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SPECTRAL GAP FOR THE INTERCHANGE PROCESS IN A BOX

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Abstract
We show that the spectral gap for the interchange process (and the symmetric exclusion process) in a d-dimensional box of side length L is asymptotic to π^2/L^2. This gives more evidence in favor of Aldous's conjecture that in any graph the spectral gap for the interchange process is the same as the spectral gap for a corresponding continuous-time random walk. Our proof uses a technique that is similar to that used by Handjani and Jungreis, who proved that Aldous's conjecture holds when the graph is a tree.

1 Introduction

1.1 Aldous's conjecture

This subsection is taken (with minor alterations) from David Aldous's web page. Consider an n-vertex graph G which is connected and undirected. Take n particles labeled $1, 2, ..., n$. In a configuration, there is one particle at each vertex. The *interchange process* is the following continuous-time Markov chain on configurations. For each edge (i, j), at rate 1 the particles at vertex i and vertex j are interchanged.
The interchange process is reversible, and its stationary distribution is uniform on all $n!$ configurations. There is a spectral gap $\lambda_{\rm IP}(G) > 0$, which is the absolute value of the largest non-zero eigenvalue of the transition rate matrix. If instead we just watch a single particle, it performs a continuous-time random walk on G (hereafter referred to simply as "the continuous-time random walk on G"), which is also reversible and hence has a spectral gap $\lambda_{\rm RW}(G) > 0$. Simple arguments (the contraction principle) show $\lambda_{\rm IP}(G) \leq \lambda_{\rm RW}(G)$.

Problem. Prove $\lambda_{\rm IP}(G) = \lambda_{\rm RW}(G)$ for all G.

Discussion. Fix m and color particles $1, 2, ..., m$ red. Then the red particles in the interchange process behave as the usual exclusion process (i.e., m particles performing the continuous-time

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random walk on G, but with moves that take two particles to the same vertex suppressed). But in the finite setting, the interchange process seems more natural.

1.2 Results

Aldous's conjecture has been proved in the case where G is a tree [7] and in the case where G is the complete graph [5]; see also [12]. In this note we prove an asymptotic version of Aldous's conjecture for G a box in $\mathbf{Z}^d$. We show that if B_L denotes a box of side length L in $\mathbf{Z}^d$ then

$$\frac{\lambda_{\mathrm{IP}}(B_L)}{\lambda_{\mathrm{RW}}(B_L)} \to 1,$$

as $L \to \infty$.

Remark: After completing a draft of this paper, I learned that Starr and Conomos had recently obtained the same result (see [14]). Their proof uses a similar approach, although the present paper is somewhat shorter. □

Connection to simple exclusion. Our result gives a bound on the spectral gap for the exclusion process. The exclusion process is a widely studied Markov chain, with connections to card shuffling [16, 1], statistical mechanics [8, 13, 2, 15], and a variety of other processes (see e.g., [10, 6]); it has been one of the major examples behind the study of convergence rates for Markov chains (see, e.g., [6, 3, 16, 1]). Our result implies that the spectral gap for the symmetric exclusion process in B_L is asymptotic to π^2/L^2. The problem of bounding the spectral gap for simple exclusion was studied in Quastel [13] and a subsequent independent paper of Diaconis and Saloff-Coste [3]. Both of these papers used a comparison to Bernoulli-Laplace diffusion (i.e., the exclusion process in the complete graph) to obtain a bound of order $1/dL^2$. Diaconis and Saloff-Coste explicitly wondered whether the factor d in the denominator is necessary; in the present paper we show that it is not.

2 Background

Consider a continuous-time Markov chain on a finite state space W with a symmetric transition rate matrix $Q(x,y)$. The spectral gap is the minimum value of $\alpha > 0$ such that

$$Qf = -\alpha f, \tag{1}$$

for some $f : W \to \mathbf{R}$. The spectral gap governs the asymptotic rate of convergence to the stationary distribution. Define

$$\mathcal{E}(f,f) = \frac{1}{2|W|} \sum_{x,y \in W} (f(x) - f(y))^2 Q(x,y),$$

and define

$$\mathrm{var}(f) = \frac{1}{|W|} \sum_{x \in W} (f(x) - \mathbf{E}(f))^2,$$

where

$$\mathbf{E}(f) = \frac{1}{|W|} \sum_{x \in W} f(x).$$

If f is a function that satisfies $Qf = -\lambda f$ for some $\lambda > 0$, then

$$\lambda = \frac{\mathcal{E}(f,f)}{\mathrm{var}(f)}. \tag{2}$$

Furthermore, if α is the spectral gap then for any non-constant $f : W \to \mathbf{R}$ we have

$$\frac{\mathcal{E}(f,f)}{\mathrm{var}(f)} \geq \alpha. \tag{3}$$

Thus the spectral gap can be obtained by minimizing the left hand side of (3) over all non-constant functions $f : W \to \mathbf{R}$.

