

A COUNTEREXAMPLE FOR THE OPTIMALITY OF KENDALL-CRANSTON COUPLING

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Abstract

We construct a Riemannian manifold where the Kendall-Cranston coupling of two Brownian particles does not maximize the coupling probability.

1 Introduction

Given two stochastic processes X_t and Y_t on a state space M , a coupling $Z_t = (Z_t^{(1)}, Z_t^{(2)})$ is a process on $M \times M$ so that $Z^{(1)}$ or $Z^{(2)}$ has the same distribution as X or Y respectively. Of particular interest in many applications is the distribution of the coupling time $T(Z) := \inf\{t > 0; Z_s^{(1)} = Z_s^{(2)} \text{ for all } s > t\}$. The goal is to make the coupling probability $\mathbb{P}[T(Z) \leq t]$ as large as possible by taking a suitable coupling. When X and Y are Brownian motions on a Riemannian manifold, Kendall [3] and Cranston [1] constructed a coupling by using the Riemannian geometry of the underlying space. Roughly speaking, under their coupling, infinitesimal motion $\Delta Y_t \in T_{Y_t}M$ at time t is given as a sort of reflection of ΔX_t via the minimal geodesic joining X_t and Y_t . Their coupling has the advantage of controlling the coupling probability by using geometric quantities such as the Ricci curvature. As a result, Kendall-Cranston coupling produces various estimates for heat kernels, harmonic maps, eigenvalues etc. under natural geometric assumptions.

On the other hand, there is the question of optimality. We say that a coupling Z of X and Y is *optimal* at time t if

$$\mathbb{P}[T(Z) \leq t] \geq \mathbb{P}[T(\tilde{Z}) \leq t]$$

holds for any other coupling \tilde{Z} . Though Kendall-Cranston coupling has a good feature as mentioned, in general there is no reason why it should be optimal.

The Kendall-Cranston coupling is optimal if the underlying space has a good symmetry. For example, in the case $M = \mathbb{R}^d$, the Kendall-Cranston coupling $(Z^{(1)}, Z^{(2)})$ is nothing but the mirror coupling. It means that $Z_t^{(2)} = \Psi(Z_t^{(1)})$ up to the time they meet, where Ψ is a reflection with respect to a hyperplane in \mathbb{R}^d so that $\Psi(X_0) = Y_0$. It is well known that the mirror coupling is optimal. Indeed, it is the only coupling which is optimal and Markovian [2]. More generally, the same result holds if there is a sort of reflection structure like a map Ψ on \mathbb{R}^d (see [4]).

In this paper, we show that the Kendall-Cranston coupling is *not* optimal in general.

Theorem 1.1 *For each $t > 0$, there is a complete Riemannian manifold M where the Kendall-Cranston coupling of two Brownian motions X and Y with specified starting points is not optimal.*

The proof of Theorem 1.1 is reduced to the case $t = 1$ by taking a scaling of Riemannian metric. We construct a manifold M in the next section and prove Theorem 1.1 in section 3.

Notation: Given a Riemannian manifold N we denote by $B_r^N(x)$ or simply $B_r(x)$ the open ball in N of radius r centered at x .

Given a Brownian motion $(X_t)_{t \geq 0}$ on N we denote by $\tau_A = \inf \{t > 0 : X_t \in A\}$ the hitting time of a set $A \subset N$. We remark that, throughout this article, τ_A always stands for the hitting time for the process $(X_t)_{t \geq 0}$ even when we consider a coupled motion $(X_t, Y_t)_{t \geq 0}$.

