Length Scales in a Rotating Stratified Fluid on the Beta Plane[1]

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ABSTRACT

The relationships between vertical and horizontal length scales in a rotating stratified fluid on the beta plane are discussed in an attempt to unify the results of previous papers. The model is steady, linear and Boussinesq, but allows for different coefficients for the horizontal and vertical eddy mixing processes. The boundary layers in previous papers together with a new physical scale are analyzed with respect to their physical balances, length scale, and existence in a parameter space. The results are summarized in a three-part schematic graph, which shows the relations between dimensionless horizontal and vertical scales, and in a table, which contains the relevant physical balances for each relation. Three internal dimensionless parameters are considered, namely S a measure of the importance of stratification relative to rotation, δ a measure of the magnitudes of vertical to horizontal mixing processes, and $\beta E^{\frac{1}{3}}$ a ratio of the length scales over which lateral friction and the beta effect are important.

1. Introduction

Many theoretical papers on the subject of rotating stratified fluids have appeared in the literature in recent years. For the most part, each author has tried to model a certain feature of oceanic circulation with the aid of boundary layer analysis. Some of these papers focused on the western boundary currents (such as the Gulf Stream and Kuroshio), the main oceanic thermocline and coastal upwelling, while others dealt with rotating fluid flows in the laboratory. Unfortunately, it is often difficult to understand how these many papers fit into an overall theory, appearing as separate entities rather than as special cases of a more general subject.

The goal of this paper is to unify much of the existing knowledge in terms of the grossest features of the fluid flow—the length scales associated with the motion and density fields. Whereas previous papers have considered a restricted region of parameter space and determined particular boundary layer scales and dynamics, we analyze an extensive region of parameter space and see where the various boundary layer dynamics are relevant. In particular, we follow the method of Blumsack (1972) and investigate the system by means of a length scale analysis, seeking the relationships between the horizontal and vertical scales of motion. The importance of such an analysis was demonstrated by Blumsack, who showed that boundary layer thickness can be a function of depth in a rotating stratified fluid.

We consider an idealized problem, the steady linearized motions of a rotating stratified Boussinesq fluid on the beta plane. We do not specify boundary conditions since we are interested only in the possible internal scales of the solution as functions of the internal parameters of the problem. The stratification parameter measures the importance of the density stratification relative to the rotation, the aspect ratio is the ratio of vertical to horizontal "viscous" scales, and the third parameter measures the relative change in the Coriolis parameter over a horizontal "viscous" scale.

The results of this paper are summarized in a table, which shows the important dynamical balances for each length scale, and in a three-part figure, which depicts the relationship between dimensionless horizontal and vertical scales for three distinct parametric conditions.

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2. Formulation

Consider the state of a Boussinesq fluid, which is a slight departure from a state of uniform rotation and density stratification, on a beta plane. Assume that the perturbation state is steady in time and the nonlinear advection terms are negligible. The linearized differential equations for the perturbation state, written in dimensional form, are

$$f_* \mathbf{k} \times \mathbf{q}_* = -\nabla_* p_* + D_*^2 \mathbf{q}_*, \tag{2.1}$$

$$0 = -\partial p_*/\partial z_* + \alpha g T_* + D_*^2 w_*, \tag{2.2}$$

$$(\sigma N^2/\alpha g) w_* = D_*^2 T_*, \tag{2.3}$$

$$\nabla_* \cdot \mathbf{q}_* + \partial w_*/\partial z_* = 0, \tag{2.4}$$

where D_*^2 is the diffusive operator,

$$D_*^2 = A_H \nabla_*^2 + A_V \partial^2/\partial z_*^2.$$

The dependent variables are the modified pressure p_*, temperature perturbation T_*, horizontal velocity $\mathbf{q}_*$, and vertical velocity w_*. We denote the vertical coordinate and unit vector by z_* and $\mathbf{k}$, and the horizontal position and gradient operator by $\mathbf{x}_*$ and ∇_*. The parameters in the problem include the horizontal and vertical coefficients for the mixing of momentum A_H and A_V, the acceleration of gravity g, coefficient of thermal expansion α, Brunt-Väisälä frequency N of the imposed stratification, single Prandtl number σ, and variable Coriolis parameter f_*.

