Elect. Comm. in Probab. **13** (2008), 352–363

ELECTRONIC COMMUNICATIONS in PROBABILITY

SHARP ESTIMATES FOR THE CONVERGENCE OF THE DENSITY OF THE EULER SCHEME IN SMALL TIME

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Submitted January 29, 2008, *accepted in final form* June 3, 2008

AMS 2000 Subject classification: 65C20 60H07 60H10 65G99 65M15 60J60
Keywords: stochastic differential equation, Euler scheme, rate of convergence, Malliavin calculus

Abstract

In this work, we approximate a diffusion process by its Euler scheme and we study the convergence of the density of the marginal laws. We improve previous estimates especially for small time.

1 Introduction

Let us consider a d-dimensional diffusion process $(X_s)_{0\leq s\leq T}$ and a q-dimensional Brownian motion $(W_s)_{0\leq s\leq T}$. X satisfies the following SDE

$$dX_s^i = b_i(s,X_s)ds + \sum_{j=1}^{q} \sigma_{ij}(s,X_s)dW_s^j, \quad X_0^i = x^i, \forall i \in \{1,\cdots,d\}. \tag{1.1}$$

We approximate X by its Euler scheme with N ($N \geq 1$) time steps, say X^N, defined as follows. We consider the regular grid $\{0 = t_0 < t_1 < \cdots < t_N = T\}$ of the interval $[0,T]$, i.e. $t_k = k\frac{T}{N}$. We put $X_0^N = x$ and for all $i \in \{1,\cdots,d\}$ we define

$$X_u^{N,i} = X_{t_k}^{N,i} + b_i(t_k, X_{t_k}^N)(u - t_k^N) + \sum_{j=1}^{q} \sigma_{ij}(t_k, X_{t_k}^N)(W_u^j - W_{t_k}^j), \text{ for } u \in [t_k, t_{k+1}[. \tag{1.2}$$

The continuous Euler scheme is an Itô process verifying

$$X_u^N = x + \int_0^u b(\varphi(s), X^N_{\varphi(s)})ds + \int_0^u \sigma(\varphi(s), X^N_{\varphi(s)})dW_s$$

where $\varphi(u) := \sup\{t_k : t_k \leq u\}$. If σ is uniformly elliptic, the Markov process X admits a transition probability density $p(0, x; s, y)$. Concerning X^N (which is not Markovian except at times $(t_k)_k$), X^N_s has a probability density $p^N(0, x; s, y)$, for any $s > 0$. We aim at proving sharp estimates of the difference $p(0, x; s, y) - p^N(0, x; s, y)$.

It is well known (see Bally and Talay [2], Konakov and Mammen [5], Guyon [4]) that this difference is of order $\frac{1}{N}$. However, the known upper bounds of this difference are too rough for small values of s. In this work, we provide tight upper bounds of $|p(0, x; s, y) - p^N(0, x; s, y)|$ in s (see Theorem 2.3), so that we can estimate quantities like

$$\mathbb{E}[f(X_T^N)] - \mathbb{E}[f(X_T)] \text{ or } \mathbb{E}\left[\int_0^T f(X^N_{\varphi(s)})ds\right] - \mathbb{E}\left[\int_0^T f(X_s)ds\right] \tag{1.3}$$

(without any regularity assumptions on f) more accurately than before (see Theorem 2.5). For other applications, see Labart [7]. Unlike previous references, we allow b and σ to be time-dependent and assume they are only C^3 in space. Besides, we use Malliavin's calculus tools.

Background results

The difference $p(0, x; s, y) - p^N(0, x; s, y)$ has been studied a lot. We can found several results in the literature on expansions w.r.t. N. First, we mention a result from Bally and Talay [2] (Corollary 2.7). The authors assume

Hypothesis 1.1. *σ is elliptic (with σ only depending on x) and b, σ are $C^\infty(\mathbb{R}^d)$ functions whose derivatives of any order greater or equal to 1 are bounded.*

By using Malliavin's calculus, they show that

$$p(0, x; T, y) - p^N(0, x; T, y) = \frac{1}{N}\pi_T(x, y) + \frac{1}{N^2}R_T^N(x, y), \tag{1.4}$$

with $|\pi_T(x, y)| + |R_T^N(x, y)| \leq \frac{K(T)}{T^\alpha}\exp(-c\frac{|x-y|^2}{T})$, where $c > 0$, $\alpha > 0$ and $K(\cdot)$ is a non decreasing function. We point out that α is unknown, which doesn't enable to deduce the behavior of $p - p^N$ when $T \to 0$.

Besides that, Konakov and Mammen [5] have proposed an analytical approach based on the so-called parametrix method to bound $p(0, x; 1, y) - p^N(0, x; 1, y)$ from above. They assume

Hypothesis 1.2. *σ is elliptic and b, σ are $C^\infty(\mathbb{R}^d)$ functions whose derivatives of any order are bounded.*

For each pair (x, y) they get an expansion of arbitrary order j of $p^N(0, x; 1, y)$. The coefficients of the expansion depend on N

$$p(0, x; 1, y) - p^N(0, x; 1, y) = \sum_{i=1}^{j-1} \frac{1}{N^i}\pi_{N,i}(0, x; 1, y) + O(\frac{1}{N^j}). \tag{1.5}$$

The coefficients have Gaussian tails : for each i they find constants $c_1 > 0$, $c_2 > 0$ s.t. for all $N \geq 1$ and all $x, y \in \mathbb{R}^d$, $|\pi_{N,i}(0,x;1,y)| \leq c_1 \exp(-c_2|x-y|^2)$. To do so, they use upper bounds for the partial derivatives of p (coming from Friedman [3]) and prove analogous results on the derivatives of p^N. Strong though this result may be, nothing is said when replacing 1 by t, for $t \to 0$. That's why we present now the work of Guyon [4].

