledge is not available, then any of the iterative methods will often fail to calculate the zero $\bar{x}$, because poor starting value is likely to be chosen. As a possible remedy, one defines a homotopy or deformation $H: \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$ such that

$$H(x, 1) = Q(x); \qquad H(x, 0) = Y(x),$$

where $Q: \mathbf{R} \rightarrow \mathbf{R}$ is a (trivial) smooth map having known zero point and H is also smooth. Typically, one may choose a convex

$$H(x, \lambda) = \lambda Q(x) + (1 - \lambda) Y(x) \tag{2.2}$$

and attempt to trace an implicitly defined curve $\Phi(z) \in H^{-1}(0)$ from a starting point $(x_1, 1)$ to a solution point $(\bar{x}, 0)$. If this succeeds, then a zero point $\bar{x}$ of Y is obtained.

The curve $\Phi(z) \in H^{-1}(0)$ can be traced numerically if it is parameterized with respect to the parameter λ, then the classical embedding methods can be applied (Allgower & George 1993).

2.2 *One-point iteration formulae*

Let $Y(x) = 0$ such that $Y : \mathbf{R} \to \mathbf{R}$ smooth map and has a solution $x = \xi$ (say). To construct iterative schemes for solving this equation, some basic definitions are to be recalled as follows:

1. The error in the kth iterate is defined as
$$\varepsilon_k = \xi - x_k.$$
2. If the sequence $\{x_k\}$ converges to $x = \xi$, then
$$\lim_{k\to\infty} x_k = \xi. \tag{2.3}$$
3. If there exists a real number $p \geq 1$ such that
$$\lim_{i\to\infty} \frac{|x_{i+1} - \xi|}{|x_i - \xi|^p} = \lim_{i\to\infty} \frac{|\varepsilon_{i+1}|}{|\varepsilon_i|^p} = K \neq 0, \tag{2.4}$$
we say that the iterative scheme is of order p at ξ. The constant K is called the asymptotic error constant. For $p = 1$, the convergence is linear; for $p = 2$, the convergence is quadratic; for $p = 3, 4, 5$, the convergence is cubic, quartic and quintic, respectively.
4. One-point iteration formulae are those which use information at only one point. Here, we shall consider only stationary one-point iteration formulae which have the form
$$x_{i+1} = R(x_i); \quad i = 0, 1, \ldots \tag{2.5}$$
5. The order of one-point iteration formulae could be determined either from:
   (a) The Taylor series of the iteration function $R(x_n)$ about ξ (e.g., Ralston & Rabinowitz 1978) or from
   (b) The Taylor series of the function $Y(x_{k+1})$ about x_k (Danby & Burkard 1983).

   On the basis of the second approach mentioned above [point (b)] it is easy to form a class of iterative formulae containing members of all integral orders (Sharaf & Sharaf 1998) to solve equation (2.1) as
$$x_{i+1} = x_i + \delta_{i,m+2}; \quad i = 0, 1, 2, \ldots; \quad m = 0, 1, 2, \ldots \tag{2.6}$$
   where
$$\delta_{i,m+2} = \frac{-Y_i}{\sum_{j=1}^{m+1} (\delta_{i,m+1})^{j-1}\, Y_i^{(j)}/j!}; \quad \delta_{i,1} = 1; \quad \forall i \geq 0, \tag{2.7}$$
$$Y_i^{(j)} \equiv \left.\frac{d^j Y(x)}{dx^j}\right|_{x=x_i}; \quad Y_i \equiv Y_i^{(0)}. \tag{2.8}$$

The convergence order is $m + 2$ as shown from the error formula
$$\varepsilon_{i+1} = -\frac{1}{(m+2)!} \frac{Y^{(m+2)}(\zeta)}{Y_i^{(1)}(\zeta_1)} \varepsilon_i^{m+2} \tag{2.9}$$
where ζ is between x_{i+1} and x_i and ζ_1 is between x_{i+1} and ξ.

3. Computational developments

3.1 *Embedding methods*

The basic idea of the embedding methods referred to at the end of subsection 2.1 is explained in the following algorithm for tracing the curve $\Phi(z) \in H^{-1}(0)$ from, say $\lambda = 1$ to $\lambda = 0$.