2 Construction of the manifold

We take three parameter $R > 0, \zeta > 0$ and $\delta > 0$ such that $\zeta < R/4$ and $\delta < \zeta/3$. Let $C = \mathbb{R} \times S^1$ be a cylinder with a flat metric such that the length of a circle S^1 equals ζ . For simplicity of notation, we write $z = (r, \theta)$ for $z \in C$ where $r \in \mathbb{R}$ and $\theta \in (-\zeta/2, \zeta/2]$ such that the Riemannian metric is written as $dr^2 + d\theta^2$. If appropriate, any $\theta \in \mathbb{R}$ will be regarded mod ζ and considered as element of $(-\zeta/2, \zeta/2]$. We put

$$M_1 := ([-R, \infty) \times S^1) \setminus B_\delta^C((0, \zeta/2)) \subset C$$

and write $\partial_{1,0} := \partial B_\delta^C((0, \zeta/2))$ as well as $\partial_{1,2} := \{-R\} \times S^1$ (see Fig.1). Let C' be a copy of C . Then we put analogously

$$M_2 := ((-\infty, R] \times S^1) \setminus B_\delta^{C'}((0, 0)) \subset C'$$

and write $\partial_{2,0} := \partial B_\delta^{C'}((0, 0))$ as well as $\partial_{2,1} := \{R\} \times S^1$. Let $M_0 = S^1 \times [-1, 1]$ be another cylinder. We write $z \in M_0$ by $z = (\varphi, r)$ where $\varphi \in (0, 2\pi]$ and $r \in [-1, 1]$. Now we define a C^∞ -manifold M (see Fig.2) by $M = M_0 \sqcup M_1 \sqcup M_2 / \sim$, where the identification “ \sim ” means

$$\begin{aligned} \partial_{1,2} \ni (-R, \theta) &\sim (R, \zeta/2 - \theta) \in \partial_{2,1} && \text{for } \theta \in (-\zeta/2, \zeta/2], \\ \partial_{1,0} \ni (\delta \cos \varphi, \zeta/2 - \delta \sin \varphi) &\sim (\varphi, -1) \in M_0 && \text{for } \varphi \in (0, 2\pi], \\ \partial_{2,0} \ni (\delta \cos \varphi, \delta \sin \varphi) &\sim (\varphi, 1) \in M_0 && \text{for } \varphi \in (0, 2\pi]. \end{aligned}$$

We endow M with a C^∞ -metric g such that (M, g) becomes a complete Riemannian manifold and:

- (i) $g|_{M_1}$ coincides with the metric on M_1 inherited from C ,

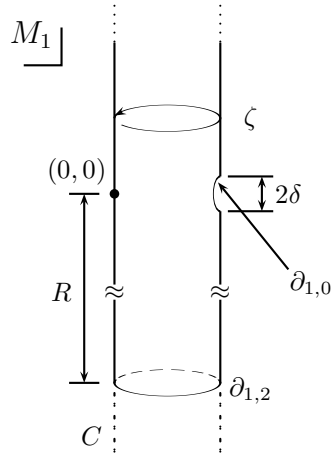


Fig.1

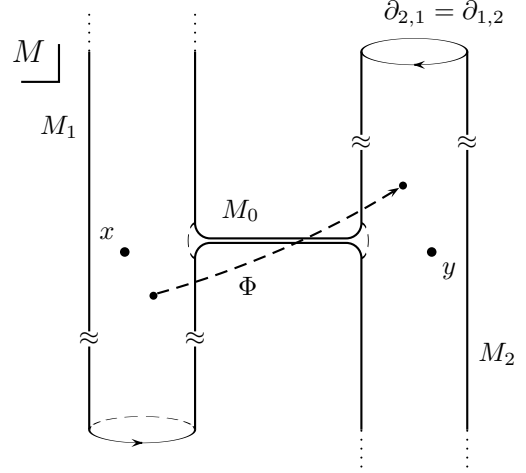


Fig.2

- (ii) $g|_{M_2}$ coincides with the metric on M_2 inherited from C' ,
- (iii) $g|_{M_0}$ is invariant under maps $(\theta, r) \mapsto (\theta, -r)$ and $(\theta, r) \mapsto (\theta + \varphi, r)$ on M_0 ,
- (iv) $d((-1, 0), (1, 0)) = \zeta$ for $z_1 = (-1, 0), z_2 = (1, 0) \in M_0$

where d is the distance function on M .

