Elect. Comm. in Probab. **13** (2008), 653–659

ELECTRONIC COMMUNICATIONS in PROBABILITY

THE LAW OF THE HITTING TIMES TO POINTS BY A STABLE LÉVY PROCESS WITH NO NEGATIVE JUMPS

GORAN PESKIR
School of Mathematics, The University of Manchester
Oxford Road, Manchester M13 9PL, United Kingdom
email: goran@maths.man.ac.uk

Submitted March 26, 2008, *accepted in final form* December 4, 2008

AMS 2000 Subject classification: Primary 60G52, 45D05. Secondary 60J75, 45E99, 26A33.
Keywords: Stable Lévy process with no negative jumps, spectrally positive, first hitting time to a point, first passage time over a point, supremum process, a Chapman-Kolmogorov equation of Volterra type, Laplace transform, the Wiener-Hopf factorisation.

Abstract

Let $X=(X_t)_{t\ge0}$ be a stable Lévy process of index $\alpha\in(1,2)$ with the Lévy measure $\nu(dx)=(c/x^{1+\alpha})I_{(0,\infty)}(x)\,dx$ for $c>0$, let $x>0$ be given and fixed, and let $\tau_x=\inf\{\,t>0:X_t=x\,\}$ denote the first hitting time of X to x. Then the density function f_{τ_x} of τ_x admits the following series representation:

$$f_{\tau_x}(t)=\frac{x^{\alpha-1}}{\pi(c\,\Gamma(-\alpha)\,t)^{2-1/\alpha}}\sum_{n=1}^{\infty}\Bigg[(-1)^{n-1}\sin(\pi/\alpha)\,\frac{\Gamma(n-1/\alpha)}{\Gamma(\alpha n-1)}\Big(\frac{x^{\alpha}}{c\,\Gamma(-\alpha)\,t}\Big)^{n-1}-\sin\Big(\frac{n\pi}{\alpha}\Big)\,\frac{\Gamma(1+n/\alpha)}{n!}\Big(\frac{x^{\alpha}}{c\,\Gamma(-\alpha)\,t}\Big)^{(n+1)/\alpha-1}\Bigg]$$

for $t>0$. In particular, this yields $f_{\tau_x}(0+)=0$ and

$$f_{\tau_x}(t)\sim\frac{x^{\alpha-1}}{\Gamma(\alpha-1)\,\Gamma(1/\alpha)}\,(c\,\Gamma(-\alpha)\,t)^{-2+1/\alpha}$$

as $t\to\infty$. The method of proof exploits a simple identity linking the law of τ_x to the laws of X_t and $\sup_{0\le s\le t}X_s$ that makes a Laplace inversion amenable. A simpler series representation for f_{τ_x} is also known to be valid when $x<0$.

1 Introduction

If a Lévy process $X=(X_t)_{t\ge0}$ jumps upwards, then it is much harder to derive a closed form expression for the distribution function of its first passage time $\tau_{(x,\infty)}$ over a strictly positive level x, and in the existing literature such expressions seem to be available only when X has no positive jumps (unless the Lévy measure is discrete). A notable exception to this rule is the recent paper [1] where an explicit series representation for the density function of $\tau_{(x,\infty)}$ was derived when X is a stable Lévy process of index $\alpha \in (1,2)$ having the Lévy measure given by $\nu(dx) = (c/x^{1+\alpha}) I_{(0,\infty)}(x)\,dx$ with $c > 0$ given and fixed. This was done by performing a time-space inversion of the Wiener-Hopf factor corresponding to the Laplace transform of $(t, y) \mapsto \mathsf{P}(S_t > y)$ where $S_t = \sup_{0 \le s \le t} X_s$ for $t > 0$ and $y > 0$.

Motivated by this development our purpose in this note is to search for a similar series representation associated with the first hitting time τ_x of X to a strictly positive level x itself. Clearly, since X jumps upwards and creeps downwards, τ_x will happen strictly after $\tau_{(x,\infty)}$, and since X reaches x by creeping through it independently from the past prior to $\tau_{(x,\infty)}$, one can exploit known expressions for the latter portion of the process and derive the Laplace transform for $(t, y) \mapsto \mathsf{P}(\tau_y > t)$. This was done in [6, Theorem 1] and is valid for any Lévy process with no negative jumps (excluding subordinators). A direct Laplace inversion of the resulting expression appears to be difficult, however, and we show that a simple (Chapman-Kolmogorov type) identity which links the law of τ_x to the laws of X_t and S_t proves helpful in this context (due largely to the scaling property of X). It enables us to connect the old result of [13] with the recent result of [1] through an additive factorisation of the Laplace transform of $(t, y) \mapsto \mathsf{P}(\tau_y > t)$. This makes the Laplace inversion possible term by term and yields an explicit series representation for the density function of τ_x.