Guyon [4] improves (1.4) and (1.5) in the following way.

Definition 1.3. *Let $\mathcal{G}_l(\mathbb{R}^d), l \in \mathbb{Z}$ be the set of all measurable functions $\pi : \mathbb{R}^d \times (0,1] \times \mathbb{R}^d \to \mathbb{R}$ s.t.*

- *for all $t \in (0,1], \pi(\cdot;t,\cdot)$ is infinitely differentiable,*
- *for all $\alpha, \beta \in \mathbb{N}^d$, there exist two constants $c_1 \geq 0$ and $c_2 > 0$ s.t. for all $t \in (0,1]$ and $x, y \in \mathbb{R}^d$,*

$$|\partial_x^\alpha \partial_y^\beta \pi(x;t,y)| \leq c_1 t^{-(|\alpha|+|\beta|+d+l)/2} \exp(-c_2|x-y|^2/t).$$

Under Hypothesis 1.2 and for $T = 1$, the author has proved the following expansions

$$p^N - p = \frac{\pi}{N} + \frac{\pi_N}{N^2}, \tag{1.6}$$

$$p^N - p = \sum_{i=1}^{j-1} \frac{\pi_{N,i}}{N^i} + \sum_{i=2}^{j} \left(t - \frac{\lfloor Nt \rfloor}{N}\right)^i \pi'_{N,i} + \frac{\pi''_{N,j}}{N^j}, \tag{1.7}$$

where $\pi \in \mathcal{G}_1(\mathbb{R}^d)$ and $(\pi_N, N \geq 1)$ is a bounded sequence in $\mathcal{G}_4(\mathbb{R}^d)$. For each $i \geq 1$, $(\pi_{N,i}, N \geq 1)$ is a bounded family in $\mathcal{G}_{2i-2}(\mathbb{R}^d)$, and $(\pi'_{N,i}, N \geq 1), (\pi''_{N,i}, N \geq 1)$ are two bounded families in $\mathcal{G}_{2i}(\mathbb{R}^d)$. These expansions can be seen as improvements of (1.4) and (1.5) : it also allows infinite differentiations w.r.t. x and y and makes precise the way the coefficients explode when t tends to 0.

As a consequence (see Guyon [4], Corollary 22), one gets

$$|p(0,x;s,y) - p^N(0,x;s,y)| \leq \frac{c_1}{N s^{\frac{d+2}{2}}} e^{-c_2 \frac{|x-y|^2}{s}}, \tag{1.8}$$

for two positive constants c_1 and c_2, and for any x, y and $s \leq 1$. This result should be compared with the one of Theorem 2.3 (when $T = 1$), in which the upper bound is tighter (s has a smaller power).

2 Main Results

Before stating the main result of the paper, we introduce the following notation

Definition 2.1. *$C_b^{k,l}$ denotes the set of continuously differentiable bounded functions $\phi : (t,x) \in [0,T] \times \mathbb{R}^d$ with uniformly bounded derivatives w.r.t. t (resp. w.r.t. x) up to order k (resp. up to order l).*

The main result of the paper, whose proof is postponed to Section 4, is established under the following Hypothesis

Hypothesis 2.2. *σ is uniformly elliptic, b and σ are in $C_b^{1,3}$ and $\partial_t\sigma$ is in $C_b^{0,1}$.*

Theorem 2.3. *Assume Hypothesis 2.2. Then, there exist a constant $c > 0$ and a non decreasing function K, depending on the dimension d and on the upper bounds of σ, b and their derivatives s.t. $\forall (s,x,y) \in]0,T] \times \mathbb{R}^d \times \mathbb{R}^d$, one has*

$$|p(0,x;s,y) - p^N(0,x;s,y)| \leq \frac{K(T)T}{Ns^{\frac{d+1}{2}}} \exp\left(-\frac{c|x-y|^2}{s}\right).$$

Corollary 2.4. *Assume Hypothesis 2.2. From the last inequality and Aronson's inequality (A.1), we deduce*

$$\left|\frac{p(0,x;T,x) - p^N(0,x;T,x)}{p(0,x;T,x)}\right| \leq \frac{K(T)}{N}\sqrt{T}. \tag{2.1}$$

This inequality yields $p(0,x;T,x) \sim p^N(0,x;T,x)$ when $T \to 0$.

Theorem 2.3 enables to bound quantities like in (1.3) in the following way

Theorem 2.5. *Assume Hypothesis 2.2. For any function f such that $|f(x)| \leq c_1 e^{c_2|x|}$, it holds*

$$\left|\mathbb{E}[f(X_T^N)] - \mathbb{E}[f(X_T)]\right| \leq c_1 e^{c_2|x|} K(T) \frac{\sqrt{T}}{N},$$
$$\left|\mathbb{E}\left[\int_0^T f(X^N_{\varphi(s)})ds\right] - \mathbb{E}\left[\int_0^T f(X_s)ds\right]\right| \leq c_1 e^{c_2|x|} K(T)\frac{T}{N}.$$

Had we used the results stated by Guyon [4] (and more precisely the one recalled in (1.8)), we would have obtained $\mathbb{E}[f(X_T^N)] - \mathbb{E}[f(X_T)] = O(\frac{1}{N})$. Intuitively, this result is not optimal: the right hand side doesn't tend to 0 when T goes to 0 while it should. Analogously, regarding $\mathbb{E}\left[\int_0^T f(X^N_{\varphi(s)})ds\right] - \mathbb{E}\left[\int_0^T f(X_{\varphi(s)})ds\right]$, we would obtain $O(\frac{T\ln N}{N})$ instead of $O(\frac{T}{N})$.