We express the problem in dimensionless form to arrive at a more compact formulation and to place (2.1)–(2.4) in a form similar to that of other papers. Suppose L is some (arbitrary) horizontal length, U a horizontal speed, and f_0 a typical value for the Coriolis parameter; we then write

$$\mathbf{x}_*=L\mathbf{x},\quad \nabla_*=L^{-1}\nabla,\quad \mathbf{q}_*=U\mathbf{q},\quad f_*=f_0f. \quad (2.5)$$

It is convenient to scale the vertical coordinate with δL, i.e.,

$$\delta=(A_V/A_H)^{\frac{1}{2}},\quad z_*=(\delta L)z,\quad w_*=(\delta U)w. \quad (2.6)$$

Finally, we scale the pressure and temperature:

$$p_*=(f_0UL)p,\quad T_*=(f_0U/\alpha g\delta)T. \quad (2.7)$$

Substitution of (2.5)–(2.7) into (2.1)–(2.4) results in the following system of dimensionless equations:

$$f\mathbf{k}\times\mathbf{q}=-\nabla p+\mathrm{E}D^2\mathbf{q}, \quad (2.8)$$

$$0=-\partial p/\partial z+T+\mathrm{E}\delta^2D^2w, \quad (2.9)$$

$$Sw=\mathrm{E}D^2T, \quad (2.10)$$

$$\nabla\cdot\mathbf{q}+\partial w/\partial z=0, \quad (2.11)$$

where D^2 is the three-dimensional Laplacian operator, S the stratification parameter, and E the Ekman number, i.e.,

$$D^2=\nabla^2+\partial^2/\partial z^2,\quad S=\sigma N^2\delta^2/f_0^2,\quad \mathrm{E}=A_H/f_0L^2. \quad (2.12)$$

The notation is similar to that of Barcilon and Pedlosky (1967) and others.

The vertical component of the vorticity plays a crucial role in rotating fluid phenomena. We use (2.8) and (2.11) to arrive at the following vorticity equation:

$$f(\partial w/\partial z)=\mathbf{q}\cdot\nabla f-\mathrm{E}D^2(\mathbf{k}\cdot\mathrm{curl}\,\mathbf{q}). \quad (2.13)$$

A new parameter, which measures the variability of f, now appears. We introduce β and its dimensional counterpart β_*,

$$\beta=|\nabla f|,\quad \beta_*=|\nabla_*f_*|. \quad (2.14)$$

We shall use (2.13) in place of (2.11) during much of the discussion that follows, particularly in situations where a geostrophic balance occurs.

There are three internal dimensionless parameters. Two of them, S and δ, appear explicitly in (2.8)–(2.11). The Ekman number E and the parameter β each depend on the length scale L. However, the combination $\beta\mathrm{E}^{\frac{1}{2}}=(A_H/f_0)^{\frac{1}{2}}(\beta_*/f_0)$ is independent of L and serves as our third dimensionless parameter. We now make a few mild assumptions concerning the magnitudes of the parameters S, δ and $\beta\mathrm{E}^{\frac{1}{2}}$ in order to focus attention on the results of other papers:

$$S\leqslant O(1),\quad \delta\leqslant O(1),\quad \beta\mathrm{E}^{\frac{1}{2}}\ll O(1),\quad f=O(1). \quad (2.15)$$

By taking $\beta\mathrm{E}^{\frac{1}{2}}\ll 1$, we guarantee that the distance over which f changes by its own magnitude is much larger than the horizontal viscous scale $(A_H/f_0)^{\frac{1}{2}}$. Notice that we make no assumptions regarding E since we wish to leave the scaling length L arbitrary. The dimensional results will, of course, be independent of L.

We now begin our quest of the generalized length scale relationships. We separate our analysis into three parts for convenience, considering first cases where the dimensionless vertical length scale h is smaller than the horizontal length scale l, then situations for which $l<h$, and finally cases where l and h have comparable magnitudes. We consider only internal scales, omitting the barotropic (z-independent) scales discussed by Beardsley (1968). As a result, the $\mathrm{E}^{\frac{1}{4}}$ scale of Stewartson (1957), the bottom friction scale, and the barotropic scale of Munk (1950) will not appear in our analysis.