2 Result and proof

1. Let $X = (X_t)_{t \ge 0}$ be a stable Lévy process of index $\alpha \in (1,2)$ whose characteristic function is given by

$$\mathsf{E} e^{i\lambda X_t} = \exp\left(t \int_0^\infty (e^{i\lambda x} - 1 - i\lambda x)\, \frac{dx}{\Gamma(-\alpha)\, x^{1+\alpha}} \right) = e^{t(-i\lambda)^\alpha} \tag{1}$$

for $\lambda \in I\!R$ and $t \ge 0$. It follows that the Laplace transform of X is given by

$$\mathsf{E} e^{-\lambda X_t} = e^{t\lambda^\alpha} \tag{2}$$

for $\lambda \ge 0$ and $t \ge 0$ (the left-hand side being $+\infty$ for $\lambda < 0$). From (2) we see that the Laplace exponent of X equals $\psi(\lambda) = \lambda^\alpha$ for $\lambda \ge 0$ and $\varphi(p) := \psi^{-1}(p) = p^{1/\alpha}$ for $p \ge 0$.

2. The following properties of X are readily deduced from (1) and (2) using standard means (see e.g. [2] and [9]): the law of $(X_{ct})_{t \ge 0}$ is the same as the law of $(c^{1/\alpha} X_t)_{t \ge 0}$ for each $c > 0$ given and fixed (scaling property); X is a martingale with $\mathsf{E} X_t = 0$ for all $t \ge 0$; X jumps upwards (only) and creeps downwards (in the sense that $\mathsf{P}(X_{\tau_{(-\infty,x)}} = x) = 1$ for $x < 0$ where $\tau_{(-\infty,x)} = \inf\{\, t > 0 : X_t < x \,\}$ is the first passage time of X over x); X has sample paths of unbounded variation; X oscillates from $-\infty$ to $+\infty$ (in the sense that $\liminf_{t\to\infty} X_t = -\infty$ and $\limsup_{t\to\infty} X_t = +\infty$ both a.s.); the starting point 0 of X is regular (for both $(-\infty, 0)$ and $(0, +\infty)$). Note that the constant $c = 1/\Gamma(-\alpha)$ in the Lévy measure $\nu(dx) = (c/x^{1+\alpha})\,dx$ of X is chosen/fixed for convenience so that X converges in law to $\sqrt{2}B$ as $\alpha \uparrow 2$ where B is a standard Brownian motion, and all the facts throughout can be extended to a general constant $c > 0$ using the scaling property of X.

3. Letting f_{X_1} denote the density function of X_1, the following series representation is known to be

valid (see e.g. (14.30) in [14, p. 88]):

$$f_{X_1}(x) = \sum_{n=1}^{\infty} \frac{\sin(n\pi/\alpha)}{\pi} \frac{\Gamma(1+n/\alpha)}{n!} x^{n-1} \tag{3}$$

for $x \in \mathbb{R}$. Setting $S_1 = \sup_{0\le t\le 1} X_t$ and letting f_{S_1} denote the density function of S_1, the following series representation was recently derived in [1, Theorem 1]:

$$f_{S_1}(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sin(\pi/\alpha)}{\pi} \frac{\Gamma(n-1/\alpha)}{\Gamma(\alpha n-1)} x^{\alpha n-2} \tag{4}$$

for $x > 0$. Clearly, the series representations (3) and (4) extend to $t \neq 1$ by the scaling property of X since $X_t \overset{\text{law}}{=} t^{1/\alpha} X_1$ and $S_t := \sup_{0\le s\le t} X_s \overset{\text{law}}{=} t^{1/\alpha} S_1$ for $t > 0$.