Proof of Theorem 2.5. Writing $\mathbb{E}[f(X_T^N)] - \mathbb{E}[f(X_T)]$ as $\int_{\mathbb{R}^d} f(y)(p^N(0,x;T,y) - p(0,x;T,y))dy$ and using Theorem 2.3 yield the first result.
Concerning the second result, we split $\mathbb{E}\left[\int_0^T (f(X^N_{\varphi(s)}) - f(X_s))ds\right]$ in two terms : $\mathbb{E}\left[\int_0^T (f(X^N_{\varphi(s)}) - f(X_{\varphi(s)}))ds\right]$ and $\mathbb{E}\left[\int_0^T (f(X_{\varphi(s)}) - f(X_s))ds\right]$. First, using Theorem 2.3 leads to

$$\left|\mathbb{E}\left[\int_0^T (f(X^N_{\varphi(s)}) - f(X_{\varphi(s)}))ds\right]\right| = \left|\int_{\mathbb{R}^d} dy \int_{\frac{T}{N}}^T ds f(y)(p^N(0,x;\varphi(s),y) - p(0,x;\varphi(s),y))\right|,$$
$$\leq \frac{K(T)T}{N} c_1 e^{c_2|x|} \int_{\frac{T}{N}}^T \frac{ds}{\sqrt{\varphi(s)}},$$

where we use the easy inequality $\int_{\mathbb{R}^d} e^{c_2|y|} \frac{e^{\frac{-c|x-y|^2}{s}}}{s^{d/2}} dy \leq K(T) e^{c_2|x|}$. Since $\varphi(s) \geq s - \frac{T}{N}$, we get $\left|\mathbb{E}\left[\int_0^T (f(X^N_{\varphi(s)}) - f(X_{\varphi(s)}))ds\right]\right| \leq \frac{K(T)T^{3/2}}{N} c_1 e^{c_2|x|}$. Second, we write

$$\left|\mathbb{E}\left[\int_0^T (f(X_{\varphi(s)}) - f(X_s))ds\right]\right| \leq c_1 e^{c_2|x|}\frac{T}{N} + \int_{\mathbb{R}^d} dy \int_{\frac{T}{N}}^T ds c_1 e^{c_2|y|} \int_{\varphi(s)}^s du |\partial_u p(0,x;u,y)|.$$

Then, Proposition A.2 yields $\left|\mathbb{E}\left[\int_0^T (f(X_{\varphi(s)}) - f(X_s))ds\right]\right| \le c_1 e^{c_2|x|}\left(\frac{T}{N} + C\int_{\frac{T}{N}}^T \ln(\frac{s}{\varphi(s)})ds\right)$. Moreover, $\int_{\frac{T}{N}}^T \ln(\frac{s}{\varphi(s)})ds = \sum_{k=1}^{N-1}\int_{t_k}^{t_{k+1}} \ln(\frac{s}{t_k})ds = \frac{T}{N}\sum_{k=1}^{N-1}((k+1)\ln(\frac{k+1}{k}) - 1) \le C\frac{T}{N}$, using a second order Taylor expansion. This gives $\left|\mathbb{E}\left[\int_0^T (f(X_{\varphi(s)}) - f(X_s))ds\right]\right| \le c_1 e^{c_2|x|}K(T)\frac{T}{N}$. □

In the next section, we give results related to Malliavin's calculus, that will be useful for the proof of Theorem 2.3.

3 Basic results on Malliavin's calculus

We refer the reader to Nualart [8], for more details. Fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ and let $(W_t)_{t\ge 0}$ be a q-dimensional Brownian motion. For $h(\cdot) \in H = \mathbb{L}^2([0,T], \mathbb{R}^q)$, $W(h)$ is the Wiener stochastic integral $\int_0^T h(t)dW_t$. Let $\mathcal{S}$ denote the class of random variables of the form $F = f(W(h_1), \cdots, W(h_n))$ where f is a C^∞ function with derivatives having a polynomial growth, $(h_1, \cdots, h_n) \in H^n$ and $n \ge 1$. For $F \in \mathcal{S}$, we define its derivative $\mathcal{D}F = (\mathcal{D}_t F := (\mathcal{D}_t^1 F, \cdots, \mathcal{D}_t^q F))_{t\in[0,T]}$ as the H valued random variable given by

$$\mathcal{D}_t F = \sum_{i=1}^n \partial_{x_i} f(W(h_1), \cdots, W(h_n))h_i(t).$$

The operator $\mathcal{D}$ is closable as an operator from $\mathbb{L}^p(\Omega)$ to $\mathbb{L}^p(\Omega; H)$, for $p \ge 1$. Its domain is denoted by $\mathbb{D}^{1,p}$ w.r.t. the norm $\|F\|_{1,p} = [\mathbb{E}|F|^p + \mathbb{E}(\|\mathcal{D}F\|_H^p)]^{1/p}$. We can define the iteration of the operator $\mathcal{D}$, in such a way that for a smooth random variable F, the derivative $\mathcal{D}^k F$ is a random variable with values on $H^{\otimes k}$. As in the case $k = 1$, the operator $\mathcal{D}^k$ is closable from $\mathcal{S} \subset \mathbb{L}^p(\Omega)$ into $\mathbb{L}^p(\Omega; H^{\otimes k}), p \ge 1$. If we define the norm

$$\|F\|_{k,p} = [\mathbb{E}|F|^p + \sum_{j=1}^k \mathbb{E}(\left\|\mathcal{D}^j F\right\|_{H^{\otimes j}}^p)]^{1/p},$$

we denote its domain by $\mathbb{D}^{k,p}$. Finally, set $\mathbb{D}^{k,\infty} = \cap_{p\ge 1}\mathbb{D}^{k,p}$, and $\mathbb{D}^\infty = \cap_{k,p\ge 1}\mathbb{D}^{k,p}$. One has the following chain rule property