Each length scale relationship is based on retaining certain terms in the equations governing our system while neglecting other terms. After obtaining the relation between h and l in this way, we must check back to verify that the neglected terms are indeed smaller than those retained. It will usually be the case that such checks restrict the validity of the relation in question to a limited region of parameter space.

Although we include no specific boundary conditions, we do make one assumption concerning the orientation of the horizontal flow. In order to estimate the size of $\mathbf{q}\cdot\nabla f$ in (2.13), we assume that the angle between $\mathbf{q}$ and ∇f is not close to 90° and write $\mathbf{q}\cdot\nabla f\approx\beta|\mathbf{q}|$. In the context of previous work, we do not analyze boundary layers near coasts that are parallel to latitudes circle.

3. Case 1: $h < l$

Suppose that the vertical scale of motion h is much smaller than the horizontal scale l. Then we can replace the Laplacian operator D^2 by $\partial^2/\partial z^2$. An Ekman layer scale can exist under certain conditions. Suppose $\mathbf{q}=O(1)$; then the Ekman scale $h_\mathrm{E}=\mathrm{E}^{\frac{1}{2}}$ is consistent whenever $p<l$. The continuity equation (2.11) gives the estimate $w\approx\mathrm{E}^{\frac{1}{2}}$, the heat equation (2.10) tells us that $T\approx S\mathrm{E}^{\frac{1}{2}}$, and the vertical momentum equation (2.9) implies $p\approx\mathrm{E}\max(S,\delta^2)$. Since both S and δ are at most order unity, the pressure gradient in (2.8) is negligible when $l>\mathrm{E}$. However, the more restrictive constraint is $h<l$. Therefore, when $l>\mathrm{E}^{\frac{1}{2}}$, there exists an Ekman scale h_E.

The other scale possible when $h<l$ is the thermocline scale discussed by Stommel and Veronis (1957) and by

Pedlosky (1969). Suppose again that $\mathbf{q}=O(1)$. The horizontal momentum balance (2.8) is geostrophic, so $p\approx l$. The vertical momentum balance (2.9) is hydrostatic, implying $T\approx lh^{-1}$. We find the vertical velocity by balancing vortex stretching with the advection of planetary vorticity, $w\approx\beta h$. Since the vertical velocity and temperature fields must be consistent with the heat equation (2.10), we have an expression for the thermocline scale h_T, i.e.,

$$h_T=(\mathrm{E}l/\beta S)^{\frac{1}{4}}. \tag{3.1}$$

The validity of this relationship depends on h being less than l and the viscous terms in (2.8), (2.9) and (2.13) being negligible. We can express these conditions as follows:

$$h<l,\quad h>\mathrm{E}^{\frac{1}{2}},\quad l>\mathrm{E}^{\frac{1}{2}}(\beta\mathrm{E}^{\frac{1}{2}})\delta^2,\quad l>hS^{\frac{1}{2}}. \tag{3.2}$$

Since the three parameters $\beta\mathrm{E}^{\frac{1}{2}}$, δ and S cannot exceed order unity, the latter two conditions are true whenever the first two inequalities are satisfied. Therefore, the thermocline relation (3.1) is valid when

$$l>\mathrm{E}^{\frac{1}{2}}(S\beta\mathrm{E}^{\frac{1}{2}})^{-\frac{1}{3}}. \tag{3.3}$$

One can show via trial and error that no other relationship between h and l, for which $h<l$, can exist. The scales h_E and h_T are represented in all three graphs in Fig. 1.

4. Case 2 : $l < h$

Now suppose that the vertical scale of motion (in our dimensionless variables) is much larger than the horizontal scale. We let n denote the horizontal coordinate in the direction of maximum change and u be the velocity component along the n axis. The other horizontal coordinate s is defined such that (n,s,z) form a right-handed system; the velocity component along the s axis is labeled v. We neglect curvature, treating (n,s) as local Cartesian coordinates, since we seek only length scale information and not detailed solutions. Eqs. (2.8)–(2.11) and (2.13) then become

$$-fv=-p_n+\mathrm{E}u_{nn}, \tag{4.1}$$

$$fu=-p_s+\mathrm{E}v_{nn}, \tag{4.2}$$

$$0=-p_z+T+\mathrm{E}\delta^2 w_{nn}, \tag{4.3}$$

$$Sw=\mathrm{E}T_{nn}, \tag{4.4}$$

$$u_n+v_s+w_z=0, \tag{4.5}$$

$$fw_z=\mathbf{q}\cdot\nabla f-\mathrm{E}v_{nnn}, \tag{4.6}$$

where we have used v_n for the vorticity in (4.6).

