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ON HOMOGENIZATION OF NON-DIVERGENCE FORM PARTIAL DIFFERENCE EQUATIONS

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Abstract

In this paper a method for proving homogenization of divergence form elliptic equations is extended to the non-divergence case. A new proof of homogenization is given when the coefficients in the equation are assumed to be stationary and ergodic. A rate of convergence theorem in homogenization is also obtained, under the assumption that the coefficients are i.i.d. and the elliptic equation can be solved by a convergent perturbation series,

1 Introduction.

In [3] we introduced a new method for proving the homogenization of elliptic equations in divergence form. The purpose of this paper is to extend the method to non-divergence form equations. As in [3] we shall be able to obtain a rate of convergence in situations where the problem is perturbative.

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and for $i = 1, ..., d$, let $a_i : \Omega \to \mathbf{R}$ be bounded measurable functions on Ω satisfying the inequality

$$\lambda_i < a_i(\omega) \le \Lambda_i, \quad \omega \in \Omega, \quad i = 1, ..., d, \tag{1.1}$$

where the $\lambda_i,\ \Lambda_i$ are positive constants. We assume that $\mathbf{Z}^d$ acts on Ω by translation operators $\tau_x : \Omega \to \Omega,\ x \in \mathbf{Z}^d$, which are measure preserving and satisfy the properties $\tau_x\tau_y = \tau_{x+y},\ \tau_0 =$ identity, $x, y \in \mathbf{Z}^d$. Let $f : \mathbf{R}^d \to \mathbf{R}$ be a smooth function with compact support. We shall be

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interested in solutions $u_\varepsilon(x,\omega)$ of the elliptic equation on the scaled lattice $\mathbf{Z}^d_\varepsilon = \varepsilon\mathbf{Z}^d$ given by

$$\sum_{i=1}^{d} a_i(\tau_{x/\varepsilon}\ \omega)\left[2u_\varepsilon(x,\omega) - u_\varepsilon(x+\varepsilon\mathbf{e}_i,\omega) - u_\varepsilon(x-\varepsilon\mathbf{e}_i,\omega)\right]/\varepsilon^2 \\ + u_\varepsilon(x,\omega) = f(x), \quad x\in\mathbf{Z}^d_\varepsilon, \quad \omega\in\Omega, \quad (1.2)$$

where the $\mathbf{e}_i \in \mathbf{R}^d$ are unit vectors with 1 in the ith entry. It is evident that (1.2) has a unique bounded solution which is bounded by the L^∞ norm of f. Letting $\lfloor\ \rfloor$ denote integer part, it is clear that for $x\in\mathbf{R}^d$, $\lim_{\varepsilon\to 0}\varepsilon\lfloor x/\varepsilon\rfloor = x$. We shall show that as $\varepsilon\to 0$ the solution $u_\varepsilon(\varepsilon\lfloor x/\varepsilon\rfloor,\omega)$ of (1.2) converges with probability 1 in Ω to the solution $u(x)$ of a homogenized equation,

$$\sum_{i=1}^{d} q_i\frac{\partial^2 u}{\partial x_i^2} + u(x) = f(x), \quad x\in\mathbf{R}^d, \tag{1.3}$$

where the coefficients q_i satisfy the inequality

$$\lambda_i \le q_i \le \Lambda_i, \quad i=1,...,d. \tag{1.4}$$

We prove the following:

Theorem 1.1 *Assume the translation operators $\tau_x,\ x\in\mathbf{Z}^d$, are ergodic. Then with probability 1,*

$$\lim_{\varepsilon\to 0}\sup_{x\in\mathbf{R}^d}|u_\varepsilon(\varepsilon\lfloor x/\varepsilon\rfloor,\ \cdot) - u(x)| = 0\ . \tag{1.5}$$

Theorem 1.1 proves that with probability 1 the solution $u_\varepsilon(\varepsilon\lfloor x/\varepsilon\rfloor,\omega)$ of the random equation (1.2) converges uniformly on $\mathbf{R}^d$ to the solution $u(x)$ of the homogenized equation (1.3). One should note that a theorem of Lawler [7] implies that $\lim_{\varepsilon\to 0}u_\varepsilon(0,\ \cdot) = u(0)$ with probability 1. Lawler's theorem [7] (see also [2]) shows that symmetric random walk in random environment converges at large time to Brownian motion. Lawler's proof follows the same lines as the continuum version of the result obtained by Papanicolaou and Varadhan [8]. An independent proof for the continuum version was also carried out by Zhikov and Sirazhudinov [11]. The papers [7, 8, 11] make use of some central theorems of probability theory, in particular the Birkhoff ergodic theorem and the martingale central limit theorem. In the proof given here of Theorem 1.1 we avoid the use of the martingale theorem. We also only apply the Birkhoff theorem to the translation operators $\tau_x,\ x\in\mathbf{Z}^d$.