3 Comparison of coupling probabilities

Let M be the manifold constructed above (with suitably chosen parameters R, ζ and δ) and fix two points $x = (0, \zeta/6) \in M_1$ and $y = (0, \zeta/3) \in M_2$.

In this paper, the construction of Kendall-Cranston coupling is due to von Renesse [5]. We will try to explain his idea briefly. His approach is based on the approximation by coupled geodesic random walks $\{\hat{X}^k\}_{k \in \mathbb{N}}$ starting in (x, y) whose sample paths are piecewise geodesic. Given their positions after $(n - 1)$ -th step, one determines its next direction ξ_n according to the uniform distribution on a small sphere in the tangent space and the other does it as the reflection of ξ_n along a minimal geodesic joining their present positions. We obtain a Kendall-Cranston coupling (X_t, Y_t) by taking the (subsequential) limit in distribution of them. We will construct another Brownian motion $(\hat{Y}_t)_{t \geq 0}$ on M starting in y , again defined on the same probability space as we construct (X_t, Y_t) such that

$$\mathbb{P}(X \text{ and } Y \text{ meet before time } 1) < \mathbb{P}(X \text{ and } \hat{Y} \text{ meet before time } 1).$$

In other words, if \mathbb{Q} denotes the distribution of (X, Y) and $\hat{\mathbb{Q}}$ denotes the distribution of (X, \hat{Y}) then

Proposition 3.1 $\mathbb{Q}[T \leq 1] < \hat{\mathbb{Q}}[T \leq 1]$.

Our construction of the process \hat{Y} will be as follows. We define a map $\Phi : M_1 \rightarrow M_2$ by $\Phi((r, \theta)) = (-r, \zeta/2 - \theta)$ and then put

- (i) $\hat{Y}_t = \Phi(X_t)$ for $t \in [0, \tau_{\partial_{1,0}} \wedge T)$;
- (ii) X and \hat{Y} move independently for $t \in [\tau_{\partial_{1,0}}, T)$ in case $\tau_{\partial_{1,0}} < T$;
- (iii) $\hat{Y}_t = X_t$ for $t \in [T, \infty)$.

Note that $\tau_{\partial_{1,2}} = T$ holds when $\tau_{\partial_{1,2}} \leq \tau_{\partial_{1,0}}$ under $\hat{\mathbb{Q}}$.

Set $H = S^1 \times \{0\} \subset M_0 \subset M$. For $z_1, z_2 \in M$ and $A \subset M$, minimal length of paths joining z_1 and z_2 which intersect A is denoted by $d(z_1, z_2; A)$. We define a constant L_0 by

$$L_0 := \inf \left\{ L \in (\delta, R]; d(z_1, z_2; H) \geq d(z_1, z_2; \partial_{1,2}) \text{ for some } z_1 = (L, \theta) \in M_1, z_2 = (L, \zeta/2 - \theta) \in M_2 \right\}.$$

Lemma 3.2 $R - \zeta < L_0 < R$.

Proof. First we show $L_0 < R$. Let $z_1 = (R, 0) \in M_1$ and $z_2 = (R, \zeta/2) \in M_2$. Obviously there is a path of length $2R$ joining z_1 and z_2 across $\partial_{1,2}$. Thus we have $d(z_1, z_2; \partial_{1,2}) \leq 2R$. By symmetry of M ,

$$d(z_1, z_2; H) = 2d(z_1, H) = 2 \left(d(z_1, \partial_{1,0}) + \frac{\zeta}{2} \right) = 2 \left(\sqrt{R^2 + \zeta^2/4} - \delta \right) + \zeta > 2R,$$

where the second equality follows from the third and fourth properties of g and the last inequality follows from the choice of δ . These estimates imply $L_0 < R$.