Proposition 3.1. *Fix $p \ge 1$. For $f \in C_b^1(\mathbb{R}^d, \mathbb{R})$, and $F = (F_1, \cdots, F_d)^*$ a random vector whose components belong to $\mathbb{D}^{1,p}$, $f(F) \in \mathbb{D}^{1,p}$ and for $t \ge 0$, one has $\mathcal{D}_t(f(F)) = f'(F)\mathcal{D}_t F$, with the notation*

$$\mathcal{D}_t F = \begin{pmatrix} \mathcal{D}_t F_1 \\ \vdots \\ \mathcal{D}_t F_d \end{pmatrix} \in \mathbb{R}^d \otimes \mathbb{R}^q.$$

We now introduce the Skorohod integral δ, defined as the adjoint operator of $\mathcal{D}$.

Proposition 3.2. *δ is a linear operator on $\mathbb{L}^2([0,T]\times\Omega, \mathbb{R}^q)$ with values in $\mathbb{L}^2(\Omega)$ s.t.*

- *the domain of δ (denoted by $Dom(\delta)$) is the set of processes $u \in \mathbb{L}^2([0,T]\times\Omega, \mathbb{R}^q)$ s.t. $|\mathbb{E}(\int_0^T \mathcal{D}_t F \cdot u_t dt)| \le c(u)|F|_{\mathbb{L}^2}$ for any $F \in \mathbb{D}^{1,2}$.*

- *If u belongs to $Dom(\delta)$, then $\delta(u)$ is the one element of $\mathbb{L}^2(\Omega)$ characterized by the integration by parts formula*

$$\forall F \in \mathbb{D}^{1,2},\ \ \mathbb{E}(F\delta(u)) = \mathbb{E}\left(\int_0^T \mathcal{D}_t F \cdot u_t dt\right).$$

Remark 3.3. *If u is an adapted process belonging to $\mathbb{L}^2([0,T]\times\Omega,\mathbb{R}^q)$, then the Skorohod integral and the Itô integral coincide : $\delta(u) = \int_0^T u_t dW_t$, and the preceding integration by parts formula becomes*

$$\forall F \in \mathbb{D}^{1,2},\ \ \mathbb{E}\left(F\int_0^T u_t dW_t\right) = \mathbb{E}\left(\int_0^T \mathcal{D}_t F \cdot u_t dt\right). \tag{3.1}$$

This equality is also called the duality formula.

This duality formula is the corner stone to establish general integration by parts formula of the form

$$\mathbb{E}[\partial^\alpha g(F)G] = \mathbb{E}[g(F)H_\alpha(F,G)]$$

for any non degenerate random variables F. We only give the formulation in the case of interest $F = X_t^N$.

Proposition 3.4. *We assume that σ is uniformly elliptic and b and σ are in $C_b^{0,3}$. For all $p > 1$, for all multi-index α s.t. $|\alpha| \le 2$, for all $t \in]0,T]$, all $u,r,s \in [0,T]$ and for any functions f and g in $C_b^{|\alpha|}$, there exist a random variable $H_\alpha \in \mathbb{L}^p$ and a function $K(T)$ (uniform in N,x,s,u,r,t,f and g) s.t.*

$$\mathbb{E}[\partial_x^\alpha f(X_t^N) g(X_u^N, X_r^N, X_s^N)] = \mathbb{E}[f(X_t^N)H_\alpha], \tag{3.2}$$

with

$$|H_\alpha|_{\mathbb{L}_p} \le \frac{K(T)}{t^{\frac{|\alpha|}{2}}}\, \|g\|_{C_b^{|\alpha|}}. \tag{3.3}$$

These results are given in the article of Kusuoka and Stroock [6]: (3.3) is owed to Theorem 1.20 and Corollary 3.7.
Another consequence of the duality formula is the derivation of an upper bound for p^N.

Proposition 3.5. *Assume σ is uniformly elliptic and b and σ are in $C_b^{0,2}$. Then, for any $x, y \in \mathbb{R}^d$, $s \in]0,T]$, one has*

$$p^N(0,x;s,y) \le \frac{K(T)}{s^{d/2}} e^{-c\frac{|x-y|^2}{s}}, \tag{3.4}$$

for a positive constant c and a non decreasing function K, both depending on d and on the upper bounds for b, σ and their derivatives.

Although this upper bound seems to be quite standard, to our knowledge such a result has not appeared in the literature before, except in the case of time homogeneous coefficients (see Konakov and Mammen [5], proof of Theorem 1.1).

Proof. The inequality (1.32) of Kusuoka and Stroock [6], Theorem 1.31 gives $p^N(0,x;s,y) \leq \frac{K(T)}{s^{d/2}}$ for any x and y. This implies the required upper bound when $|x-y| \leq \sqrt{s}$. Let us now consider the case $|x-y| > \sqrt{s}$. Using the same notations as in Kusuoka and Stroock [6], we denote $\psi(y) = \rho(\frac{|y-x|}{r})$ where $r > 0$ and ρ is a C_b^∞ function such that $\mathbf{1}_{\{[3/4,\infty[\}} \leq \rho \leq \mathbf{1}_{\{[1/2,\infty[\}}$. Then, combining inequality (1.33) of Kusuoka and Stroock [6], Theorem 1.31 and Corollary 3.7 leads to

$$\sup_{|y-x|\geq r} p^N(0,x;s,y) \leq K(T)\frac{e^{-c\frac{r^2}{s}}}{s^{d/2}}\left(1+\sqrt{\frac{s}{r^2}}\right),$$

where we use $\left\|\psi(X_s^N)\right\|_{1,q} \leq K(T)e^{-c\frac{r^2}{s}}\left(1+\sqrt{\frac{s}{r^2}}\right)$. This easily completes the proof in the case $|x-y| \geq \sqrt{s}$. □