Another key fact required in the proof of Theorem 1.1 is the Alexsandrov-Bakelman-Pucci (ABP) inequality [2, 5, 6]. If $1-\lambda_i/\Lambda_i$ is sufficiently small, $i=1,...,d$, one can also avoid the use of this inequality. To see how the ABP inequality occurs we consider the representation for the homogenized coefficients q_i of (1.3). Let $<\cdot>$ denote expectation w.r. to $(\Omega,\mathcal{F},\mu)$. Then

$$q_i = \Big\langle a_i(\cdot)\Phi(\cdot)\Big\rangle, \quad i=1,..,d, \tag{1.6}$$

where $\Phi(\omega)$ is the invariant measure for a Markov chain on Ω. The generator of the chain is $-\mathcal{L}$ where the operator $\mathcal{L}$ on functions $v:\Omega\to\mathbf{R}$ is given by

$$\mathcal{L}v(\omega) = \sum_{i=1}^{d} a_i(\omega)\left[2v(\omega) - v(\tau_{\mathbf{e}_i}\ \omega) - v(\tau_{-\mathbf{e}_i}\ \omega)\right]\ . \tag{1.7}$$

The adjoint $\mathcal{L}^*$ of $\mathcal{L}$ is the operator satisfying

$$\Big\langle u(\cdot)\mathcal{L}v(\cdot)\Big\rangle = \Big\langle \mathcal{L}^* u(\cdot)v(\cdot)\Big\rangle, \quad u,v \in L^2(\Omega).$$

The function Φ of (1.6) is the unique solution in $L^1(\Omega)$ to the equation

$$\mathcal{L}^*\Phi(\omega) = 0, \quad \omega \in \Omega, \quad < \Phi(\cdot) >= 1. \tag{1.8}$$

The ABP inequality enables us to prove existence and uniqueness of the solution to (1.8) by constructing the solution Φ in $L^{d/(d-1)}(\Omega)$.
We turn now to the situation where (1.2) can be solved by a converging perturbation expansion. Homogenization using this method was proved in [1]. Here we obtain a rate of convergence result.

Theorem 1.2 *Suppose the $a_i(\tau_x \cdot),\ x \in \mathbf{Z}^d,\ i = 1,...,d$ are independent and $\gamma = \sup_{1\le i\le d}[1-\lambda_i/\Lambda_i]$ is sufficiently small. Then if $g : \mathbf{R}^d \to \mathbf{R}$ is C^∞ of compact support, there exists $\beta > 0$ and a constant C such that*

$$\left\langle \left\{ \int_{\mathbf{Z}^d_\varepsilon} g(x)\left[u_\varepsilon(x,\cdot) - \langle u_\varepsilon(x,\cdot)\rangle\right] dx \right\}^2 \right\rangle \le C\varepsilon^\beta,$$

where $\beta > 0$ is a constant depending only on γ, and C only on γ, g, f. The number β can be taken arbitrarily close to d if $\gamma > 0$ is taken sufficiently small.

Theorem 1.2 is the analogue of Theorem 1.3 of [3]. The proof follows the same lines as the proof in [3]. In Section 3 we outline the argument and refer the reader to [3] for further details.

2 Proof of Theorem 1.1

We first state a discrete version of the ABP inequality. The proof can be found in [2, 6, 7]. Let $a_i : \mathbf{Z}^d \to \mathbf{R},\ i = 1,...,d$, be functions satisfying the inequality,

$$\lambda_i \le a_i(x) \le \Lambda_i, \quad x \in \mathbf{Z}^d, \quad i = 1,...,d. \tag{2.1}$$

Suppose $D \subset \mathbf{Z}^d$ is a finite set of lattice points. An interior point of D is a point, all of whose nearest neighbors are also in D. Let $\mathrm{Int}(D)$ be the set of interior points of D and the boundary of D be $\partial D = D\backslash Int(D)$. Consider now the Dirichlet problem on D,

$$\sum_{i=1}^d a_i(x)\Big[2u(x) - u(x+\mathbf{e}_i) - u(x-\mathbf{e}_i)\Big] = f(x),\ x \in Int(D), \quad u(x) = 0, \quad x \in \partial D. \tag{2.2}$$

It is easy to see by the maximum principle that (2.2) has a unique solution $u(x)$ and it satisfies the inequality,

$$\|u\|_\infty \le C[diam(D)]^2\ \|f\|_\infty \Big/ \sum_{i=1}^d \lambda_i, \tag{2.3}$$

where $diam(D)$ is the diameter of D and C is a universal constant. The ABP inequality for (2.2) is given by

$$\|u\|_\infty \le C_d[diam(D)]\ \|f\|_d \Big/ (\lambda_1 \cdots \lambda_d)^{1/d}, \tag{2.4}$$

where C_d is a constant depending only on the dimension d, and $\|f\|_d$ is the norm of f on $L^d(\mathbf{Z}^d)$. Evidently (2.4) implies (2.3) modulo the ratio of the arithmetic mean of the $\lambda_i,\ i = 1, ...d$, to the geometric. The fact that the geometric mean occurs in (2.4) illustrates some of the subtlety of the inequality.

Let $\mathcal{L}$ be the operator (1.7) on functions on Ω. For any $\eta > 0$ we define an operator T_η on functions $f : \Omega \to \mathbf{R}$ by

$$T_\eta f(\omega) = \eta \left[\mathcal{L} + \eta\right]^{-1} f(\omega), \quad \omega \in \Omega. \tag{2.5}$$

It is easy to see that T_η is a bounded operator on $L^\infty(\Omega)$ with norm at most 1. We also define for $n \in \mathbf{Z}^d$ the operator τ_n on $L^\infty(\Omega)$ induced by the translation τ_n on Ω by

$$\tau_n f(\omega) = f(\tau_n \omega), \quad \omega \in \Omega. \tag{2.6}$$

Lemma 2.1 *Suppose $f \in L^\infty(\Omega)$ and $R \geq 1$. Then there is a constant C depending only on $d,\ \lambda_i, \Lambda_i,\ i = 1, ...d$, such there is the inequality,*

$$\limsup_{\eta \to 0} \ \left[\ \sup\{\|\tau_n T_\eta f\|_\infty \ : \ n \in \mathbf{Z}^d,\ |n| \leq R/\sqrt{\eta} \ \} \ \right] \leq CR \ \|f\|_d \ . \tag{2.7}$$

Proof. Define a continuous time random walk on Ω as follows:

(a) The waiting time at $\omega \in \Omega$ is exponential with parameter $2 \ \sum_{j=1}^d \ a_j(\omega)$.