We take l to be the scale of variation in the direction n and h the vertical scale; variations along s are much smaller than along n. We enumerate the many possible relationships, indicate the relevant dynamical balances, and discuss the existence of each relationship in parameter space.

We begin our list with the three relationships that do not contain h explicitly.

a. *Carrier-Munk scale:* $l_M=(\mathrm{E}/\beta)^{\frac{1}{3}}$

The determining balance is between the "beta" and viscous terms in the vorticity equation (4.6), which is identical to the dynamics of the layer considered by Munk (1950) and Beardsley (1968) for a barotropic fluid. Existence of this scale is confirmed by first setting $v\approx 1$. We note that $l_M=\mathrm{E}^{\frac{1}{2}}(\beta\mathrm{E}^{\frac{1}{2}})^{-\frac{1}{3}}$ is larger than $\mathrm{E}^{\frac{1}{2}}$; Eq. (4.1) reduces to a geostrophic balance, $p\approx l$. After using (4.4) to find $T\approx(S\mathrm{E}^{-1}l^2)w$, we employ (4.3) to estimate the vertical velocity, $w\approx h^{-1}\min(\mathrm{E}S^{-1}l^{-1}, l^3\mathrm{E}^{-1}\delta^{-2})$. Therefore, the vortex stretching term is negligible when $h>\mathrm{E}^{\frac{1}{2}}\min[(\beta\mathrm{E}^{\frac{1}{2}})^{-\frac{1}{3}}S^{-\frac{1}{2}}, \delta^{-1}(\beta\mathrm{E}^{\frac{1}{2}})^{-1}]$. Note that there is no upper limit on h for the validity of the Carrier-Munk scale.

b. *Upwelling scale:* $l_U=\mathrm{E}^{\frac{1}{2}}$

The controlling dynamics entail the neglect of the pressure gradient terms in (4.1) and (4.2). This ageostrophic layer was used first by Pedlosky (1968) in a homogeneous fluid and later by Pedlosky (1969) and Blumsack (1972) in a fluid with density stratification. The criteria for the existence of this scale are found by letting $u,v\approx 1$, deducing $w\approx h\mathrm{E}^{-\frac{1}{2}}$ from continuity, $T\approx Sh\mathrm{E}^{-\frac{1}{2}}$ from the heat equation, and $p\approx h^2\mathrm{E}^{-\frac{1}{2}}\times\max(S,\delta^2)$ from the vertical momentum equation. Neglect of the pressure gradient term in (4.1) is allowable only when $h<\mathrm{E}^{\frac{1}{2}}\min(S^{-\frac{1}{2}},\delta^{-1})$. In addition, we require $h>\mathrm{E}^{\frac{1}{2}}$ so that $l<h$.

c. *Buoyancy scale:* $l_B=\mathrm{E}^{\frac{1}{2}}(\delta^2/S)^{\frac{1}{4}}$

Veronis (1967) found this scale by neglecting the vertical pressure gradient term in (4.3). The validity of this nonrotating scale is derived by taking $w\approx 1$ and $T\approx Sl^2\mathrm{E}^{-1}$. From continuity, $u\approx lh^{-1}$ and from (4.1), (4.2) the pressure is estimated to be $p\approx\mathrm{E}h^{-1}\max(1,\delta^2S^{-1})$. The vertical scale h must satisfy the following: $h>\mathrm{E}^{\frac{1}{2}}\max(\delta^{-\frac{1}{2}}S^{-\frac{1}{4}},\delta^{\frac{1}{2}}S^{-\frac{3}{4}})$. As for the Carrier-Munk scale, there is no upper limit on h, only a lower limit.