4 Proof of Theorem 2.3

In the following, $K(\cdot)$ denotes a generic non decreasing function (which may depend on d, b and σ). To prove Theorem 2.3, we take advantage of Propositions 3.4 and 3.5. The scheme of the proof is the following

- Use a PDE and Itô's calculus to write the difference $p^N(0,x;s,y) - p(0,x;s,y)$

$$\begin{aligned} &= \int_0^s \mathbb{E}\Bigg[\sum_{i=1}^d (b_i(\varphi(r),X^N_{\varphi(r)}) - b_i(r,X^N_r))\partial_{x_i}p(r,X_r^N;s,y) \\ &+\frac{1}{2}\sum_{i,j=1}^d (a_{ij}(\varphi(r),X^N_{\varphi(r)}) - a_{ij}(r,X^N_r))\partial^2_{x_ix_j}p(r,X_r^N;s,y)\Bigg]\,dr := E_1 + E_2. \end{aligned} \tag{4.1}$$

- Prove the intermediate result $\forall (r,x,y) \in [0,s[\times\mathbb{R}^d\times\mathbb{R}^d$ and $c>0$

$$\mathbb{E}\left[\exp\left(-c\frac{|y-X_r^N|^2}{s-r}\right)\right] \leq K(T)\left(\frac{s-r}{s}\right)^{\frac{d}{2}}\exp\left(-c'\frac{|x-y|^2}{s}\right), \tag{4.2}$$

  where $c' > 0$.

- Use Malliavin's calculus, Proposition 3.5 and the intermediate result, to show that each term E_1 and E_2 (see (4.1)) is bounded by $\frac{K(T)T}{N}\frac{1}{s^{\frac{d+1}{2}}}\exp(-c\frac{|x-y|^2}{s})$.

Definition 4.1. *We say that a term $E(x,s,y)$ satisfies property $\mathcal{P}$ if $\forall (x,s,y) \in \mathbb{R}^d\times]0,T]\times\mathbb{R}^d$*

$$|E(x,s,y)| \leq \frac{K(T)T}{N}\frac{1}{s^{\frac{d+1}{2}}}\exp\left(-c\frac{|x-y|^2}{s}\right). \;(\mathcal{P})$$

4.1 Proof of equality (4.1)

First, the transition density function $(r,x) \longmapsto p(r,x;s,y)$ satisfies the PDE

$$(\partial_r + \mathcal{L}_{(r,x)})p(r,x;s,y) = 0, \;\; \forall r \in [0,s[, \forall x \in \mathbb{R}^d,$$

where $\mathcal{L}_{(r,x)}$ is defined by $\mathcal{L}_{(r,x)} = \sum_{i,j} a_{ij}(r,x)\partial^2_{x_i x_j} + \sum_i b_i(r,x)\partial_{x_i}$, and $a_{ij}(r,x) = \frac{1}{2}[\sigma\sigma^*]_{ij}(r,x)$. The function, as well as its first derivatives, are uniformly bounded by a constant depending on ϵ for $|s-r| \geq \epsilon$ (see Appendix A).
Second, since $p^N(0,x;s,y)$ is a continuous function in s and y (convolution of Gaussian densities), we observe that

$$p^N(0,x;s,y) - p(0,x;s,y) = \lim_{\epsilon\to 0} \mathbb{E}[p(s-\epsilon, X^N_{s-\epsilon}; s,y) - p(0,x;s,y)].$$

Then, for any $\epsilon > 0$, Itô's formula leads to

$$\begin{aligned}\mathbb{E}[p(s-\epsilon, X^N_{s-\epsilon}; s,y) - p(0,x;s,y)] =& \mathbb{E}\left[\int_0^{s-\epsilon} \partial_r p(r, X^N_r; s,y)dr\right] \\ &+ \mathbb{E}\left[\int_0^{s-\epsilon} \sum_{i=1}^d b_i(\varphi(r), X^N_{\varphi(r)})\partial_{x_i} p(r, X^N_r; s,y)dr \right.\\ &\left.+\frac{1}{2}\int_0^{s-\epsilon} \sum_{i,j=1}^d a_{ij}(\varphi(r), X^N_{\varphi(r)})\partial^2_{x_i x_j} p(r, X^N_r; s,y)dr\right].\end{aligned}$$

From the PDE, the above equality becomes

$$\begin{aligned}&\mathbb{E}[p(s-\epsilon, X^N_{s-\epsilon}; s,y) - p(0,x;s,y)] = \\ &\qquad \mathbb{E}\left[\int_0^{s-\epsilon} \sum_{i=1}^d (b_i(\varphi(r), X^N_{\varphi(r)}) - b_i(r, X^N_r))\partial_{x_i} p(r, X^N_r; s,y)dr\right] \\ &\qquad + \frac{1}{2}\mathbb{E}\left[\int_0^{s-\epsilon} \sum_{i,j=1}^d (a_{ij}(\varphi(r), X^N_{\varphi(r)}) - a_{ij}(r, X^N_r))\partial^2_{x_i x_j} p(r, X^N_r; s,y)dr\right], \\ &\qquad := \int_0^{s-\epsilon} \mathbb{E}[\phi(r)]dr,\end{aligned}$$

where $\phi(r) = \sum_{i=1}^d (b_i(\varphi(r), X^N_{\varphi(r)}) - b_i(r, X^N_r))\partial_{x_i} p(r, X^N_r; s,y) + \frac{1}{2}\sum_{i,j=1}^d (a_{ij}(\varphi(r), X^N_{\varphi(r)}) - a_{ij}(r, X^N_r))\partial^2_{x_i x_j} p(r, X^N_r; s,y)$. To get (4.1), it remains to prove that $\mathbb{E}(\phi(r))$ is integrable over $[0,s]$. We check it by looking at the rest of the proof.