(b) The particle jumps from ω to $\tau_{\mathbf{e}_i}\ \omega$ or $\tau_{-\mathbf{e}_i}\ \omega$ with equal probability $a_i(\omega) \Big/ 2 \sum_{j=1}^d a_j(\omega),\ i = 1, ..., d$.

If $\omega(t),\ t > 0$, is the position of the walk at time $t > 0$ then one has

$$[\mathcal{L} + \eta]^{-1} \ f(\omega) = E\left[\int_0^\infty e^{-\eta t} \ f\Big(\omega(t)\Big) dt \ \Big| \omega(0) = \omega\right].$$

Consider now the random walk on $\mathbf{Z}^d$ with transition probabilities depending on $\omega \in \Omega$:

(a) The waiting time at $x \in \mathbf{Z}^d$ is exponential with parameter $2 \ \sum_{j=1}^d \ a_j(\tau_x\ \omega)$.

(b) The particle jumps from x to $x + \mathbf{e}_i$ or $x - \mathbf{e}_i$ with equal probability $a_i(\tau_x\ \omega) \Big/ 2 \sum_{j=1}^d a_j(\tau_x\ \omega),\ i = 1, ..., d$.

Let $X_\omega(t), t > 0$, denote the random walk on $\mathbf{Z}^d$ starting at 0 with these transition probabilities. Then it is easy to see that

$$E\left[f(\omega(t))|\omega(0) = \omega\right] = E\left[f\left(\tau_{X_\omega(t)}\ \omega\right)\right].$$

For $r = 1, 2, ...$ let $\tau_{r,\eta}$ be the first exit time for the walk $X_\omega(t)$ from the ball $|x| < r/\sqrt{\eta}$. Then by (2.4) there is the inequality,

$$E\left[\int_0^\infty e^{-\eta t} f\left(\tau_{X_\omega(t)}\ \omega\right) dt\right] \leq$$
$$C \sum_{r=0}^\infty E\left[e^{-\eta \tau_{r,\eta}}\right] \frac{(r+1)}{\sqrt{\eta}} \left[\sum_{|x|<(r+1)/\sqrt{\eta}} |f(\tau_x \omega)|^d\right]^{1/d},$$

for a constant C depending only on $d, \lambda_1, ..., \lambda_d$.
Now one can see that

$$E\left[e^{-\eta\tau_{r,\eta}}\right] \le \exp[-Kr], \quad r = 0, 1, 2, .., \tag{2.8}$$

where the constant K depends only on $d, \Lambda_1, .., \Lambda_d$. In fact the left hand side of (2.8) is bounded above by the same expectation, but with the stopping time $\tau_{r,\eta}$ replaced by the time $\tau'_{r,\eta}$ for the walk to exit the the cube $\{x : |x_j| < r/\sqrt{d\eta},\ 1 \le j \le d\}$. If $\tau'_{r,\eta,j}$ denotes the time for the walk to exit the region $\{x : |x_j| < r/\sqrt{d\eta}\}$, then there is the inequality,

$$E\left[e^{-\eta\tau'_{r,\eta}}\right] \le \sum_{j=1}^{d} E\left[e^{-\eta\tau'_{r,\eta,j}}\right] \ . \tag{2.9}$$

Each term on the RHS of (2.9) is bounded above by the same expectation, but with the parameter in (a) for the waiting time replaced by $2(\Lambda_1 + .. + \Lambda_d)$. This new expectation is the solution to a one dimensional explicitly solvable finite difference problem, which is bounded above by the right hand side of (2.8).
It follows from (2.8) that

$$\sup\{|\tau_n T_\eta f(\omega)| \ : \ n \in \mathbf{Z}^d,\ |n| \le R/\sqrt{\eta}\ \} \ \le$$
$$C \ \sum_{r=0}^{\infty} \exp[-Kr] \ (r+1)\sqrt{\eta} \left[\sum_{|x|<(r+R+1)/\sqrt{\eta}} |f(\tau_x\omega)|^d \right]^{1/d}, \quad \omega \in \Omega.$$

The result follows now from the last inequality by the Birkhoff ergodic theorem [4] on letting $\eta \to 0$. □

Lemma 2.2 *There is a unique (up to scalar multiplication) non-trivial solution to the equation* $\mathcal{L}^*\Phi = 0$ *in* $L^1(\Omega)$*. The function* $\Phi \in L^{d/(d-1)}(\Omega)$ *and can be chosen such that* $\Phi(\omega) > 0$ *with probability 1 in* ω*.*

Proof. We can assume wlog that the sigma field $\mathcal{F}$ is the smallest sigma field such that the functions $a_i,\ 1 \le i \le d$, are measurable on $(\Omega, \mathcal{F})$ and the translation operators $\tau_x,\ x \in \mathbf{Z}^d$, on Ω are also measurable. Then there is a countable set S of subsets $E \subset \Omega$ such that the span of the characteristic functions $\chi_E,\ E \in S$, is dense in $L^d(\Omega)$. Since the norm of T_η on $L^\infty(\Omega)$ is at most 1, we can choose a sequence $\eta_k,\ k \ge 1$, with $\lim_{k\to\infty} \eta_k = 0$ such that the limit,

$$T(f) = \lim_{k\to\infty} \ \langle T_{\eta_k} f(\cdot)\rangle \tag{2.10}$$

exists provided $f = \chi_E$ with $E \in S$. Now by (2.7) the linear functional T can be uniquely extended to a bounded linear functional on $L^d(\Omega)$. Hence by the Riesz representation theorem [9] there is a unique $\Phi \in L^{d/(d-1)}(\Omega)$ such that

$$T(f) = \langle f\Phi\rangle\,, \qquad f \in L^d(\Omega). \tag{2.11}$$

Now it follows from (2.7) that the limit (2.10) holds for all $f \in L^\infty(\Omega)$. Hence by (2.11) we have for any $h \in L^\infty(\Omega)$,

$$\langle(\mathcal{L}h)\Phi\rangle = \lim_{k\to\infty} \left[\eta_k < h > -\eta_k^2 \left\langle [\mathcal{L} + \eta_k]^{-1} h\right\rangle\right] = 0.$$