The ordering of these three scales, namely l_M, l_U and l_B, provides the key for understanding the regions of validity for the remaining scales. We note first that $\beta\mathrm{E}^{\frac{1}{2}}<1$ implies $l_U<l_M$. Therefore, there are only three distinct orderings for l_M, l_U, and l_B. Fig. 1a depicts the results when $l_U<l_M<l_B$, which is appropriate when $S<(\beta\mathrm{E}^{\frac{1}{2}})^{\frac{4}{3}}\delta^2$. Fig. 1b applies when $l_U<l_B<l_M$, or $(\beta\mathrm{E}^{\frac{1}{2}})^{\frac{4}{3}}\delta^2<S<\delta^2$. Finally, Fig. 1c shows the length scale relationships when $l_B<l_U<l_M$, or $\delta^2<S<1$. We note here that geostrophy in (4.1) requires $l>l_U$, a hydrostatic balance is (4.3) is possible only when $l>l_B$, and the neglect of the viscous term in the vorticity equation is consistent with $l>l_M$.

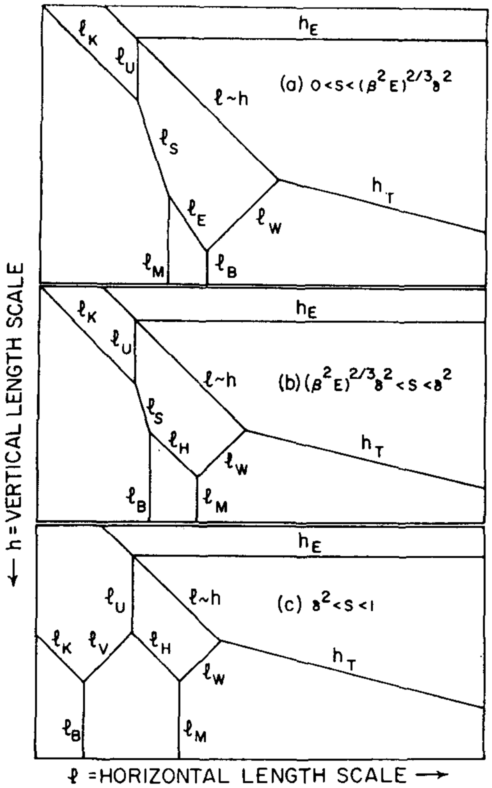

FIG. 1. Schematic graphs depicting the relationships between the vertical and horizontal length scales for the three regions in parameter space.

We now present the remaining scales for which $l<h$. Each of the following horizontal scales depends explicitly on h as well as on the dimensionless parameters S, δ, β and E.

d. Stewartson scale: $l_S=(\mathrm{E}\delta h)^{\frac{1}{3}}$

The relevant balances for this scale, which was first analyzed by Stewartson (1957), omit the viscous term in (4.1), the buoyancy term in (4.3), and the beta in (4.6). Thus, we must have $l>l_U$, $l<l_B$, and $l<l_M$. Since we have taken $l_U<l_M$, the only important restriction is that $l_U<l_B$. As a result, the Stewartson scale does not appear in Fig. 1c. The condition $l_S<l_B$ can be expressed as $S^3h^4<\mathrm{E}^2\delta^2$, which reduces to the criterion $S<\mathrm{E}^{\frac{2}{3}}$ of Barcilon and Pedlosky (1967) when we set $\delta=1$, $h=1$.

e. Hydrostatic-Lineykin scale: $l_H=hS^{\frac{1}{2}}$

Lineykin (1955), Barcilon and Pedlosky (1967), Blumsack and Barcilon (1971), and Blumsack (1972) discuss the dynamics and role of this scale. By omitting the viscous terms in (4.1) and (4.3) and the beta term in (4.6), we require $l>l_U$, $l>l_B$, and $l<l_M$. These conditions are not possible simultaneously when $l_M<l_B$, which explains the absence of l_H in Fig. 1a.

f. Viscous-hydrostatic scale: $l_V=\mathrm{E}h^{-1}S^{-\frac{1}{2}}$

Blumsack and Barcilon (1971) introduced this scale which neglects the Coriolis acceleration in (4.1), the viscous term in (4.3), and the middle term of the continuity equation (4.5). The requirements $l<l_U$ and $l>l_B$ are simultaneously possible only when $l_B<l_U<l_M$; the scale l_V appears only in Fig. 1c.

g. Stokes' scale: $l_K=\delta h$

Also introduced by Blumsack and Barcilon (1971), the dynamics of this scale include the neglect of the Coriolis acceleration in (4.1), the buoyancy term in (4.3), and the middle term of (4.5). Therefore, $l<l_U$ and $l<l_B$, which place no restriction on the ordering of l_M, l_U, and l_B. The Stokes' scale appears in all three parts of Fig. 1.