3.1.1 *Computational algorithm 1*

Purpose: To solve $Y(x) = 0$ by embedding method.
Input:

(1) The function $Q(x)$ with defined root x_1 such that $H(x_1, 1) = 0$,
(2) positive integer m.

Output: Solution x of $Y(x) = 0$.

Computational sequence:

(1) Set $x = x_1$, $\lambda = (m-1)/m$, $\Delta\lambda = 1/m$.
(2) For $i := 1$ to m, do
  begin
    Solve $H(y, \lambda) = \lambda Q(y) + (1-\lambda)Y(y) = 0$ iteratively for y using x as starting value.
    $x = y$
    $\lambda = \lambda - \Delta\lambda$
  end.

3.2 *Solution of the hyperbolic form of Kepler's equation*

Recalling equation (1.1) as

$$Y(G) = e \sinh G - G - M = 0. \tag{3.1}$$

Two notes are to be recorded as follows:

(1) From equation (2.9) it is clear that an iterative scheme for solving equation (3.1) includes derivatives of Y as much as the order of the scheme. On the other hand, the higher the order of an iterative scheme, the higher its accuracy and rate of convergence will be. Regarding this last fact, the remarkable simplicity of the derivative formulae of Y which are

$$Y^{(1)} = e \cosh G - 1, \tag{3.2.1}$$

$$Y^{(2k)} = e \sinh G; \quad k \geq 1, \tag{3.2.2}$$

$$Y^{(2k+1)} = e \cosh G; \quad k \geq 1, \tag{3.2.3}$$

enables us to find derivatives of $Y(G)$ as many as we need.

(2) Homotopy continuation method (see subsection 2.1) is very powerful technique for solving $Y(G) = 0$ without prior knowledge of the initial guess.
From these two notes, we can now establish for the solution of equation (3.1), an iterative algorithm of any positive integer order $l \geq 2$. Moreover, the algorithm

does not need prior knowledge of the initial guess. According to equation (2.9), the algorithm is of a dynamic nature in the sense that it includes iterative schemes up to the lth order such that, going from one scheme to the subsequent one, only additional instruction is needed.

This algorithm is illustrated in what follows with algorithm 1 augmented to it, together with the Q function of the homotopy H (equation 2.2) as $Q(x) = x - 1$, so that $H(x_1, 1) = 0$, where $x_1 = 1$.

3.2.1 *Computational algorithm 2*

Purpose: To solve the hyperbolic form of Kepler's equation by iterative schemes of quadratic up to lth convergence orders without prior knowledge of the initial guess using homotopy continuation method with $Q(G) = G - 1$.
Input: M, e, l, m (positive integer), *Tol* (specified tolerance)
Computational Sequence:

1. Set $G = 1$, $\Delta\lambda = 1/m$, $\lambda = 1 - \Delta\lambda$
2. For $i := 1$ to m do
   begin $\{i\}$
   $\Phi = 1 - \lambda$
   $Y = \lambda(G - 1) + \Phi(e \sinh G - G - M)$
   $Y^{(1)} = \lambda + \Phi(e \cosh G - 1)$
   $\Delta G = -Y/Y^{(1)}$
   If $l = 2$ go to step 3
   $Y^{(2)} = \Phi e \sinh G$
   $H = Y^{(1)} + \Delta G * Y^{(2)}/2$
   $\Delta G = -Y/H$
   If $l = 3$ go to step 3
   $Y^{(3)} = \Phi e \cosh G$
   $H = Y^{(1)} + \Delta G * Y^{(2)}/2 + (\Delta G)^2 * Y^{(3)}/6$
   $\Delta G = -Y/H$
   If $l = 4$ go to step 3
   $L = l - 1$
   For k : 4 to L do
   begin $\{k\}$
   Set $Y^{(k)} = Y^{(k-2)}$; $n = k - 1$; $H = Y^{(1)}$; $B = 1$
   For $j := 1$ to n do
   begin $\{j\}$
   $B = \Delta G * B/(j + 1)$
   $H = H + B * Y^{(j+1)}$
   end $\{i\}$
   $\Delta G = -Y/H$
   end $\{k\}$
3. If $|\Delta G| \leq Tol$ go to step 4
   $G = G + \Delta G$
   $\lambda = \lambda - \Delta\lambda$
   end $\{j\}$
4. end.