We conclude that $\mathcal{L}^*\Phi = 0$. Since $T(1) = 1$ we also have $< \Phi >= 1$ whence Φ is non-trivial. It is evident that T is a positive functional whence $\Phi(\omega) \geq 0$ with probability 1 in ω. To see that $\Phi(\omega) > 0$ with probability 1 in ω we just observe that the set $E = \{\omega \in \Omega : \Phi(\omega) = 0\}$ is invariant under translation whence $\mu(E) = 0$. The invariance follows from $\mathcal{L}^*\Phi(\omega) = 0,\ \omega \in E$, which implies $\tau_{\mathbf{e}_i} E \subset E,\ i = 1, ..., d$.
To get uniqueness, suppose $\Phi \in L^1(\Omega)$ such that $\mathcal{L}^*\Phi = 0$. Let $\Phi = \Phi^+ - \Phi^-$ where $\Phi^+ = \sup\{\Phi, 0\}$ and assume Φ^+, Φ^- are not identically zero. Arguing as before, there is a sequence $\eta_k \to 0$ such that

$$\lim_{k\to\infty} \left\langle T_{\eta_k} f(\cdot)\Phi^+(\cdot)\right\rangle = \left\langle f(\cdot)\Psi^+(\cdot)\right\rangle, \quad f \in L^\infty(\Omega),$$

for some unique $\Psi^+ \in L^{d/(d-1)}(\Omega)$ satisfying $\Psi^+ \geq 0$, $\langle \Psi^+\rangle = \langle \Phi^+\rangle$ and $\mathcal{L}^*\Psi^+ = 0$. We can assume that for the same subsequence $\eta_k,\ k \geq 1$,

$$\lim_{k\to\infty} \left\langle T_{\eta_k} f(\cdot)\Phi^-(\cdot)\right\rangle = \left\langle f(\cdot)\Psi^-(\cdot)\right\rangle, \quad f \in L^\infty(\Omega),$$

for unique $\Psi^- \in L^{d/(d-1)}(\Omega)$ satisfying $\Psi^- \geq 0$, $\langle \Psi^-\rangle = \langle \Phi^-\rangle$, $\mathcal{L}^*\Psi^- = 0$. Since one also has that

$$\langle T_\eta f(\cdot)\Phi(\cdot)\rangle = \langle f(\cdot)\Phi(\cdot)\rangle, \quad f \in L^\infty(\Omega),$$

it follows that $\Phi = \Psi^+ - \Psi^-$. Now we have already observed that $\Psi^- > 0$ with probability 1 since $\mathcal{L}^*\Psi^- = 0$ and $\Psi^- \geq 0$. Hence $\Psi^+ > \Phi^+$ with positive probability whence $\langle \Psi^+\rangle > \langle \Phi^+\rangle$ contradicting the identity $\langle \Psi^+\rangle = \langle \Phi^+\rangle$. $\square$

For $\zeta \in [-\pi, \pi]^d$ we define an operator $\mathcal{L}_\zeta$ on functions $\Psi : \Omega \to \mathbf{C}$ by

$$\mathcal{L}_\zeta\ \Psi(\omega) = \sum_{i=1}^d a_i(\omega)\left[2\Psi(\omega) - e^{-i\mathbf{e}_i\cdot\zeta}\Psi(\tau_{\mathbf{e}_i}\ \omega) - e^{i\mathbf{e}_i\cdot\zeta}\Psi(\tau_{-\mathbf{e}_i}\ \omega)\right].$$

Evidently for $\zeta = 0$, $\mathcal{L}_\zeta$ coincides with the operator $\mathcal{L}$ of (1.7). We generalize the operator $T_\eta,\ \eta > 0$ on $L^\infty(\Omega)$ of (2.5) to an operator $T_{\eta,\zeta},\ \eta > 0, \zeta \in [-\pi, \pi]^d$ on $L^\infty(\Omega)$ defined by

$$T_{\eta,\zeta} f(\omega) = \eta\left[\mathcal{L}_\zeta + \eta\right]^{-1} f(\omega), \quad \omega \in \Omega. \tag{2.12}$$

Evidently $T_{\eta,0} = T_\eta$ and $T_{\eta,\zeta}$ is a bounded operator on $L^\infty(\Omega)$ with norm at most 1.

Lemma 2.3 *Let $\Phi \in L^{d/(d-1)}(\Omega)$ be the unique solution of $\mathcal{L}^*\Phi = 0$ satisfying $\Phi \geq 0$, $< \Phi >= 1$. Suppose $f \in L^\infty(\Omega)$ and $< f\Phi >= 0$. Then there is the limit,*

$$\lim_{(\eta,\zeta)\to(0,0)} \left[\ \sup\{\|\tau_n T_{\eta,\zeta}\ f\|_\infty\ :\ n \in \mathbf{Z}^d,\ |n| \leq R/\sqrt{\eta}\ \}\ \right] = 0\ . \tag{2.13}$$