h. Western scale: $l_w=\mathrm{E}(\beta h^2S)^{-1}$

The balances are similar to the thermocline scale since we omit the viscous terms in (4.1), (4.3) and (4.6). Here, however, horizontal diffusion of heat compensates for the advection of the basic stratification. This scale is consistent when $l>l_U$, $l>l_M$, and $l>l_B$; there are no restrictions on the ordering of l_U, l_M, and l_B, and the western scale appears in all three graphs of Fig. 1. The name "western" is used because the boundary layer associated with l_w is used to satisfy a boundary condition at a western boundary, similar to Stommel's (1948) bottom friction layer.

i. Eastern scale: $l_\mathrm{E}=(\mathrm{E}\beta\delta^2h^2)^{\frac{1}{3}}$

The determining dynamics omit the viscous terms in (4.1) and (4.6) and the buoyancy term in (4.3). Hence, we require $l>l_U$, $l>l_M$, $l<l_B$, and the eastern scale can appear only in Fig. 1a. The associated boundary layer can adjust two boundary conditions at an east coast and only one at a west coast, just the reverse of the Carrier-Munk layer.

5. Case 3: $l \approx h$

We complete the length scale analysis by considering dynamics for which l and h are comparable in magnitude. The easiest way to deduce the criteria for such scales is to consider limiting cases of the dynamics discussed in Sections 3 and 4.

Consider the Ekman dynamics as l decreases through $\mathrm{E}^{\frac{1}{2}}$. Horizontal mixing becomes as important as vertical mixing. When $l<\mathrm{E}^{\frac{1}{2}}$, the Ekman scale no longer applies. Instead, horizontal mixing balances vertical mixing, resulting in $l\sim h$.

Next, consider the thermocline relationship as l decreases and the western scale as h decreases. When $h=(\mathrm{E}/\beta S)^{\frac{1}{3}}$, the vertical and horizontal scales for each of these become identical. For smaller values of h, vertical and horizontal diffusion of heat provide the

TABLE 1. Summary of length scales.

Name	Symbol	Thickness: Non-dimensional	Thickness: Dimensional	Dynamical balances: Horizontal momentum	Dynamical balances: Vertical momentum	Dynamical balances: Vertical vorticity
Ekman	h_E	$E^{\frac{1}{2}}$	$(A_V/f_0)^{\frac{1}{2}}$	$\mathbf{k}\times\mathbf{q}=E\mathbf{q}_{zz}$	—	—
Thermocline	h_T	$(El/\beta S)^{\frac{1}{4}}$	$(A_V f_0^2 l_*/\sigma\beta_* N^2)^{\frac{1}{4}}$	$v=p_n$	$p_z=T$	$w_z=\beta v$
Carrier-Munk	l_M	$(E/\beta)^{\frac{1}{3}}$	$(A_H/\beta_*)^{\frac{1}{3}}$	—	—	$\beta v=Ev_{nnn}$
Upwelling	l_U	$E^{\frac{1}{2}}$	$(A_H/f_0)^{\frac{1}{2}}$	$\mathbf{k}\times\mathbf{q}=E\mathbf{q}_{nn}$	—	—
Buoyancy	l_B	$(E^2\delta^2/S)^{\frac{1}{4}}$	$\sigma^{-\frac{1}{4}}(A_H/N)^{\frac{1}{2}}$	—	$0=T+E\delta^2 w_{nn}$	—
Stewartson	l_S	$(E\delta h)^{\frac{1}{3}}$	$(A_H h_*/f_0)^{\frac{1}{3}}$	$v=p_n$	$p_z=E\delta^2 w_{nn}$	$w_z+Ev_{nnn}=0$
Hydrostatic Lineykin	l_H	$S^{\frac{1}{2}}h$	$\sigma^{\frac{1}{2}}(N/f_0)h_*$	$v=p_n$	$p_z=T$	$w_z+Ev_{nnn}=0$
Viscous-hydrostatic	l_V	$Eh^{-1}S^{-\frac{1}{2}}$	$\sigma^{-\frac{1}{2}}(A_H/h_* N)$	$p_n=Eu_{nn}$	$p_z=T$	—
Stokes	l_K	δh	h_*	$p_n=Eu_{nn}$	$p_z=E\delta^2 w_{nn}$	—
Western	l_W	$E(\beta h^2 S)^{-1}$	$f_0^2 A_H/\sigma N^2\beta_* h_*^2$	$v=p_n$	$p_z=T$	$w_z=\beta v$
Eastern	l_E	$(\beta E\delta^2 h^2)^{\frac{1}{3}}$	$(A_H h_*^2\beta_*/f_0^2)^{\frac{1}{3}}$	$v=p_n$	$p_z=E\delta^2 w_{nn}$	$w_z=\beta v$