4.2 Proof of the intermediate result (4.2)

We prove inequality (4.2). $\mathbb{E}[\exp(-c\frac{|y-X^N_r|^2}{s-r})] = \int_{\mathbb{R}^d} \exp(-c\frac{|y-z|^2}{s-r})p^N(0,x;r,z)dz$. Using Proposition 3.5, we get

$$\begin{aligned}\mathbb{E}\left[\exp\left(-c\frac{|y-X^N_r|^2}{s-r}\right)\right] &\leq \frac{K(T)}{r^{\frac{d}{2}}}\int_{\mathbb{R}^d} \exp\left(-c\frac{|y-z|^2}{s-r}\right)\exp\left(-c'\frac{|x-z|^2}{r}\right)dz \\ &\leq K(T)\Pi_{i=1}^d \int_{\mathbb{R}} \frac{1}{\sqrt{r}}\exp\left(-c\frac{|y_i-z_i|^2}{s-r}\right)\exp\left(-c'\frac{|x_i-z_i|^2}{r}\right)dz_i,\end{aligned}$$

and $\displaystyle\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\frac{(s-r)}{2c}}}\exp(-c\frac{|y_i-z_i|^2}{s-r})\frac{1}{\sqrt{2\pi\frac{r}{2c'}}}\exp(-c'\frac{|x_i-z_i|^2}{r})dz_i$ is the convolution product of the density of two independant Gaussian random variables $\mathcal{N}(-x_i, \frac{r}{2c'})$ and $\mathcal{N}(y_i, \frac{s-r}{2c})$

computed at 0. Hence, the integral is equal to $\frac{1}{\sqrt{2\pi(\frac{r}{2c'}+\frac{s-r}{2c})}}\exp\left(-\frac{|x_i-y_i|^2}{\frac{r}{c'}+\frac{s-r}{c}}\right)$. Then,

$$\int_{\mathbb{R}} \frac{1}{\sqrt{r}} \exp\left(-c\frac{|y_i-z_i|^2}{s-r}\right) \exp\left(-c'\frac{|x_i-z_i|^2}{r}\right) dz_i \le C\left(\frac{s-r}{s}\right)^{\frac{1}{2}} \exp\left(-c''\frac{|x_i-y_i|^2}{s}\right)$$

and (4.2) follows.

4.3 Upper bound for E_1

We recall that $E_1 = \int_0^s \mathbb{E}\left[\sum_{i=1}^d (b_i(\varphi(r), X^N_{\varphi(r)}) - b_i(r, X^N_r))\partial_{x_i} p(r, X^N_r; s, y)\right] dr$. For each i, we apply Itô's formula to $b_i(u, X^N_u)$ between $u = \varphi(r)$ and $u = r$. We get

$$b_i(\varphi(r), X^N_{\varphi(r)}) - b_i(r, X^N_r) = \int_{\varphi(r)}^r \alpha^i_u du + \int_{\varphi(r)}^r \sum_{k=1}^q \beta^{i,k}_u dW^k_u, \tag{4.3}$$

where α^i_u depends on $\partial_t b, \partial_x b, \partial^2_x b, \sigma$, and $\beta^i_u = -\nabla_x b_i(u, X^N_u)\sigma(\varphi(r), X^N_{\varphi(r)})$. Since b, σ belong to $C^{1,3}_b$, α^i and $(\beta^{i,k})_{1\le k\le q}$ are uniformly bounded. Using (4.3) and the duality formula (3.1) yield

$$\begin{aligned} E_1 &= \sum_{i=1}^d \int_0^s \{\mathbb{E}[\int_{\varphi(r)}^r \partial_{x_i} p(r, X^N_r; s, y)\alpha^i_u du + \mathbb{E}[\int_{\varphi(r)}^r \mathcal{D}_u(\partial_{x_i} p(r, X^N_r; s, y)) \cdot \beta^i_u du]\}dr \\ &:= E_{11} + E_{12}, \end{aligned} \tag{4.4}$$

where β^i_u is a row vector of q components. We upper bound E_{11} and E_{12}.

Bound for $E_{11} = \sum_{i=1}^d \int_0^s \mathbb{E}[\int_{\varphi(r)}^r \partial_{x_i} p(r, X^N_r; s, y)\alpha^i_u du]dr$.

Since $|\sum_{i=1}^d \partial_{x_i} p(r, X^N_r; s, y)\alpha^i_u| \le |\alpha_u||\partial_x p(r, X^N_r; s, y)|$ and α_u is uniformly bounded in u, we have

$$|E_{11}| \le C\frac{T}{N}\int_0^s \mathbb{E}|\partial_x p(r, X^N_r; s, y)|dr.$$

Besides that, from Proposition A.2, $|\partial_x p(r, X^N_r; s, y)| \le \frac{K(T)}{(s-r)^{\frac{d+1}{2}}}\exp\left(-c\frac{|y-X^N_r|^2}{s-r}\right)$. Then,

$$|E_{11}| \le K(T)\frac{T}{N}\int_0^s \frac{1}{(s-r)^{\frac{d+1}{2}}}\mathbb{E}\left[\exp\left(-c\frac{|y-X^N_r|^2}{s-r}\right)\right] dr.$$

Using the intermediate result (4.2) yields

$$|E_{11}| \le K(T)\frac{T}{N}\int_0^s \frac{1}{\sqrt{s-r}}\frac{1}{s^{\frac{d}{2}}}\exp\left(-c\frac{|x-y|^2}{s}\right) dr \le K(T)\frac{T}{N}\frac{1}{s^{\frac{d-1}{2}}}\exp\left(-c\frac{|x-y|^2}{s}\right)$$

and thus, E_{11} satisfies property $\mathcal{P}$ (see Definition 4.1).