Proof. Observe that $f \in L^d(\Omega)$ is orthogonal to the null space of $\mathcal{L}^*$ as an operator on $L^{d/(d-1)}(\Omega)$. Hence [9] for any $\delta > 0$ there exists $g_\delta \in L^d(\Omega)$ such that $\|f - \mathcal{L}g_\delta\|_d < \delta$. Since $L^\infty(\Omega)$ is dense in $L^d(\Omega)$ we may assume wlog that $g_\delta \in L^\infty(\Omega)$. Writing

$$T_{\eta,\zeta}\ \mathcal{L}g_\delta = \eta g_\delta + T_{\eta,\zeta}[\ (\mathcal{L} - \mathcal{L}_\zeta)g_\delta - \eta g_\delta\ ]\ ,$$

we see that

$$\lim_{(\eta,\zeta)\to(0,0)} \|T_{\eta,\zeta}\ \mathcal{L}g_\delta\|_\infty = 0\ .$$

The result follows from the previous inequality and Lemma 2.1 by observing that for any $h \in L^\infty(\Omega)$,

$$|T_{\eta,\zeta}\ h(\omega)| \le T_\eta\ |h|(\omega), \quad \omega \in \Omega.$$

□

Lemma 2.4 *Suppose $f : \mathbf{Z}^d_\varepsilon \to \mathbf{R}$ has finite support in the set $\{x = (x_1, ..., x_d) \in \mathbf{Z}^d_\varepsilon \ : \ |x| < R\}$. Let $u_\varepsilon(x, \omega)$ be the solution to (1.2) and $\alpha(k)$ be defined by*

$$\alpha(k) = 2\{\cosh k\varepsilon - 1\}/\varepsilon^2\ .$$

Then if $1 \le j \le d$ and $k > 0$ satisfies $\Lambda_j \alpha(k) < 1$, there is the inequality,

$$|u_\varepsilon(x, \omega)| \le \exp[\ kR - k|x_j|\ ]\ \|f\|_\infty / [1 - \Lambda_j \alpha(k)] \quad x \in \mathbf{Z}^d_\varepsilon,\ \omega \in \Omega. \tag{2.14}$$

Proof. We may assume wlog that the function f is nonnegative whence $u_\varepsilon(x, \omega)$ is also nonnegative. We write $u_\varepsilon(x, \omega) = e^{-kx_j} u_{\varepsilon,k}(x, \omega)$ where $1 \le j \le d$. Then from (1.2) $u_{\varepsilon,k}(x, \omega)$ satisfies the equation,

$$\begin{aligned}\sum_{i=1}^d\ a_i(\tau_{x/\varepsilon}\ \omega)\,[2u_{\varepsilon,k}(x, \omega) - u_{\varepsilon,k}(x + \varepsilon\mathbf{e}_i, \omega) - u_{\varepsilon,k}(x - \varepsilon\mathbf{e}_i, \omega)]\,/\varepsilon^2 \\ + a_j(\tau_{x/\varepsilon}\ \omega)\ (e^{k\varepsilon} - 1)\{u_{\varepsilon,k}(x, \omega) - u_{\varepsilon,k}(x - \varepsilon\mathbf{e}_j, \omega)\}/\varepsilon^2 \\ - a_j(\tau_{x/\varepsilon}\ \omega)\ (1 - e^{-k\varepsilon})\{u_{\varepsilon,k}(x, \omega) - u_{\varepsilon,k}(x + \varepsilon\mathbf{e}_j, \omega)\}/\varepsilon^2 \\ + [\ 1 - a_j(\tau_{x/\varepsilon}\ \omega)\alpha(k)\ ]u_{\varepsilon,k}(x, \omega) = e^{kx_j} f(x), \quad x \in \mathbf{Z}^d_\varepsilon, \quad \omega \in \Omega.\end{aligned}$$

Now if $\Lambda_j \alpha(k) < 1$ and $x \in \mathbf{Z}^d_\varepsilon$ is the point at which $u_{\varepsilon,k}(x, \omega)$ takes its maximum then all terms on the LHS of the previous expression are nonnegative except for the third term. This is however less in absolute value than the first term. Hence the last term is less than the RHS. The inequality (2.14) follows. □

Lemma 2.4 enables us to take the Fourier transform of the equation (1.2). Proceeding as in [3] we write $u_\varepsilon(x, \omega) = v_\varepsilon(x, \tau_{x/\varepsilon}\ \omega)$ whence (1.2) becomes

$$\begin{aligned}\sum_{i=1}^d a_i(\tau_{x/\varepsilon}\ \omega)\Big[2v_\varepsilon(x, \tau_{x/\varepsilon}\ \omega) - v_\varepsilon(x + \varepsilon\mathbf{e}_i, \tau_{\mathbf{e}_i}\tau_{x/\varepsilon}\ \omega) \\ - v_\varepsilon(x - \varepsilon\mathbf{e}_i, \tau_{-\mathbf{e}_i}\tau_{x/\varepsilon}\ \omega)\Big]\Big/\varepsilon^2 + v_\varepsilon(x, \tau_{x/\varepsilon}\ \omega) = f(x),\ x \in \mathbf{Z}^d_\varepsilon, \quad \omega \in \Omega.\end{aligned}$$

We conclude from the previous equation that v_ε satisfies the equation,

$$\begin{aligned}\sum_{i=1}^d a_i(\omega)\Big[2v_\varepsilon(x, \omega) - v_\varepsilon(x + \varepsilon\mathbf{e}_i, \tau_{\mathbf{e}_i}\ \omega) - v_\varepsilon(x - \varepsilon\mathbf{e}_i, \tau_{-\mathbf{e}_i}\ \omega)\Big]\Big/\varepsilon^2 \\ + v_\varepsilon(x, \omega) = f(x), \quad x \in \mathbf{Z}^d_\varepsilon, \quad \omega \in \Omega.\end{aligned} \tag{2.15}$$