3 Main result

Before specializing to the interchange process, we first prove a general proposition relating the eigenvalues of a certain function of a Markov chain to the eigenvalues of the Markov chain itself. Let X_t be a continuous-time Markov chain on a finite state space W with a symmetric transition rate matrix $Q(x,y)$. Let T be another space and let $g : W \to T$ be a function on W such that if $g(x) = g(y)$ and $U = g^{-1}(u)$ for some u, then $\sum_{u' \in U} Q(x,u') = \sum_{u' \in U} Q(y,u')$. Note that $g(X_t)$ is a Markov chain. Let W' denote the collection of subsets of W of the form $g^{-1}(u)$ for some $u \in T$. We can identify the states of $g(X_n)$ with elements of W'. Let Q' denote the transition rate matrix for $g(X_n)$. Note that if $U, U' \in W'$, with $U = g^{-1}(u)$ for some $u \in T$ and $U \neq U'$, then $Q'(U,U') = \sum_{y \in U'} Q(u,y)$.
We shall need the following proposition, which generalizes Lemma 2 of [7].

Proposition 1. *Let X_t, g and Q' be as defined above. Suppose $f : W \to \mathbf{R}$ is an eigenvector of Q with corresponding eigenvalue $-\lambda$ and define $h : W' \to \mathbf{R}$ by $h(U) = \sum_{x \in U} f(x)$. Then $Q'h = -\lambda h$. That is, either h is an eigenvector of Q' with corresponding eigenvalue $-\lambda$, or h is identically zero.*

Proof: Note that for all $U' \in W'$ we have

$$\begin{aligned}
(Q'h)(U') &= \sum_{U \in W'} h(U) Q'(U,U') \\
&= \sum_{U \in W'} \sum_{x \in U} f(x) \sum_{y \in U'} Q(x,y) \\
&= \sum_{y \in U'} (Qf)(y) \\
&= -\lambda \sum_{y \in U'} f(y) \\
&= -\lambda h(U'),
\end{aligned}$$

so $Q'h = -\lambda h$. □

The following Lemma is a weaker version of Aldous's conjecture. The proof is similar to the proof of Theorem 1 in [7].

Lemma 2. *Let G be a connected, undirected graph with vertices labeled $1, \ldots, n$. For $2 \le k \le n$ let G_k be the subgraph of G induced by the vertices $1, 2, \ldots, k$. Let $\lambda_{\mathrm{RW}}(G_k)$ be the spectral gap for the continuous-time random walk on G_k, and define $\alpha_k = \min_{2 \le j \le k} \lambda_{\mathrm{RW}}(G_j)$. Then*

$$\lambda_{\mathrm{IP}}(G) \ge \alpha_n.$$

Proof: Our prooof will be by induction on the number of vertices n. The base case $n = 2$ is trivial, so assume $n > 2$. Let W and Q be the state space and transition rate matrix, respectively, for the interchange process on G. Let $f : W \to \mathbf{R}$ be a function that satisfies $Qf = -\lambda f$. We shall show that $\lambda \ge \alpha_n$. Note that a configuration of the interchange process can be identified with a permutation π in S_n, where if particle i is in vertex j, then $\pi(i) = j$. For positive integers m and k with $m, k \le n$, we write $f(\pi(m) = k)$ for

$$\sum_{\pi : \pi(m) = k} f(\pi).$$

We consider two cases.

Case 1: For some m and k we have $f(\pi(m) = k) \ne 0$. Define $h : V \to \mathbf{R}$ by $h(j) = f(\pi(m) = j)$. Then h is not identically zero, and using Proposition 1 with g defined by $g(\pi) = \pi(m)$ gives that if Q' is the transition rate matrix for continuous time random walk on G, then $Q'h = -\lambda h$. It follows that λ is an eigenvalue of Q' and hence $\lambda \ge \lambda_{\mathrm{RW}}(G) = \alpha_n$.

Case 2: For all m and k we have $f(\pi(m) = k) = 0$. Define the *suppressed process* as the interchange process with moves involving vertex n suppressed. That is, the Markov chain with the following transition rule:

For every edge e not incident to n, at rate 1 switch the particles at the endpoints of e.