Bound for $E_{12} = \sum_{i=1}^d \int_0^s \mathbb{E}[\int_{\varphi(r)}^r \mathcal{D}_u(\partial_{x_i} p(r, X^N_r; s, y)) \cdot \beta^i_u du]dr$.

To rewrite E_{12}, we use the expression of β_u^i and Proposition 3.1, which gives $\mathcal{D}_u(\partial_{x_i} p(r, X_r^N; s, y)) = \nabla_x(\partial_{x_i} p(r, X_r^N; s, y))\sigma(\varphi(r), X_{\varphi(r)}^N)$. Then,

$$E_{12} = -\int_0^s dr \int_{\varphi(r)}^r \sum_{i,k=1}^d \mathbb{E}[\partial^2_{x_i x_k} p(r, X_r^N; s, y)[(\sigma\sigma^*)(\varphi(r), X_{\varphi(r)}^N)(\nabla_x b_i(u, X_u^N))^*]_k]du. \quad (4.5)$$

Using the integration by parts formula (3.2), we get that

$$E_{12} = -\int_0^s dr \int_{\varphi(r)}^r \sum_{i,k=1}^d \mathbb{E}[\partial_{x_i} p(r, X_r^N; s, y) H_{e_k}(i)]du$$

where e_k is a vector whose k-th component is 1 and other components are 0. From (3.3), we deduce $\mathbb{E}[|H_{e_k}(i)|^p]^{1/p} \leq C\frac{K(T)}{r^{1/2}}$, where C only depends on $|\sigma|_\infty$, $|\partial_x \sigma|_\infty$, $|\partial_x b|_\infty$, $|\partial^2_{xx} b|_\infty$. By the Hölder inequality, it follows that

$$|E_{12}| \leq K(T)\int_0^s dr \int_{\varphi(r)}^r \frac{1}{r^{1/2}} \mathbb{E}[|\partial_x p(r, X_r^N; s, y)|^{\frac{d+1}{d}}]^{\frac{d}{d+1}} du.$$

Using Proposition A.2 leads to $|\partial_x p(r, X_r^N; s, y)| \leq \frac{K(T)}{(s-r)^{\frac{d+1}{2}}} \exp(-c\frac{|y-X_r^N|^2}{s-r})$, and combining this inequality with the intermediate result (4.2) yields

$$\mathbb{E}[|\partial_x p(r, X_r^N; s, y)|^{\frac{d+1}{d}}]^{d/(d+1)} \leq \frac{K(T)}{(s-r)^{\frac{d+1}{2}}} \left(\frac{s-r}{s}\right)^{\frac{d^2}{2(d+1)}} \exp\left(-c\frac{|y-x|^2}{s}\right). \quad (4.6)$$

Hence, E_{12} is bounded by

$$\frac{K(T)}{s^{\frac{d^2}{2(d+1)}}} \frac{T}{N} \exp\left(-c\frac{|y-x|^2}{s}\right) \int_0^s \frac{1}{r^{1/2}} \frac{1}{(s-r)^{\frac{d+1}{2} - \frac{d^2}{2(d+1)}}} dr.$$

The above integral equals $s^{\frac{1}{2} - \frac{d+1}{2} + \frac{d^2}{2(d+1)}} B(\frac{1}{2}, \frac{1}{2(d+1)})$ where B is the function Beta. Thus $|E_{12}| \leq \frac{K(T)}{s^{d/2}} \frac{T}{N} \exp(-c\frac{|y-x|^2}{s})$, and E_{12} satisfies property $\mathcal{P}$.

4.4 Upper bound for E_2

We recall $E_2 = \frac{1}{2}\int_0^s \mathbb{E}[\sum_{i,j=1}^d (a_{ij}(\varphi(r), X_{\varphi(r)}^N) - a_{ij}(r, X_r^N))\partial^2_{x_i x_j} p(r, X_r^N; s, y)]dr$. As we did for E_1, we apply Itô's formula to $a_{ij}(u, X_u^N)$ between $\varphi(r)$ and r. We get $a_{ij}(\varphi(r), X_{\varphi(r)}^N) - a_{ij}(r, X_r^N) = \int_{\varphi(r)}^r \gamma_u^{ij} du + \int_{\varphi(r)}^r \delta_u^{ij} dW_u$, where γ_u^{ij} depends on $\sigma, \partial_t\sigma, \partial_x\sigma, b, \partial^2_{xx}\sigma$ and δ_u^{ij} is a row vector of size q, with l-th component $(\delta_u^{ij})_l = -\sum_{k=1}^d \partial_{x_k} a_{ij}(u, X_u^N)\sigma_{kl}(\varphi(r), X_{\varphi(r)}^N)$. Then, the duality formula (3.1) leads to

$$\begin{aligned} E_2 = \sum_{i,j=1}^d \int_0^s \{\mathbb{E}[\int_{\varphi(r)}^r \partial^2_{x_i x_j} p(r, X_r^N; s, y)\gamma_u^{ij} du + \mathbb{E}[\int_{\varphi(r)}^r \mathcal{D}_u(\partial^2_{x_i x_j} p(r, X_r^N; s, y)) \cdot \delta_u^{ij} du]\}dr \\ := E_{21} + E_{22}. \end{aligned}$$