We put now

$$\hat{v}_\varepsilon(\xi, \omega) = \int_{\mathbf{Z}^d_\varepsilon} v_\varepsilon(x, \omega) e^{ix\cdot\xi} dx, \quad \xi \in \left[\frac{-\pi}{\varepsilon}, \frac{\pi}{\varepsilon}\right]^d,$$

where the integral of a summable function $g : \mathbf{Z}_\varepsilon^d \to \mathbf{C}$ is defined by

$$\int_{\mathbf{Z}_\varepsilon^d} g(x)\ dx = \sum_{x \in \mathbf{Z}_\varepsilon^d} \varepsilon^d g(x).$$

Then (2.15) yields the equation,

$$\frac{1}{\varepsilon^2} \sum_{i=1}^d a_i(\omega) \Big[2\hat{v}_\varepsilon(\xi,\omega) - e^{-i\varepsilon \mathbf{e}_i \cdot \xi} \hat{v}_\varepsilon(\xi, \tau_{\mathbf{e}_i}\ \omega) - e^{i\varepsilon \mathbf{e}_i \cdot \xi} \hat{v}_\varepsilon(\xi, \tau_{-\mathbf{e}_i}\ \omega) \Big] + \hat{v}_\varepsilon(\xi,\omega) = \hat{f}_\varepsilon(\xi), \quad (2.16)$$

where $\hat{f}_\varepsilon$ denotes the Fourier transform of f as a function on $\mathbf{Z}_\varepsilon^d$. It follows from (2.12) and (2.16) that

$$\hat{v}_\varepsilon(\xi,\omega) = \hat{f}_\varepsilon(\xi)\ T_{\varepsilon^2,\varepsilon\xi}\ 1(\omega), \quad \omega \in \Omega, \quad (2.17)$$

where $1 : \Omega \to \mathbf{R}$ is the constant function with value 1.

Lemma 2.5 *Let $f \in L^\infty(\Omega)$, Φ be the unique function of Lemma 2.2 and $h : \mathbf{R}^d \to \mathbf{R}$ be given by the formula,*

$$h(\xi) = 1 \Big/ \left[1 + \sum_{j=1}^d |\xi_j|^2 < a_j \Phi > \right], \quad \xi \in \mathbf{R}^d. \quad (2.18)$$

Then for any $R, R' \geq 1$ there is the limit,

$$\lim_{\varepsilon \to 0} \left[\ \sup\{ \| \tau_n T_{\varepsilon^2,\varepsilon\xi}\ f\ -\ h(\xi) < f\Phi > \|_\infty\ :\ |n| \leq R/\varepsilon,\ \ |\xi| \leq R' \}\ \right] = 0\ . \quad (2.19)$$

Proof. This follows from Lemma 2.3 once we observe that

$$[\mathcal{L}_{\varepsilon\xi} + \varepsilon^2]1 = \sum_{j=1}^d \left[2 - e^{-i\varepsilon \mathbf{e}_j \cdot \xi} - e^{i\varepsilon \mathbf{e}_j \cdot \xi} \right] a_j + \varepsilon^2.$$

Hence we have

$$1 = \sum_{j=1}^d \frac{1}{\varepsilon^2} \left[2 - e^{-i\varepsilon \mathbf{e}_j \cdot \xi} - e^{i\varepsilon \mathbf{e}_j \cdot \xi} \right] T_{\varepsilon^2,\varepsilon\xi}\ a_j + T_{\varepsilon^2,\varepsilon\xi} 1.$$

On writing $a_j = [a_j - < a_j\Phi > 1] + < a_j\Phi > 1$ the result for $f \equiv 1$ follows from the previous identity and Lemma 2.3. For general $f \in L^\infty(\Omega)$ we write $f = [f - < f\Phi > 1] + < f\Phi > 1$ and use Lemma 2.3 and the fact that we have already established (2.19) for $f \equiv 1$. □

Proof. [Proof of Theorem 1.1] By the Fourier inversion theorem we have that

$$u_\varepsilon(\varepsilon\lfloor x/\varepsilon \rfloor,\ \omega) = \frac{1}{(2\pi)^d} \int_{[-\pi/\varepsilon,\pi/\varepsilon]^d} \hat{v}_\varepsilon\left(\xi, \tau_{\lfloor x/\varepsilon \rfloor}\ \omega\right) \exp\left[-i\varepsilon\lfloor x/\varepsilon \rfloor \cdot \xi\right] d\xi. \quad (2.20)$$

For $x \in \mathbf{R}^d$ let $u(x)$ be given by,

$$u(x) = \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} \frac{\hat{f}(\xi) e^{-ix\cdot\xi}}{\sum_{j=1}^d |\xi_j|^2\ < a_j\Phi > +1} d\xi.$$

By Lemma 2.4 it is sufficient to prove that for any $R \geq 1$ one has with probability 1 the limit,

$$\lim_{\varepsilon \to 0} \sup_{|x| \leq R} |u_\varepsilon(\varepsilon \lfloor x/\varepsilon \rfloor, \ \cdot) - u(x)| = 0 \ . \tag{2.21}$$

Since $f : \mathbf{R}^d \to \mathbf{R}$ is C^∞ of compact support there is for any $\delta > 0$ an $R' \geq 1$ such that

$$\limsup_{\varepsilon \to 0} \int_{\xi \in [-\pi/\varepsilon, \pi/\varepsilon]^d, \ |\xi| \geq R'} |\hat{f}_\varepsilon(\xi)| d\xi \leq \delta.$$