Next, let $z'_1 = (R - \zeta, \theta) \in M_1$ and $z'_2 = (R - \zeta, \zeta/2 - \theta) \in M_2$. In the same way as observed above, we have

$$d(z'_1, z'_2; H) = 2 \left(\sqrt{(R - \zeta)^2 + \theta^2} - \delta \right) + \zeta \leq 2R - 2\delta.$$

Note that the length of a path joining z'_1 and z'_2 which intersects both of $\partial_{1,2}$ and H is obviously greater than $d(z'_1, z'_2; H)$. Thus, in estimating $d(z'_1, z'_2; \partial_{1,2})$, it is sufficient to consider all paths joining z'_1 and z'_2 across $\partial_{1,2}$ which do not intersect H . Such a path must intersect both $\{\delta\} \times S^1 \subset M_1$ and $\{-\delta\} \times S^1 \subset M_1$ (see Fig.3). Thus we have

$$\begin{aligned} d(z'_1, z'_2; \partial_{1,2}) &\geq d(z'_1, \{\delta\} \times S^1) + d(\{-\delta\} \times S^1, \partial_{1,2}) + d(\partial_{2,1}, z'_2) \\ &\geq (R - \zeta - \delta) + (R - \delta) + \zeta \\ &= 2R - 2\delta. \end{aligned}$$

Hence, the conclusion follows. \square

Set $M'_1 := M_1 \cap [-L_0, L_0] \times S^1 \subset C$ and $M'_2 := M_2 \cap [-L_0, L_0] \times S^1 \subset C'$. We define a submanifold $M' \subset M$ with boundary by $M' = M_0 \sqcup M'_1 \sqcup M'_2 / \sim$ (see Fig.4). Let $\Psi : M' \rightarrow M'$ be the reflection with respect to H . For instance, for $z = (r, \theta) \in M'_1$, $\Psi(z) = (r, \zeta/2 - \theta) \in M'_2$. Note that Ψ is an isometry, $\Psi \circ \Psi = \text{id}$ and $\{z \in M'; \Psi(z) = z\} = H$.

Let X' be the given Brownian motion starting in x and now stopped at $\partial M'$, i.e. $X'_t = X_{t \wedge \tau_{\partial M'}}$. Define a stopped Brownian motion starting in y by $Y'_t = \Psi(X'_t)$ for $t < \tau_H$ and by $Y_t = X_t$ for $t \geq \tau_H$ (that is, the two Brownian particles coalesce after τ_H). Then we can prove the following lemma.

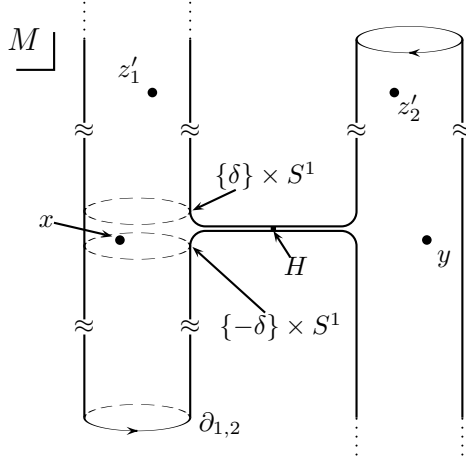


Fig.3

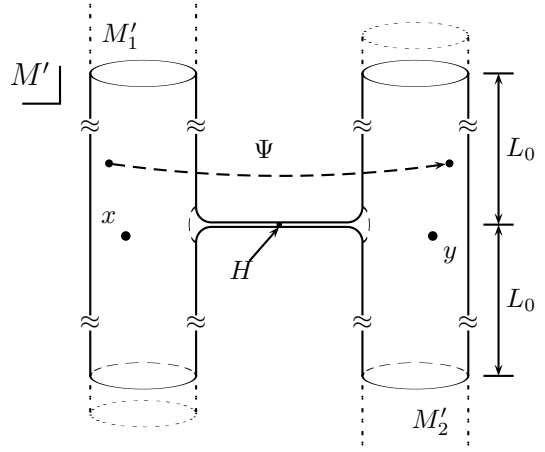


Fig.4

Lemma 3.3 *The law of $(X_{t \wedge \tau_{\partial M'}}, Y_{t \wedge \tau_{\partial M'}})_{t \geq 0}$ coincides with that of $(X'_t, Y'_t)_{t \geq 0}$.*