important dynamical balance, overwhelming any induced vertical advection of the imposed stratification. Therefore, there is a scale $l\approx h$ when $h<(E/\beta S)^{\frac{1}{3}}$. This is consistent with the result of Blumsack (1972), who noted that the potential vorticity was associated with equal horizontal and vertical scales when $\beta=0$.

In summary then, when $E^{\frac{1}{2}}<h<E^{\frac{1}{2}}(S\beta E^{\frac{1}{2}})^{-\frac{1}{3}}$, there is one independent scale satisfying $l\approx h$, and there are a total of three such scales when $h<E^{\frac{1}{2}}$.

6. Conclusions

We have seen how to unify much of the work on the steady linear theory of rotating stratified fluids by means of length scale relationships. Fig. 1 and the accompanying Table 1 contain the important results of this paper. We note that, in Fig. 1, there are four degrees of freedom for each value of h and each type of boundary (eastern or western), allowing us to satisfy four boundary conditions.

Fig. 1 incorporates all possible scales for which a vertical scale exists; the barotropic layers, which have been investigated by Beardsley (1968), are omitted.

We conclude now with an example of how to use Fig. 1 and the table to calculate dimensional length scales. Suppose we take $f_0=10^{-4}$ sec^{-1}, $\beta_*=10^{-13}$ cm^{-1} sec^{-1}, $A_H=10^6$ cm^2 sec^{-1}, $A_v=10^2$ cm^2 sec^{-1}, $N=10^{-3}$ sec^{-1} and $\sigma=1$. We calculate first the three internal dimensionless parameters: $\delta=10^{-2}$, $S=10^{-2}$ and $\beta E^{\frac{1}{2}}=10^{-4}$. The three reference horizontal scales are $l_{*U}=1$ km, $l_{*B}=0.3$ km, and $l_{*M}=20$ km; the Ekman scale is $h_{*E}=10$ m.

For this example, Fig. 1c applies since $S>\delta^2$. The upwelling scale is valid where h is less than $E^{\frac{1}{2}}S^{-\frac{1}{2}}$, which is ten times an Ekman depth. Thus, $l_{*U}=1$ km when 10 m$<h_*<$100 m. The Stokes' scale is valid until $l_K=l_B$, which occurs at $h=E^{\frac{1}{2}}(\delta^2 S)^{-\frac{1}{4}}$ or 30 times an Ekman depth; we have a Stokes' scale when $h_*<$300 m. Then we note from Fig. 1c that the scale l_v is relevant when 100 m$<h_*<$300 m and the buoyancy scale $l_{*B}=0.3$ km when $h_*>$300 m.

The hydrostatic-Lineykin scale ceases to be valid when $l_H=l_M$, or $h=E^{\frac{1}{2}}S^{-\frac{1}{2}}(\beta E^{\frac{1}{2}})^{-\frac{1}{3}}$ which is 200 times an Ekman depth. The scale l_{*H} exists for vertical scales between 100 m and 2000 m. Note that the Carrier-Munk scale is valid for vertical scales larger than 2000 m.

The thermocline scale has its minimum vertical scale when $l=E^{\frac{1}{2}}(S\beta E^{\frac{1}{2}})^{-\frac{1}{3}}$, or about 100 times an upwelling scale. The scale h_T is valid for horizontal scales larger than 100 km and has a minimum vertical scale of 100 times an Ekman depth, i.e., 1000 m. Then we use Fig. 1c to deduce the validity of the western scale when 1000 m$<h_*<$2000 m, and the existence of equal dimensionless scales, $l_*=100\ h_*$, when $h_*<$1000 m.

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