From (2.17) and the fact that the operator $T_{\eta,\zeta}$ has norm at most 1 on $L^\infty(\Omega)$ it follows that $|\hat{v}_\varepsilon(\xi, \omega)| \leq |\hat{f}(\xi)|, \ \xi \in [-\pi/\varepsilon, \pi/\varepsilon]^d, \ \omega \in \Omega$. Hence we may replace the representation (2.20) for $u_\varepsilon(\varepsilon\lfloor x/\varepsilon \rfloor, \ \omega)$ by an integral over $|\xi| \leq R'$ at a cost of at most $\delta/(2\pi)^d$. With this replacement the limit (2.21) follows from Lemma 2.5. Now (2.21) without the cutoff follows by letting $\delta \to 0$. □

3 Proof of Theorem 1.2

Proof. As in [3] we decompose the function $\hat{v}_\varepsilon(\xi, \omega)$ into the sum of its mean and a part orthogonal to the constant,

$$\hat{v}_\varepsilon(\xi, \omega) = \hat{u}_\varepsilon(\xi) + \hat{\psi}_\varepsilon(\xi, \omega), \quad \left\langle \hat{\psi}_\varepsilon(\xi, \cdot) \right\rangle = 0.$$

On taking the expectation in (2.16) we have that

$$\sum_{i=1}^d < a_i(\cdot) > \frac{1}{\varepsilon^2}\left[2 - e^{-i\varepsilon \mathbf{e}_i \cdot \xi} - e^{i\varepsilon \mathbf{e}_i \cdot \xi}\right]\hat{u}_\varepsilon(\xi) + \hat{u}_\varepsilon(\xi)$$
$$+ \sum_{i=1}^d \left\langle a_i(\cdot)\frac{1}{\varepsilon^2}\left[2\hat{\psi}_\varepsilon(\xi, \cdot) - e^{-i\varepsilon \mathbf{e}_i \cdot \xi}\hat{\psi}_\varepsilon(\xi, \tau_{\mathbf{e}_i} \ \cdot) - e^{i\varepsilon \mathbf{e}_i \cdot \xi}\hat{\psi}_\varepsilon(\xi, \tau_{\mathbf{e}_i} \ \cdot)\right]\right\rangle = \hat{f}_\varepsilon(\xi). \tag{3.1}$$

If we subtract (3.1) from (2.16) we obtain the equation,

$$P \ \sum_{i=1}^d a_i(\omega)\frac{1}{\varepsilon^2}\left[2\hat{\psi}_\varepsilon(\xi, \omega) - e^{-i\varepsilon \mathbf{e}_i \cdot \xi}\hat{\psi}_\varepsilon(\xi, \tau_{\mathbf{e}_i} \ \omega) - e^{i\varepsilon \mathbf{e}_i \cdot \xi}\hat{\psi}_\varepsilon(\xi, \tau_{-\mathbf{e}_i} \ \omega)\right] + \hat{\psi}_\varepsilon(\xi, \omega)$$
$$+ \sum_{i=1}^d \ \frac{1}{\varepsilon^2}\left[2 - e^{-i\varepsilon \mathbf{e}_i \cdot \xi} - e^{i\varepsilon \mathbf{e}_i \cdot \xi}\right]\hat{u}_\varepsilon(\xi) P \ a_i(\omega) = 0, \tag{3.2}$$

where P is the projection operator orthogonal to the constant. For $1 \leq j \leq d, \ \zeta \in [-\pi, \pi]^d$, $\eta > 0$ consider the equation,

$$P\mathcal{L}_\zeta \Psi_j(\zeta, \eta, \omega) + \eta \Psi_j(\zeta, \eta, \omega) + P \ a_j(\omega) = 0. \tag{3.3}$$

for the function $\Psi_j(\zeta, \eta, \omega)$. If $\gamma = \sup_{1 \leq i \leq d}[1 - \lambda_i/\Lambda_i]$ is sufficiently small then (3.3) can be solved uniquely in $L^2(\Omega)$ by a convergent perturbation expansion. Evidently $< \Psi_j(\zeta, \eta, \cdot) >= 0$. It is also clear that the function

$$\hat{\psi}_\varepsilon(\xi, \omega) = \hat{u}_\varepsilon(\xi) \sum_{j=1}^d \left[2 - e^{-i\varepsilon \mathbf{e}_j \cdot \xi} - e^{i\varepsilon \mathbf{e}_j \cdot \xi}\right] \Psi_j(\varepsilon\xi, \varepsilon^2, \omega) \tag{3.4}$$

is the solution to (3.2). It follows now from (2.17),that

$$|\hat{u}_\varepsilon(\xi)| \le |\hat{f}_\varepsilon(\xi)|, \quad \xi \in [-\pi/\varepsilon, \pi/\varepsilon]^d. \tag{3.5}$$

We can assume wlog that $\Lambda_i = 1,\ i = 1, ..., d$. Consider now the equation

$$\sum_{i=1}^d \left[2\psi(\omega) - e^{-i\mathbf{e}_i\cdot\zeta}\psi(\tau_{\mathbf{e}_i}\ \omega) - e^{i\mathbf{e}_i\cdot\zeta}\psi(\tau_{-\mathbf{e}_i}\ \omega)\right] + \eta\ \psi(\omega) = \varphi(\omega), \quad \omega \in \Omega. \tag{3.6}$$