Proof. Note that the minimal geodesic in M joining z and $\Psi(z)$ must intersect H for every $z \in M'$ by virtue of the choice of L_0 . Thus, by the symmetry of M' with respect to H , coupled geodesic random walks $\hat{\Xi}^k$ are in E defined by

$$E := \left\{ (z^{(1)}, z^{(2)}) \in C([0, \infty) \rightarrow M \times M) ; z_t^{(2)} = \Psi(z_t^{(1)}) \text{ before } z_t^{(1)} \text{ exits from } M' \right\}$$

(cf. Theorem 5.1 in [4]). Since E is closed in $C([0, \infty) \rightarrow M \times M)$, $(X, Y) \in E$ holds \mathbb{P} -almost surely by taking a (subsequential) limit in distribution of $\{\hat{\Xi}^k\}_{k \in \mathbb{N}}$. Thus the conclusion follows. \square

We now begin to show Proposition 3.1. First we give a lower estimate of $\hat{\mathbb{Q}}[T \leq 1]$. Let

$$\begin{aligned} \gamma(a) &:= \{(x_1, x_2) \in \mathbb{R}^2 ; x_2 = a\}, \quad a \in \mathbb{R}, \\ A(\delta) &:= \bigcup_{n \in \mathbb{Z}} B_\delta^{\mathbb{R}^2} \left(\left(\zeta \left(n + \frac{1}{3} \right), 0 \right) \right). \end{aligned}$$

The remark after the definition of $\hat{\mathbb{Q}}$ implies

$$\hat{\mathbb{Q}}[T \leq 1] \geq \hat{\mathbb{Q}}[T \leq 1, \tau_{\partial_{1,2}} < \tau_{\partial_{1,0}}] = \hat{\mathbb{Q}}[\tau_{\partial_{1,2}} \leq 1 \wedge \tau_{\partial_{1,0}}].$$

By lifting X_t to \mathbb{R}^2 , the universal cover of C ,

$$\begin{aligned} \hat{\mathbb{Q}}[\tau_{\partial_{1,2}} \leq 1 \wedge \tau_{\partial_{1,0}}] &= \mathbb{P}^{\mathbb{R}^2}[\tau_{\gamma(R)} \leq 1 \wedge \tau_{A(\delta)}] \\ &\geq \mathbb{P}^{\mathbb{R}^2}[\tau_{\gamma(R)} \leq 1, \tau_{A(\delta)} > 1] \\ &\geq \mathbb{P}^{\mathbb{R}}[\tau_R \leq 1] - \mathbb{P}^{\mathbb{R}^2}[\tau_{A(\delta)} \leq 1]. \end{aligned} \tag{3.1}$$

Here $\mathbb{P}^{\mathbb{R}^2}$ and $\mathbb{P}^{\mathbb{R}}$ denote the usual Wiener measure for Brownian motion (starting at the origin) on \mathbb{R}^2 or \mathbb{R} , resp. For simplicity, we write τ_R instead of $\tau_{\{R\}}$.

Next we give an upper estimate of $\mathbb{Q}[T \leq 1]$. Let $E := \{\tau_{\partial_{1,0}} < 1 \wedge \tau_{\partial M'}\}$. Then

$$\mathbb{Q}[E] = \mathbb{P}[E] \leq \mathbb{P}^{\mathbb{R}^2}[\tau_{A(\delta)} < 1].$$

Note that, on $\{T \leq 1\} \cap E^c$, X must hit $\partial M'$ before T . It means

$$\mathbb{Q}[\{T \leq 1\} \cap E^c] = \mathbb{Q}[\{\tau_{\partial M'} < T \leq 1\} \cap E^c].$$