Bound for $E_{21} = \sum_{ij=1}^d \int_0^s \mathbb{E}[\int_{\varphi(r)}^r \partial^2_{x_i x_j} p(r, X_r^N; s, y)\gamma_u^{ij} du]dr$**.**
As $\sigma, b, \partial_t\sigma, \partial_x\sigma, \partial_x^2\sigma$ are C_b^1 in space, γ_u^{ij} has the same smoothness properties as the term $[(\sigma\sigma^*)(\varphi(r), X^N_{\varphi(r)})(\nabla_x b_i(u, X_u^N))^*]_k$ appearing in (4.5). Thus, E_{21} can be treated as E_{12} and satisfies to the same estimate.

Bound for $E_{22} = \sum_{i,j=1}^d \int_0^s \mathbb{E}[\int_{\varphi(r)}^r \mathcal{D}_u(\partial^2_{x_i x_j} p(r, X_r^N; s, y)) \cdot \delta_u^{ij} du]dr.$

To rewrite E_{22}, we use the expression of δ_u^{ij} and Proposition 3.1, which asserts $\mathcal{D}_u(\partial^2_{x_i x_j} p(r, X_r^N; s, y)) = \nabla_x(\partial^2_{x_i x_j} p(r, X_r^N; s, y))\sigma(\varphi(r), X^N_{\varphi(r)})$. Thus,

$$E_{22} = -\sum_{i,j,k=1}^d \int_0^s dr \int_{\varphi(r)}^r \mathbb{E}[\partial^3_{x_i x_j x_k} p(r, X_r^N; s, y)[(\sigma\sigma^*)(\varphi(r), X^N_{\varphi(r)})(\nabla_x a_{ij}(u, X_u^N))^*]_k]du.$$

To complete this proof, we split E_{22} in two terms : E_{22}^1 (resp E_{22}^2) corresponds to the integral in r from 0 to $\frac{s}{2}$ (resp. from $\frac{s}{2}$ to s).

- On $[0, \frac{s}{2}]$, E_{22}^1 is bounded by $C\frac{T}{N}\int_0^{\frac{s}{2}} \mathbb{E}[|\partial^3_{x_i x_j x_k} p(r, X_r^N; s, y)|]dr$. Using Proposition A.2 and (4.2), it gives

$$|E_{22}^1| \le \frac{K(T)T}{N}\frac{1}{s^{d/2}}\exp\left(-c\frac{|x-y|^2}{s}\right)\int_0^{\frac{s}{2}} \frac{1}{(s-r)^{3/2}}dr.$$

  Hence, E_{22} satisfies $\mathcal{P}$.

- On $[\frac{s}{2}, s]$, we use the integration by parts formula (3.2) of Proposition 3.4, with $|\alpha| = 2$.

$$E_{22}^2 = -\sum_{i,j,k=1}^d \int_{\frac{s}{2}}^s dr \int_{\varphi(r)}^r \mathbb{E}[\partial_{x_i} p(r, X_r^N; s, y) H_{e_{jk}}(i)]du,$$

  where e_{jk} is a vector full of zeros except the j-th and the k-th components. Using Hölder's inequality and (3.3) (remember that $\sigma \in C_b^{1,3}$), we obtain

$$|E_{22}^2| \le K(T)\frac{T}{N}\int_{\frac{s}{2}}^s \frac{1}{r}\mathbb{E}[|\partial_x p(r, X_r^N; s, y)|^{\frac{d+1}{d}}]^{\frac{d}{d+1}}dr. \tag{4.7}$$

  By applying (4.6), we get

$$|E_{22}^2| \le K(T)\frac{T}{N}\frac{1}{s^{1+\frac{d^2}{2(d+1)}}}\exp\left(-c\frac{|x-y|^2}{s}\right)\int_{\frac{s}{2}}^s \frac{1}{(s-r)^{\frac{2d+1}{2d+2}}}dr,$$

  and the result follows.

A Bounds for the transition density function and its derivatives

We bring together classical results related to bounds for the transition probability density of X defined by (1.1).

Proposition A.1 (Aronson [1]). *Assume that the coefficients σ and b are bounded measurable functions and that σ is uniformly elliptic. There exist positive constants K, α_0, α_1 s.t. for any x, y in $\mathbb{R}^d$ and any $0 \le t < s \le T$, one has*

$$\frac{K^{-1}}{(2\pi\alpha_1(s-t))^{\frac{d}{2}}} e^{-\frac{|x-y|^2}{2\alpha_1(s-t)}} \le p(t,x;s,y) \le K \frac{1}{(2\pi\alpha_2(s-t))^{\frac{d}{2}}} e^{-\frac{|x-y|^2}{2\alpha_2(s-t)}}. \tag{A.1}$$

Proposition A.2 (Friedman [3]). *Assume that the coefficients b and σ are Hölder continuous in time, C_b^2 in space and that σ is uniformly elliptic. Then, $\partial_x^{m+a}\partial_y^b p(t,x;s,y)$ exist and are continuous functions for all $0 \le |a| + |b| \le 2, |m| = 0, 1$. Moreover, there exist two positive constants c and K s.t. for any x, y in $\mathbb{R}^d$ and any $0 \le t < s \le T$, one has*

$$|\partial_x^{m+a}\partial_y^b p(t,x;s,y)| \le \frac{K}{(s-t)^{(|m|+|a|+|b|+d)/2}} \exp\left(-c\frac{|y-x|^2}{s-t}\right).$$

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