4. Consider the first hitting time of X to x given by

$$\tau_x = \inf\{\, t > 0 : X_t = x \,\} \tag{5}$$

for $x > 0$. Then it is known (see (2.16) in [6]) that the time-space Laplace transform equals

$$\int_0^\infty e^{-\lambda x}\, \mathsf{E}(e^{-p\tau_x})\, dx = \frac{1}{\lambda-\varphi(p)} + \frac{1}{\varphi'(p)\,(p-\psi(\lambda))} = \frac{1}{\lambda - p^{1/\alpha}} + \frac{\alpha}{p^{-1+1/\alpha}(p-\lambda^\alpha)} \tag{6}$$

for $\lambda > 0$ and $p > 0$. Note that this can be rewritten as follows:

$$\int_0^\infty e^{-pt} dt \int_0^\infty e^{-\lambda x}\, \mathsf{P}(\tau_x > t)\, dx = \frac{1}{\lambda p} + \frac{1}{p(p^{1/\alpha}-\lambda)} - \frac{\alpha}{p^{1/\alpha}(p-\lambda^\alpha)} \tag{7}$$

for $\lambda > 0$ and $p > 0$.

Let $\mathbb{L}_p^{-1}$ denote the inverse Laplace transform with respect to p. Using that $1/(p(p^{1/\alpha}-\lambda)) = \sum_{n=1}^\infty \lambda^{n-1}/p^{1+n/\alpha}$ and $\mathbb{L}_p^{-1}[1/p^a] = t^{a-1}/\Gamma(a)$ for $a > 0$, it is easily verified that

$$\mathbb{L}_p^{-1}\Big[\frac{1}{p(p^{1/\alpha}-\lambda)}\Big](t) = \frac{1}{\lambda}\Big[E_{1/\alpha}(\lambda t^{1/\alpha}) - 1\Big] \tag{8}$$

for $t > 0$ where $E_a(x) = \sum_{n=0}^\infty x^n/\Gamma(an+1)$ denotes the Mittag-Leffler function. On the other hand, by (3) in [8, p. 238] we find

$$\mathbb{L}_p^{-1}\Big[\frac{1}{p^{1/\alpha}(p-\lambda^\alpha)}\Big](t) = \frac{1}{\Gamma(1/\alpha)} \frac{e^{\lambda^\alpha t}}{\lambda} \gamma(1/\alpha, \lambda^\alpha t) \tag{9}$$

for $t > 0$ where $\gamma(a,x) = \int_0^x y^{a-1} e^{-y}\, dy$ denotes the incomplete gamma function. Combining (7) with (8) and (9) we get

$$\begin{aligned}\int_0^\infty e^{-\lambda x}\, \mathsf{P}(\tau_x > t)\, dx &= \frac{1}{\lambda} E_{1/\alpha}(\lambda t^{1/\alpha}) - \frac{\alpha}{\Gamma(1/\alpha)} \frac{e^{\lambda^\alpha t}}{\lambda} \gamma(1/\alpha, \lambda^\alpha t) \\ &= \frac{\alpha}{\lambda}\Big[\frac{\alpha}{\Gamma(1/\alpha)} e^{\lambda^\alpha t} \int_{\lambda t^{1/\alpha}}^\infty e^{-z^\alpha} dz - e^{\lambda^\alpha t} + \frac{1}{\alpha} E_{1/\alpha}(\lambda t^{1/\alpha})\Big]\end{aligned} \tag{10}$$

for $\lambda > 0$ and $t > 0$.
The first and the third term on the right-hand side of (10) may now be recognised as the Laplace transforms of particular functions considered in [1] and [13] respectively (recall also (2.2) above). The proof of the following theorem provides a simple probabilistic argument (of Chapman-Kolmogorov type) for this additive factorisation (see Remark 1 below).

Theorem 1. *Let $X = (X_t)_{t\geq 0}$ be a stable Lévy process of index $\alpha \in (1,2)$ with the Lévy measure $\nu(dx) = (c/x^{1+\alpha})I_{(0,\infty)}(x)\,dx$ for $c > 0$, let $x > 0$ be given and fixed, and let τ_x denote the first hitting time of X to x. Then the density function f_{τ_x} of τ_x admits the following series representation:*

$$f_{\tau_x}(t) = \frac{x^{\alpha-1}}{\pi(c\,\Gamma(-\alpha)\,t)^{2-1/\alpha}} \sum_{n=1}^{\infty} \Bigg[(-1)^{n-1} \sin(\pi/\alpha)\, \frac{\Gamma(n-1/\alpha)}{\Gamma(\alpha n-1)} \Big(\frac{x^\alpha}{c\,\Gamma(-\alpha)\,t} \Big)^{n-1} - \sin\Big(\frac{n\pi}{\alpha}\Big) \frac{\Gamma(1+n/\alpha)}{n!} \Big(\frac{x^\alpha}{c\,\Gamma(-\alpha)\,t} \Big)^{(n+1)/\alpha-1} \Bigg] \tag{11}$$