By Lemma 3.3, $Y_{\tau_{\partial M'}} = \Psi(X_{\tau_{\partial M'}})$ on E^c under \mathbb{Q} . In order to collide two Brownian motions starting at $X_{\tau_{\partial M'}}$ and $\Psi(X_{\tau_{\partial M'}})$, either of them must escape from the flat cylinder of length $2(L_0 - \delta)$ where its starting point has distance $L_0 - \delta$ from the boundary. This observation together with the strong Markov property yields

$$\begin{aligned} \mathbb{Q}[\{\tau_{\partial M'} < T \leq 1\} \cap E^c] &= \mathbb{Q}\left[\mathbb{Q}_{(X_{\tau_{\partial M'}}, \Psi(X_{\tau_{\partial M'}}))}[T \leq 1 - s] \Big|_{s=\tau_{\partial M'}}; \tau_{\partial M'} < 1 \wedge \tau_{\partial_{1,0}}\right] \\ &\leq 2\mathbb{Q}\left[\mathbb{P}^{\mathbb{R}^2}[\tau_{-(L_0-\delta)} \wedge \tau_{L_0-\delta} < 1 - s] \Big|_{s=\tau_{\partial M'}}; \tau_{\partial M'} < 1 \wedge \tau_{\partial_{1,0}}\right] \\ &\leq 4\mathbb{Q}\left[\mathbb{P}^{\mathbb{R}^2}[\tau_{L_0-\delta} < 1 - s] \Big|_{s=\tau_{\partial M'}}; \tau_{\partial M'} < 1 \wedge \tau_{\partial_{1,0}}\right]. \end{aligned}$$

By Lemma 3.2 and the definition of ζ and δ , we have $L_0 - \delta \geq R - \zeta - \delta > 2R/3$. Thus

$$\begin{aligned} \mathbb{Q}\left[\mathbb{P}^{\mathbb{R}^2}[\tau_{L_0-\delta} < 1 - s] \Big|_{s=\tau_{\partial M'}}; \tau_{\partial M'} < 1 \wedge \tau_{\partial_{1,0}}\right] &\leq 2 \exp\left(-\frac{(L_0 - \delta)^2}{2}\right) \mathbb{P}[\tau_{\partial M'} < 1 \wedge \tau_{\partial_{1,0}}] \\ &\leq 2 \exp\left(-\frac{2R^2}{9}\right) \mathbb{P}[\tau_{\partial M'} < 1 \wedge \tau_{\partial_{1,0}}]. \end{aligned}$$

By lifting X_t to \mathbb{R}^2 , we have

$$\mathbb{P}[\tau_{\partial M'} < 1 \wedge \tau_{\partial_{1,0}}] \leq \mathbb{P}^{\mathbb{R}^2}[\tau_{\gamma(L_0)} \wedge \tau_{\gamma(-L_0)} < 1 \wedge \tau_{A(\delta)}] \leq 2\mathbb{P}^{\mathbb{R}^2}[\tau_{L_0} < 1] \leq 2\mathbb{P}^{\mathbb{R}^2}[\tau_{R-\zeta} < 1].$$

Here the last inequality follows from Lemma 3.2. Consequently, we obtain

$$\mathbb{Q}[T \leq 1] \leq \mathbb{P}^{\mathbb{R}^2}[\tau_{A(\delta)} < 1] + 16 \exp\left(-\frac{2R^2}{9}\right) \mathbb{P}^{\mathbb{R}^2}[\tau_{R-\zeta} < 1]. \quad (3.2)$$

Now take $R > 3\sqrt{2\log 2}$. After that we choose ζ so small that $\mathbb{P}^{\mathbb{R}^2}[\tau_{R-\zeta} < 1] \approx \mathbb{P}^{\mathbb{R}^2}[\tau_R < 1]$. Finally we choose δ so small that $\mathbb{P}^{\mathbb{R}^2}[\tau_{A(\delta)} < 1] \approx 0$. Then Proposition 3.1 follows from (3.1) and (3.2).

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