Then the solution to (3.6) is given by the formula

$$\psi(\omega) = \sum_{x \in \mathbf{Z}^d} G_\eta(x) e^{-ix\cdot\zeta}\varphi(\tau_x\ \omega),$$

where G_η is the Green's function for the standard random walk on $\mathbf{Z}^d$,

$$-\Delta\ G_\eta(x) + \eta\ G_\eta(x) = \delta(x), \quad x \in \mathbf{Z}^d.$$

For $1 \le k \le d,\ \zeta \in [-\pi, \pi]^d, \eta > 0$, we define an operator $T_{k,\zeta,\eta}$ on $L^2(\Omega)$ by

$$T_{k,\zeta,\eta}\varphi(\omega) = 2\psi(\omega) - e^{-i\mathbf{e}_k\cdot\zeta}\psi(\tau_{\mathbf{e}_k}\ \omega) - e^{i\mathbf{e}_k\cdot\zeta}\psi(\tau_{-\mathbf{e}_k}\ \omega), \quad \omega \in \Omega,$$

where ψ is the solution to (3.6). It is easy to see that $T_{k,\zeta,\eta}$ is a bounded operator on $L^2(\Omega)$ and $\|T_{k,\zeta,\eta}\| \le 1$. Putting $b_k(\omega) = 1 - a_k(\omega)$, $1 \le k \le d$, we see that equation (3.3) is equivalent to the equations,

$$\varphi(\omega) - P\sum_{k=1}^d b_k(\omega) T_{k,\zeta,\eta}\varphi(\omega) = P\ b_j(\omega), \quad \omega \in \Omega, \quad \Psi_j(\zeta, \eta, \omega) = \psi(\omega), \tag{3.7}$$

where ψ is the solution of (3.6) with φ as the solution to the first equation of (3.7). This has a unique solution in $L^2(\Omega)$ given by the perturbation expansion,

$$\varphi(\omega) = \sum_{m=0}^\infty \left[P\sum_{k=1}^d b_k(\cdot)T_{k,\zeta,\eta}\right]^m P\ b_j(\omega), \quad \omega \in \Omega,$$

provided

$$\sum_{k=1}^d |b_k(\omega)|^2 < 1, \qquad \omega \in \Omega.$$

Thus if $\gamma < 1/\sqrt{d}$ then (3.3) is uniquely solvable in $L^2(\Omega)$ for $\Psi_j(\zeta, \eta, \omega)$.
Now from (3.4) the quantity we need to estimate as $\varepsilon \to 0$ is given by

$$\left\langle \left[\int_{\mathbf{Z}^d_\varepsilon} dx g(x)\psi_\varepsilon(x, \tau_{x/\varepsilon}\ \cdot)\right]^2 \right\rangle \le d\sum_{k=1}^d \left\langle \left| \int_{\mathbf{Z}^d_\varepsilon} dx g(x)\frac{1}{(2\pi)^d}\int_{\left[\frac{-\pi}{\varepsilon}, \frac{-\pi}{\varepsilon}\right]^d} d\xi e^{-ix\cdot\xi} \right.\right.$$
$$\left.\left. \hat{u}_\varepsilon(\xi)\left[2 - e^{-i\varepsilon\mathbf{e}_k\cdot\xi} - e^{i\varepsilon\mathbf{e}_k\cdot\xi}\right]\Psi_k(\varepsilon\xi, \varepsilon^2, \tau_{x/\varepsilon}\ \cdot)\right|^2 \right\rangle. \tag{3.8}$$

For $n \in \mathbf{Z}^d,\ \xi, \xi' \in [\frac{-\pi}{\varepsilon}, \frac{\pi}{\varepsilon}]^d$ let $h_\varepsilon(n, \xi, \xi')$ be the function,

$$h_\varepsilon(n, \xi, \xi') = \int_{\mathbf{Z}^d_\varepsilon} dx g(x) g(x - \varepsilon n) \exp\left[ix \cdot (\xi' - \xi) - i\varepsilon n \cdot \xi'\right].$$

Similarly let $\chi_{\varepsilon,k}(n, \xi, \xi')$ be the function

$$\chi_{\varepsilon,k}(n, \xi, \xi') = \varepsilon^4 \left\langle \Psi_k(\varepsilon\xi, \varepsilon^2, \tau_n \cdot) \overline{\Psi_k(\varepsilon\xi', \varepsilon^2, \cdot)} \right\rangle.$$

Then the RHS of (3.8) is the same as

$$4d\varepsilon^d \sum_{k=1}^d \int_{[\frac{-\pi}{\varepsilon}, \frac{\pi}{\varepsilon}]^d} d\xi \int_{[\frac{-\pi}{\varepsilon}, \frac{\pi}{\varepsilon}]^d} d\xi' \ \hat{u}_\varepsilon(\xi)[1 - \cos \varepsilon \mathbf{e}_k \cdot \xi]/\varepsilon^2$$
$$\overline{\hat{u}_\varepsilon(\xi')}[1 - \cos \varepsilon \mathbf{e}_k \cdot \xi']/\varepsilon^2 \sum_{n \in \mathbf{Z}^d} h_\varepsilon(n, \xi, \xi') \chi_{\varepsilon,k}(n, \xi, \xi'). \quad (3.9)$$

From (3.5) the integral in (3.9) is bounded uniformly as $\varepsilon \to 0$, provided we can obtain a bound on the sum over $n \in \mathbf{Z}^d$. Since g has compact support it is easy to see that for any r', $1 \le r' \le \infty$, h_ε is in $L^{r'}(\mathbf{Z}^d)$ with norm $\|h_\varepsilon\|_{r'} \le C\varepsilon^{-d/r'}$ for some constant C. We may argue now exactly as in [3] that $\chi_{\varepsilon,k}$ is in $L^r(\mathbf{Z}^d)$ for some r, $1 < r < \infty$, if γ is sufficiently small. In fact r can be taken arbitrarily close to 1 for small γ. Choosing r' so that $1/r' + 1/r = 1$, we have then from (3.9) that the variance on the RHS is bounded by $C\varepsilon^{d(1-1/r')}$ for some constant C. $\square$

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