for $t > 0$. In particular, this yields:

$$f_{\tau_x}(t) = o(1) \quad as \;\; t \downarrow 0\,; \tag{12}$$

$$f_{\tau_x}(t) \sim \frac{x^{\alpha-1}}{\Gamma(\alpha-1)\,\Gamma(1/\alpha)}\,(c\,\Gamma(-\alpha)\,t)^{-2+1/\alpha} \quad as \;\; t \uparrow \infty\,. \tag{13}$$

Proof. It is no restriction to assume below that $c = 1/\Gamma(-\alpha)$ as the general case follows by replacing t in (11) with $c\,\Gamma(-\alpha)\,t$ for $t > 0$.

Since X creeps downwards, we can apply the strong Markov property of X at τ_x, use the additive character of X, and exploit the scaling property of X to find

$$\begin{aligned} \mathsf{P}(S_1 > x) &= \mathsf{P}(S_1 > x, X_1 > x) + \mathsf{P}(S_1 > x, X_1 \leq x) \\ &= \mathsf{P}(X_1 > x) + \int_0^1 \mathsf{P}(X_1 \leq x \,|\, \tau_x = t)\, F_{\tau_x}(dt) \\ &= \mathsf{P}(X_1 > x) + \int_0^1 \mathsf{P}(x + X_{1-t} \leq x)\, F_{\tau_x}(dt) \\ &= \mathsf{P}(X_1 > x) + \int_0^1 \mathsf{P}((1-t)^{1/\alpha} X_1 \leq 0)\, F_{\tau_x}(dt) \\ &= \mathsf{P}(X_1 > x) + (1/\alpha)\,\mathsf{P}(\tau_x \leq 1) \end{aligned} \tag{14}$$

where we also use that $\mathsf{P}(X_1 \leq 0) = 1/\alpha$ and F_{τ_x} denotes the distribution function of τ_x. Note that the second equality in (14) represents a Chapman-Kolmogorov equation of Volterra type (see [11, Section 2] for a formal justification and a brief historical account of the argument). Since $\tau_x \overset{\text{law}}{=} x^\alpha\, \tau_1$ by the scaling property of X, we find that (14) reads

$$\mathsf{P}(S_1 > x) = \mathsf{P}(X_1 > x) + (1/\alpha)\, F_{\tau_1}(1/x^\alpha) \tag{15}$$

for $x > 0$. Hence we see that F_{τ_1} is absolutely continuous (cf. [10] for a general result on the absolute continuity) and by differentiating in (15) we get

$$f_{\tau_1}(1/x^\alpha) = x^{1+\alpha}\,\big[f_{S_1}(x) - f_{X_1}(x)\big] \tag{16}$$

for $x > 0$. Letting $t = 1/x^\alpha$ we find that

$$f_{\tau_1}(t) = t^{-1-1/\alpha}\big[f_{S_1}(t^{-1/\alpha}) - f_{X_1}(t^{-1/\alpha})\big] \tag{17}$$

for $t > 0$. Hence (11) with $x=1$ follows by (3) and (4) above. Moreover, since $\tau_x \overset{\text{law}}{=} x^\alpha \tau_1$ we see that $f_{\tau_x}(t) = x^{-\alpha} f_{\tau_1}(t x^{-\alpha})$ and this yields (11) with $x > 0$.

It is known that $f_{X_1}(x) \sim c\, x^{-1-\alpha}$ as $x \to \infty$ (see e.g. (14.34) in [14, p. 88]) and likewise $f_{S_1}(x) \sim c\, x^{-1-\alpha}$ as $x \to \infty$ (see [1, Corollary 3] and [7] for a proof). From (16) we thus see that $f_{\tau_1}(0+) = 0$ and hence $f_{\tau_x}(0+) = 0$ for all $x > 0$ as claimed in (12). The asymptotic relation (13) follows directly from (11) using the reflection formula $\Gamma(1-z)\Gamma(z) = \pi/\sin \pi z$ for $z \in \mathbb{C} \setminus \mathbb{Z}$. This completes the proof. □