3.3 *Numerical applications*

3.3.1 *Hyperbolic limitations on mean anomaly*

Because a trajectory in hyperbolic orbit approaches the asymptotes at infinite distance, the true anomaly ν is limited within a certain range. These values of ν are

$$-180^\circ + \cos^{-1}\left(\frac{1}{e}\right) < \nu < 180^\circ - \cos^{-1}\left(\frac{1}{e}\right),$$

from this inequality and the known equation

$$\tanh\left(\frac{G}{2}\right) = \sqrt{\frac{e-1}{e+1}}\tan\left(\frac{\nu}{2}\right),$$

we get using equation (1.1) Table 1, which shows some representative values of the limitations on the mean anomaly of a hyperbolic orbit whose eccentricity varies.

3.3.2 *Numerical results*

For a given values of M and e (as obtained in Table 1) there exists an optimum pair (m^*, l^*) that can determine the solution of equation (1.1) more accurately than that determined from other pairs. Consequently we may define an acceptable solution set to hyperbolic form of Kepler's equation for a given M and e as

$$G = \{G : Tol \leq \varepsilon, m = m^*; l = l^*\},$$

where Tol is a given tolerance.

The above computational algorithm 2 is applied for

$$e = 1.5, 2.0, 3.0, 4.0, 5.0, 9.0, 10.5, 13.5, 16.0, 19.0, 21.0, 25.5$$

and some values of M with $Tol = 10^{-8}$. The elements of the acceptable solution set of the present method are listed in Table 2 for the given M and e.

Table 1. Hyperbolic limitations on mean anomaly.

Eccentricity	Minimum M	Maximum M
1.5	− 11171.2	11171.2
2.0	− 17311.2	17311.2
3.0	− 28274.9	28274.9
4.0	− 38720.5	38720.5
5.0	− 48980.4	48980.4
9.0	− 89433.3	89433.3
10.5	−104513	104513
13.5	−134620	134620
16	−159678	159678
19	−189727	189727
21	−209752	209752
25.5	−254794	254794

Table 2. Elements of acceptable solution set of the hyperbolic form of Kepler's equation with $Tol = 10^{-8}$.

M	m^*	l^*	G
Case 1: $e = 1.5$			
−11151.0	20	3	−9.60783
11171.0	5	3	9.60962
Case 2: $e = 2.0$			
6311.0	6	3	8.75144
−17000.0	12	3	−9.74154
Case 3: $e = 3.0$			
2827.0	8	3	7.54417
−3500.0	18	3	−7.75727
Case 4: $e = 4.0$			
3700.2	7	3	7.52503
−370.2	6	4	−5.23497
Case 5: $e = 5.0$			
48970.4	7	3	9.88288
−3200.0	13	3	−7.15685
Case 6: $e = 9.0$			
89333.3	7	3	9.89616
−103.8	8	4	−3.17024
Case 7: $e = 10.5$			
145.31	19	3	3.34464
−104511.0	14	3	−9.89891
Case 8: $e = 13.5$			
1345.21	9	3	5.29872
−124520.0	17	3	−9.82276
Case 9: $e = 16.0$			
11154.2	6	3	7.24078
−154.2	9	4	−2.98053
Case 10: $e = 19.0$			
1997.5	7	3	5.35106
−180.0	9	4	−2.96066
Case 11: $e = 21.0$			
17500.5	6	3	7.41903
−4582.51	16	3	−6.07996
Case 12: $e = 25.5$			
12.85	14	3	0.502235
−1000.98	12	3	−4.36772

4. Conclusion

In concluding the paper, an efficient and simple algorithm for the solution of the hyperbolic form of Kepler's equation has been established. Its efficiency is due to factors such as:

- It is of a dynamical nature, in the sense that it includes iterative schemes of any positive integer order ≥ 2 such that, moving from one scheme to the subsequent one, only one additional instruction is needed.

- Does not depend on initial guess – a property that avoids it from falling in critical situations between divergent and very slow convergent solutions that may exist in other numerical methods which depend on initial guess.
- It includes two controllable parameters only, m and l by means of which one can certainly determine an optimum solution to any desired accuracy by what we call the acceptable solution set.

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