For $1 \le k \le n$, let $W_k = \{\pi \in W : \pi^{-1}(n) = k\}$. Note that the W_k are the irreducible classes of the suppressed process, and that for each k the restriction of the suppressed process to W_k can be identified with the interchange process on G_{n-1}. For k with $1 \le k \le n$, define

$$\mathcal{E}_k(f, f) = \frac{1}{2(n-1)!} \sum_{\pi_1, \pi_2 \in W_k} (f(\pi_1) - f(\pi_2))^2 Q(\pi_1, \pi_2),$$

and define

$$\mathrm{var}_k(f) = \frac{1}{(n-1)!} \sum_{\pi \in W_k} f(\pi)^2.$$

(Note that for every k we have $\sum_{\pi \in W_k} f(x) = 0$.)
By the induction hypothesis, the spectral gap for the interchange process on G_{n-1} is at least α_{n-1}. Hence for every k with $1 \le k \le n$ we have

$$\mathcal{E}_k(f, f) \ge \alpha_{n-1} \mathrm{var}_k(f) \ge \alpha_n \mathrm{var}_k(f).$$

It follows that

$$\begin{aligned} n!\mathcal{E}(f,f) &\geq \tfrac{1}{2}\sum_{k=1}^{n}\sum_{\pi_1,\pi_2\in W_k}(f(\pi_1)-f(\pi_2))^2 Q(\pi_1,\pi_2) &(4)\\ &= \sum_{k=1}^{n}(n-1)!\mathcal{E}_k(f,f) &(5)\\ &\geq \sum_{k=1}^{n}\alpha_n(n-1)!\mathrm{var}_k(f) &(6)\\ &= \alpha_n\sum_{k=1}^{n}\sum_{\pi\in W_k} f(\pi)^2 &(7)\\ &= \alpha_n n!\mathrm{var}(f). &(8) \end{aligned}$$

Combining this with equation (2) gives $\lambda \geq \alpha_n$. □

Remark: Theorem 2 is optimal if the vertices are labeled in such a way that $\lambda_{\mathrm{RW}}(G_k)$ is nonincreasing in k, in which case it gives $\lambda_{\mathrm{IP}}(G) = \lambda_{\mathrm{RW}}(G)$. Since any tree can be built up from smaller trees (with larger spectral gaps), we recover the result proved in [7] that $\lambda_{\mathrm{IP}}(T) = \lambda_{\mathrm{RW}}(T)$ if T is a tree. □

Our main application of Lemma 2 is the following asymptotic version of Aldous's conjecture in the special case where G is a box in $\mathbf{Z}^d$.

Corollary 3. *Let $B_L = \{0,\dots,L\}^d$ be a box of side length L in $\mathbf{Z}^d$. Then the spectral gap for the interchange process on B_L is asymptotic to π^2/L^2.*

Proof: In order to use Lemma 2 we need to label the vertices of B_L in some way. Our goal is to label in such a way that for every k the quantity $\lambda_{\mathrm{RW}}(G_k)$ (i.e., the spectral gap corresponding to the subgraph of B_L induced by the vertices $1,\dots,k$) is not too much smaller than $\lambda_{\mathrm{RW}}(B_L)$. So our task is to build B_L, one vertex at a time, in such a way that the spectral gaps of the intermediate graphs don't get too small.
We shall build B_L by inductively building B_{L-1} and then building B_L from B_{L-1}. Since $\lambda_{\mathrm{RW}}(B_L) \downarrow 0$, it is enough to show that

$$\frac{\beta_L}{\lambda_{\mathrm{RW}}(B_L)} \to 1,$$

where β_L is the minimum spectral gap for any intermediate graph between B_{L-1} and B_L.
For a graph H, let $V(H)$ denote the set of vertices in H. For $j \geq 0$, let $\mathcal{L}_j = \{0,\dots,j\}$ be the line graph with $j+1$ vertices. Define $\gamma_L = \lambda_{\mathrm{RW}}(\mathcal{L}_L)$. It is well known that γ_L is decreasing in L and asymptotic to π^2/L^2 as $L \to \infty$. It is also well known that if H and H' are graphs and $\times$ denotes Cartesian product, then $\lambda_{\mathrm{RW}}(H \times H') = \min(\lambda_{\mathrm{RW}}(H), \lambda_{\mathrm{RW}}(H'))$. Since $B_L = \mathcal{L}_L^d$, it follows that $\lambda_{\mathrm{RW}}(B_L) = \gamma_L$.
We construct B_L from B_{L-1} using intermediate graphs $H_0,\dots,H_d$, where for k with $1 \leq k \leq d$ we define $H_k = \mathcal{L}_L^k \times \mathcal{L}_{L-1}^{d-k}$. Note that $H_0 = B_{L-1}$ and $H_d = B_L$. We obtain H_k from H_{k-1} by adding vertices to lengthen H_{k-1} by one unit in direction k. The order in which the vertices in $V(H_k) - V(H_{k-1})$ are added is arbitrary.