Remark 1. Note that (14) can be rewritten as follows:

$$(1/\alpha)\,\mathsf{P}(\tau_x > 1) = 1/\alpha + F_{S_1}(x) - F_{X_1}(x) = F_{S_1}(x) - \big(F_{X_1}(x) - F_{X_1}(0)\big) \tag{18}$$

for $x > 0$, and from (2.30) in [1] we know that

$$\int_0^\infty e^{-\lambda x} f_{S_1}(x)\, dx = e^{\lambda^\alpha} \int_\lambda^\infty e^{-z^\alpha}\, dz \tag{19}$$

for $\lambda > 0$. In view of (10) this implies that

$$\int_0^\infty e^{-\lambda x} f_{X_1}(x)\, dx = e^{\lambda^\alpha} - \frac{1}{\alpha} E_{1/\alpha}(\lambda) \tag{20}$$

for $\lambda > 0$. Recalling (2) we see that (20) is equivalent to

$$\int_{-\infty}^0 e^{-\lambda x} f_{X_1}(x)\, dx = \frac{1}{\alpha} E_{1/\alpha}(\lambda) \tag{21}$$

for $\lambda > 0$. An explicit series representation for f in place of f_{X_1} in (21) was found in [13] (see also [12]) and this expression coincides with (3) above when $x < 0$. (Note that (21) holds for all $\lambda \in \mathbb{R}$ and substitute $y = -x$ to connect to [13].) This represents an analytic argument for the additive factorisation addressed following (10) above.

Remark 2. In contrast to (12) note that

$$f_{\tau_{(x,\infty)}}(0+) = \frac{c}{\alpha\, x^\alpha} \tag{22}$$

for $x > 0$. This is readily derived from $\mathsf{P}(\tau_{(x,\infty)} \le t) = \mathsf{P}(S_t \ge x)$ using $S_t \overset{\text{law}}{=} t^{1/\alpha} S_1$ and $f_{S_1}(x) \sim c\, x^{-1-\alpha}$ for $x \to \infty$ as recalled in the proof above.

Remark 3. If $x < 0$ then applying the same arguments as in (14) above with $I_t = \inf_{0 \le s \le t} X_s$ we find that

$$\begin{aligned} \mathsf{P}(I_t \le x) &= \mathsf{P}(I_t \le x, X_t \le x) + \mathsf{P}(I_t \le x, X_t > x) \\ &= \mathsf{P}(X_t \le x) + \int_0^t \mathsf{P}(x + X_{t-s} > x)\, F_{\tau_x}(ds) \\ &= \mathsf{P}(X_t \le x) + (1 - 1/\alpha)\, \mathsf{P}(\tau_x \le t) \end{aligned} \tag{23}$$

for $t>0$. In this case, moreover, we also have $\mathsf{P}(I_t \le x) = \mathsf{P}(\sigma_x \le t)$ since X creeps through x, so that (23) yields

$$\mathsf{P}(\tau_x \le t) = \alpha\, \mathsf{P}(X_t \le x) \tag{24}$$

for $x<0$ and $t>0$. Since $X_t =^{\text{law}} t^{1/\alpha} X_1$ this implies

$$f_{\tau_x}(t) = -x\, t^{-1-1/\alpha} F_{X_1}(x\, t^{-1/\alpha}) = -\sum_{n=1}^{\infty} \frac{\sin(n\pi/\alpha)}{\pi} \frac{\Gamma(1+n/\alpha)}{n!} \frac{x^n}{t^{1+n/\alpha}} \tag{25}$$

for $t>0$ upon using (3) above. Replacing t in (25) by $c\,\Gamma(-\alpha)t$ we get a series representation for f_{τ_x} in the case when $c>0$ is a general constant. The first identity in (25) is known to hold in greater generality (see [4] and [2, p. 190] for different proofs).

Remark 4. If $c = 1/2\Gamma(-\alpha)$ and $\alpha \uparrow 2$ then the series representations (11) and (25) with $t/2$ in place of t reduce to the known expressions for the density function f_{τ_x} of $\tau_x = \inf\{\, t>0 : B_t = x\,\}$ where $B = (B_t)_{t\ge 0}$ is a standard Brownian motion:

$$f_{\tau_x}(t) = \frac{|x|}{\sqrt{2\pi t^3}}\, e^{-x^2/2t} = \frac{|x|}{\sqrt{2\pi t^3}} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} \frac{x^{2n}}{t^n} \tag{26}$$

for $t>0$ and $x \in \mathbb{R}\setminus\{0\}$.

Remark 5. Duality theory for Markov/Lévy processes (see [3, Chap. VI] and [2, Chap. II and Corollary 18 on p. 64]) implies that

$$\mathsf{E} e^{-p\tau_x} = \frac{\int_0^\infty e^{-pt} f_{X_t}(x)\, dt}{\int_0^\infty e^{-pt} f_{X_t}(0)\, dt} \tag{27}$$

from where the following identity can be derived (see [2, Lemma 13, p. 230]):

$$\mathsf{P}(\tau_x \le t) = \frac{1}{\Gamma(1-1/\alpha)\,\Gamma(1/\alpha)\, f_{X_1}(0)} \int_0^t \frac{f_{X_s}(x)}{(t-s)^{1-1/\alpha}}\, ds \tag{28}$$

for $x \in \mathbb{R}$ and $t>0$ (being valid for any stable Lévy process). By the scaling property of X we have $f_{X_s}(x) = s^{-1/\alpha} f_{X_1}(xs^{-1/\alpha})$ for $s \in (0,t)$ and $x \in \mathbb{R}$. Recalling the particular form of the series representation for f_{X_1} given in (3), we see that it is not possible to integrate term by term in (28) in order to obtain an explicit series representation.

Remark 6. The density function f_{X_1} from (3) can be expressed in terms of the Fox functions (see [15]), and the density function f_{S_1} from (4) can be expressed in terms of the Wright functions (see [5, Sect. 12] and the references therein). In view of the identity (17) and the fact that $f_{\tau_x}(t) = x^{-\alpha} f_{\tau_1}(t\, x^{-\alpha})$, these facts can be used to provide alternative representations for the density function f_{τ_x} from (11) above. We are grateful to an anonymous referee for bringing these references to our attention.

References

[1] BERNYK, V. DALANG, R. C. *and* PESKIR, G. (2008). The law of the supremum of a stable Lévy process with no negative jumps. *Ann. Probab.* 36 (1777–1789). MR2440923

[2] BERTOIN, J. (1996). *Lévy Processes.* Cambridge Univ. Press. MR1406564

[3] BLUMENTHAL, R. M. *and* GETOOR, R. K. (1968). *Markov Processes and Potential Theory.* Academic Press. MR0264757

[4] BOROVKOV, K. *and* BURQ, Z. (2001). Kendall's identity for the first crossing time revisited. *Electron. Comm. Probab.* 6 (91–94). MR1871697

[5] BRAAKSMA, B. L. J. (1964). Asymptotic expansions and analytic continuations for a class of Barnes-integrals. *Compositio Math.* 15 (239–341). MR not available. MR0167651

[6] DONEY, R. A. (1991). Hitting probabilities for spectrally positive Lévy processes. *J. London Math. Soc.* 44 (566–576). MR1149016

[7] DONEY, R. A. (2008). A note on the supremum of a stable process. *Stochastics* 80 (151–155). MR2402160

[8] ERDÉLYI, A. (1954). *Tables of Integral Transforms*, Vol. 1. McGraw-Hill.

[9] KYPRIANOU, A. E. (2006). *Introductory Lectures on Fluctuations of Lévy Processes with Applications.* Springer-Verlag. MR2250061

[10] MONRAD, D. (1976). Lévy processes: absolute continuity of hitting times for points. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* 37 (43–49). MR0423550

[11] PESKIR, G. (2002). On integral equations arising in the first-passage problem for Brownian motion. *J. Integral Equations Appl.* 14 (397–423). MR1984752

[12] POLLARD, H. (1946). The representation of $e^{-x^{\lambda}}$ as a Laplace integral. *Bull. Amer. Math. Soc.* 52 (908–910). MR0018286

[13] POLLARD, H. (1948). The completely monotonic character of the Mittag-Leffler function $E_a(-x)$. *Bull. Amer. Math. Soc.* 54 (1115–1116). MR0027375

[14] SATO, K. (1999). *Lévy Processes and Infinitely Divisible Distributions.* Cambridge Univ. Press. MR1739520

[15] SCHNEIDER, W. R. (1986). Stable distributions: Fox functions representation and generalization. *Proc. Stoch. Process. Class. Quant. Syst.* (Ascona 1985), Lecture Notes in Phys. 262, Springer (497–511). MR0870201