Fix k with $1 \le k \le d$, and define $G' = G'(L,k)$ as follows. Let

$$V' = V(H_k), \qquad E' = \{(u,v) : \text{either } u \text{ or } v \text{ is a vertex in } H_{k-1}\},$$

and let $G' = (V', E')$. It is well known and easily shown that if H is a graph, then adding edges to H cannot decrease $\lambda_{\mathrm{RW}}(H)$, nor can removing pendant edges. Since each intermediate graph $\tilde{G}$ between H_{k-1} and H_k can be obtained from G' by adding edges and removing pendant edges, it follows that for any such graph $\tilde{G}$ we have $\lambda_{\mathrm{RW}}(\tilde{G}) \ge \lambda_{\mathrm{RW}}(G')$. Thus, it is enough to bound $\lambda_{\mathrm{RW}}(G')$ from below. We shall show that for any $\epsilon > 0$ we have $\lambda_{\mathrm{RW}}(G'(L,k)) \ge (1-\epsilon)\gamma_L$ if L is sufficiently large.
Let e_k be the unit vector in direction k. Let

$$S = V(H_{k-1}); \qquad \partial S = V' - S.$$

Let X_t be the continuous-time random walk on G', with transition rate matrix Q. Fix $f : V' \to \mathbf{R}$ with $Qf = -\lambda f$ for some $\lambda > 0$. For $x \in \mathbf{Z}^d$, let $g_k(x)$ denote the component of x in the kth coordinate. Note that $g_k(X_t)$ is the continuous-time random walk on $\mathcal{L}_L$. Let Q' be the transition rate matrix for $g_k(X_t)$. Proposition 1 implies that if $h : \{0,\dots,L\} \to \mathbf{R}$ is defined by $h(j) = \sum_{\substack{x \in V' \\ g_k(x)=j}} f(x)$, then $Q'h = -\lambda h$. Thus if $\lambda < \gamma_L$, then g is identically zero and hence $\sum_{x\in S} f(x) = 0$. Define

$$\mathcal{E}(f,f) = \frac{1}{2|V'|} \sum_{x,y \in V'} (f(x) - f(y))^2 Q(x,y),$$

and let $\mathcal{E}_S(f,f)$ be defined analogously, but with only vertices in S included in the sum. Note that $\mathcal{E}(f,f) \ge \mathcal{E}_S(f,f)$. Since $\sum_{x\in S} f(x) = 0$, we have

$$\frac{\mathcal{E}_S(f,f)}{\sum_{x\in S} f(x)^2} \ge \lambda_{\mathrm{RW}}(H_{k-1}) \ge \gamma_L, \tag{9}$$

where the second inequality follows from the fact that H_{k-1} is a Cartesian product of d graphs, each of which is either $\mathcal{L}_{L-1}$ or $\mathcal{L}_L$.
Fix $\epsilon > 0$ and let M be a positive integer large enough so that $(1 - 4M^{-1})^{-1} \le (1-\epsilon)^{-1/2}$. For each $x \in \partial S$, say that x is *good* if there is a $y \in S$ such that $x = y + ie_k$ for some $i \le M$ and $|f(y)| \le |f(x)|/2$. Otherwise say that x is *bad*. Let $\mathcal{G}$ and $\mathcal{B}$ denote the set of good and bad vertices, respectively, in ∂S. Note that if x is bad and $M \le L$ then $f(x)^2 \le \frac{4}{M}\sum_{j=1}^{M} f(x - je_k)^2$. Summing this over bad x gives

$$\sum_{x\in\mathcal{B}} f(x)^2 \le \frac{4}{M} \sum_{x\in V'} f(x)^2 \tag{10}$$

Note that if x is good, then there must be an $x' \in S$ of the form $x - ie_k$ such that $|f(x') - f(x'+e_k)| > f(x)/2M$. It follows that

$$\frac{\mathcal{E}(f,f)}{\sum_{x\in\mathcal{G}} f(x)^2} \ge 1/4M^2. \tag{11}$$

Since $V' = S \cup \mathcal{B} \cup \mathcal{G}$, combining equations (11), (9) and (10) gives

$$\sum_{x\in V'} f(x)^2 \le (\gamma_L^{-1} + 4M^2)\mathcal{E}(f,f) + 4M^{-1} \sum_{x\in V'} f(x)^2,$$

and hence

$$\sum_{x\in V'} f(x)^2 \leq (1-4M^{-1})^{-1}(\gamma_L^{-1}+4M^2)\mathcal{E}(f,f). \tag{12}$$

Recall that $(1-4M^{-1})^{-1} \leq (1-\epsilon)^{-\frac{1}{2}}$, and note that since $\gamma_L \to 0$ as $L \to \infty$, we have $\gamma_L^{-1}+4M^2 \leq (1-\epsilon)^{-\frac{1}{2}}\gamma_L^{-1}$ for sufficiently large L. Combining this with equation (12) gives

$$\frac{\mathcal{E}(f,f)}{\sum_{x\in V'} f(x)^2} \geq (1-\epsilon)\gamma_L,$$

for sufficiently large L. It follows that $\lambda_{\mathrm{RW}}(G') \geq (1-\epsilon)\gamma_L$ for sufficiently large L and so the proof is complete. □

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